ME231
Measurements Laboratory
Spring 1999

Dynamic Response of Measurement Systems

Edmundo Corona ©

The handout "Getting Ready to Measure" presented a couple of examples where the results of a static calibration were not sufficient to ensure accurate results when using the instruments in dynamic applications. When the measured quantity changes rapidly, it is important to consider the dynamic response of measuring instruments. This handout addresses the response of instruments under such operating conditions.

In general, the response of measurement instruments under dynamic conditions can be complex. The fundamental concepts of dynamic response, however, can be understood by studying relatively simple mathematical models.

Mathematical Models of System Response

We will consider three mathematical models for dynamic system response: zeroth, first and second order systems.

Zeroth Order Systems

Imagine a thermometer that measures the temperature in a room of an office building. For practical purposes, the thermometer will indicate the current temperature at the location where it has been installed. The fact that the output of the instrument follows the input "exactly" is the defining characteristic of zeroth order systems.

Mathematically, if we let $f(t)$ be the input to the system as a function of time and $y(t)$ be the output, then the relationship between them is

$$y(t) = Kf(t). \tag{1}$$

In the example above $f(t)$ would be the actual temperature of the room and $y(t)$ would be the indicated temperature. K is a constant that multiplies the input to generate the output. If $y(t)$ is the temperature as displayed in a readout device and the thermometer is calibrated correctly, then K would ideally be equal to one. On the other hand, if the output of the thermometer is, for example, an electrical signal, then K would be a constant with units of Volts per degree Fahrenheit. The electrical signal could be used to actuate a valve that directs either cold or hot air from the air-conditioning system to the room. K is usually called the *static sensitivity*. Note that the output of zeroth order systems is not affected by the speed at which $f(t)$ changes. Equation (1) is always valid, so the results of static calibration are sufficient to characterize the response of the system.

First Order Systems

Let's consider the example of an oral thermometer used at a clinic to measure body temperature in patients. Prior to use, the thermometer is at room temperature. When the thermometer is put in the patient's mouth it experiences a sudden increase in temperature. Generally, we have to wait for a while before reading the temperature. Unlike the case of a thermometer monitoring the examination room's temperature, the situation at hand cannot be represented by

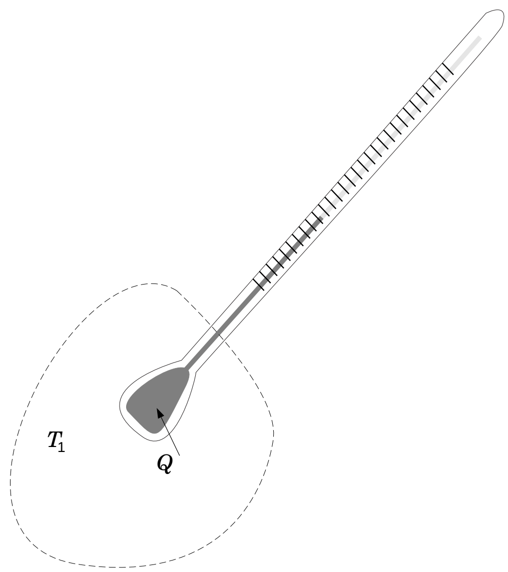

Figure 1: Glass bulb thermometer.

a zeroth order model. Why? Let's consider a common glass bulb thermometer to explain.

The thermometer in Fig. 1 was originally at room temperature, which will be denoted by T_o. It is then put in the mouth of a patient—represented by the area inside the dashed line— which is at temperature T_1. In order for the thermometer to work, the mercury in the bulb must be heated to T_1. The thermal expansion of the mercury will cause the column of mercury in the stem of the thermometer to increase in length. Measuring this length with the scale marked in the glass gives a temperature reading $T(t)$. It takes a while, however, for the temperature of the mercury to reach the value T_1, so we must wait that long before the thermometer indicates the correct temperature. In order to write an equation to model the response of the thermometer we need a little background in thermodynamics and heat transfer.

- Heat is a form of energy. We will represent it by Q. It flows from a hot place to a cold one. The energy in the mercury in the bulb of the thermometer, which we will call E, increases as heat travels from the mouth of the patient to the bulb. Due to conservation of energy, the rate of change of energy in the bulb with respect to time is equal to how fast heat is flowing in. In mathematical terms:

$$\frac{dE}{dt} = \frac{dQ}{dt}, \tag{2}$$

where t represents time.

- As E increases, the temperature of the mercury, T, rises in proportion. How fast the temperature increases depends on how much mercury the bulb holds (the mass, m, of the mercury) and a property of mercury called the 'specific heat,' c_v. The increase in energy is related to the increase in temperature by

$$\frac{dE}{dt} = mc_v\frac{dT}{dt}. \tag{3}$$

- Heat must travel through the glass walls of the bulb on its way from the patient's mouth to the mercury. How fast heat can flow through the walls of the bulb depends on a property of glass called the convection heat transfer coefficient, h, the surface area of the bulb, A, and the current temperature difference between the mercury and the mouth of the patient. In equation form,

$$\frac{dQ}{dt} = hA[T_1 - T]. \tag{4}$$

- Substituting (4) into (2) and the result into (3) we obtain

$$mc_v\frac{dT}{dt} = hA[T_1 - T], \tag{5}$$

which can be re-written as

$$\frac{mc_v}{hA}\frac{dT}{dt} + T = T_1. \tag{6}$$

This is a differential equation that governs what the temperature of the mercury is at any time. Since the length of the column of mercury is proportional the temperature, the equation also governs what the indicated temperature is.

In general, the equation of a first order system is given by

$$\tau \frac{dy}{dt} + y = Kf(t). \tag{7}$$

For the example above, we have

$$y = T,$$

$$f(t) = T_1,$$

$$\tau = \frac{mc_v}{hA}$$

and

$$K = 1.$$

As you can see, the equation governing the behavior of a first order system is a first order differential equation, so called because the highest derivative of the output variable in the equation is the first with respect to time.

Second Order Systems

When you go to the grocery store, note that weight scales are available at the produce department so you can get an idea how many pounds of potatoes you will have to pay for. The scales generally have a pointer, which indicates weight on a big dial. If you drop a bag with potatoes on the scale, chances are that the pointer will oscillate a bit before settling and indicating the correct weight. The reason for the oscillation of the pointer is that the scale has mass, hence

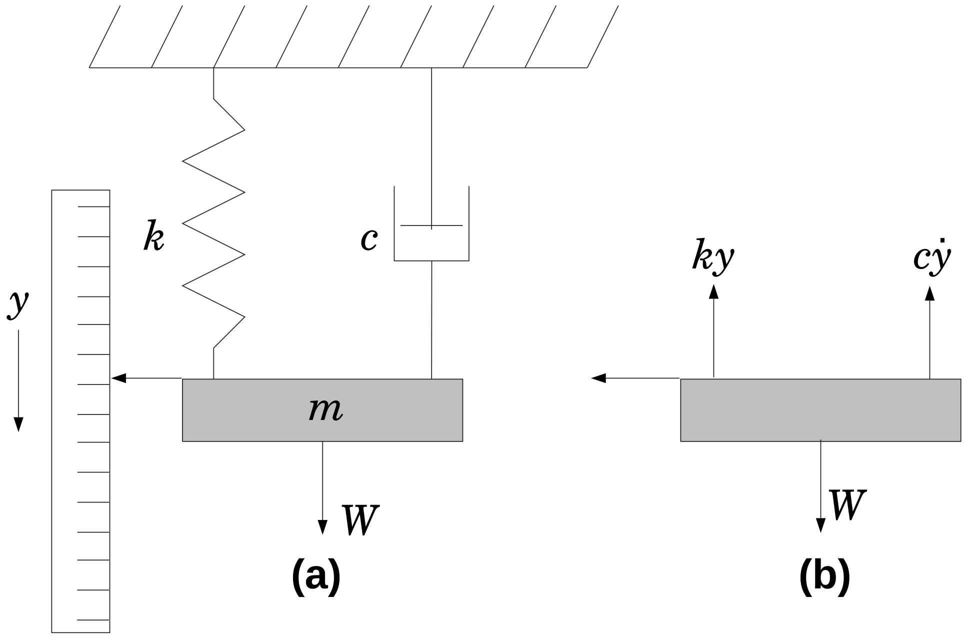

Figure 2: (a) Model of weight balance. (b) Free body diagram.

inertia, and its behavior is dictated by Newton's laws of motion. You may now see where the second order label comes from if you recall that acceleration is the second derivative of displacement ... Let's analyze the balance and see what we get.

The balance can be represented by the mass-spring-dashpot system shown in Fig. 2(a). To simplify things a bit, the dial has been replaced by the straight scale at the left. We can write the equation of motion for the balance using basic dynamics as follows:

- The mass m represents the object weighted so that the weight W is equal to mg where g is the acceleration of gravity.

- The spring, attached near the left edge of the mass, is a mechanical element which develops force in proportion to how much it has been stretched. Since the top of the spring is fixed to the ceiling, the force is proportional to the displacement y of the mass. If the zero value of y corresponds to the position of the spring when it is unloaded, then the force F_s required to stretch the spring a distance y is given by

$$F_s = ky \tag{8}$$

where k is called the *spring constant*.

- The balance needs a way to dampen the oscillations of the pointer after a weight is dropped. The damping is provided by the dashpot, which is attached to the mass near the right. Dashpots are like shock absorbers in cars. You can imagine one as a piston inside a closed cylinder. The cylinder is filled with a viscous fluid. As the piston moves, a small gap between the piston and the cylinder allows the fluid to flow from one side of the piston to the other. A force is needed to move the piston due to the viscosity of the fluid. The net result is that dashpots produce a force, F_d which is proportional to the speed of the piston relative to the cylinder. This force-speed relation can be written as

$$F_d = c\frac{dy}{dt} \tag{9}$$

where c is called the *damping coefficient*.

- The free body diagram of the mass is shown in Fig. 2(b). Applying Newton's second law we obtain the equation

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = W, \tag{10}$$

which can be re-written as

$$\frac{m}{k}\frac{d^2y}{dt^2} + \frac{c}{k}\frac{dy}{dt} + y = \frac{1}{k}W. \tag{11}$$

This is the differential equation that governs the motion of the scale. Since the weight is indicated by the displacement of the scale, the equation also governs the indicated weight.

In general, the equation for a second order system is given by

$$\frac{1}{\omega_n^2}\frac{d^2y}{dt^2} + \frac{2\zeta}{\omega_n}\frac{dy}{dt} + y = Kf(t). \tag{12}$$

In the example of the scale above we have

$$\omega_n = \sqrt{\frac{k}{m}},$$

$$\zeta = \frac{c}{2\sqrt{km}}$$

and

$$K = \frac{1}{k}.$$

Solution to the Differential Equations

Writing down the differential equations (7) and (12) for first and second order systems was nice, but in order to see how these systems behave for a given input we need to solve them. The solutions to the differential equations depend, of course, on how the input varies with time. Does it increase, decrease or cycle? Smoothly or abruptly? If smoothly, are the changes slow or fast? To make the long story short, although an infinite number of possible inputs exist, we will consider only two, which are most telling about the characteristics of the system response. These two inputs are: the step input and the harmonic input.

We will not go over the method of solution of the differential equations, instead we will just quote the results. If you know how to solve differential equations, it is reasonably straight forward to obtain the solutions. If not, there is no need to learn now, just make sure you can use the solutions presented below.

First Order Systems

Step input response

A step input is used to represent situations when the input changes from one, constant, value to another constant value in a very short period of time. The two examples above—the thermometer in a patients' mouth and the potatoes dropped on the weight scale—can be thought of as step inputs. Mathematically a step input in $f(t)$ from F_o to F_1 applied at $t = 0$ is given by:

$$f(t) = \begin{cases} F_o & : \quad t < 0 \\ F_1 & : \quad t \geq 0 \end{cases} . \tag{13}$$

The application of the step input causes the output to change from a initial value y_o to a final value y_1. Then, the solution to (7) is

$$y(t) = KF_1 - (KF_1 - y_o)e^{-t/\tau}. \tag{14}$$

Note that at $t = 0$, $y(t) = y_o$, and that as $t \to \infty$, $y(t) \to KF_1 = y_1$. The transition between the initial and the final values of $y(t)$ is exponential as shown in Fig. 3. The parameter τ is called the *time constant* of the system. It indicates how fast the system responds to a change in the input. Note that after a time of one time constant, $y(t)$ is 63.2% of the way to its final value. Generally, $y(t)$ is considered to have achieved its final value after five time constants. As an example, suppose that an oral thermometer takes at least 3 minutes to indicate a patient's temperature. Then we can guess that the time constant of the thermometer is in the order of 0.6 minutes.

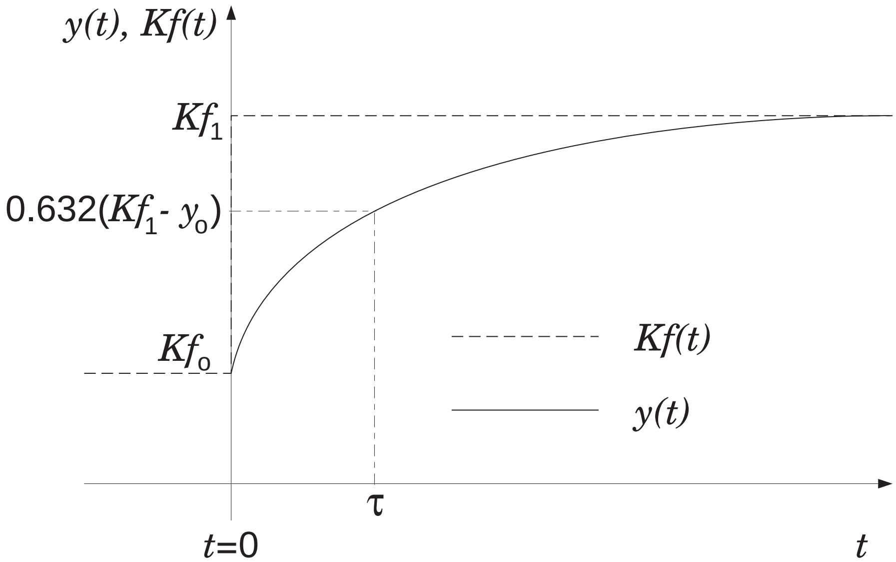

Figure 3: Response of a first order system to a step input response.

Harmonic input response

The word harmonic always makes me think of music, but in the context of dynamic system response it has a different meaning. It means that the input is a sine wave with constant frequency and amplitude. In other words the input is of the form(s)

$$f(t) = A \sin \frac{2\pi t}{T} = A \sin 2\pi f t = A \sin \omega t. \tag{15}$$

All three definitions are equivalent. In the first, T is the period in seconds. In the second, f is the frequency in cycles per second or Hertz. In the third, ω is the circular frequency in radians per second. The relations between T, f and ω can be easily deduced from (15).

Why should we be concerned with the response of measurement systems to harmonic inputs? The answer is that many natural and man-made processes are periodic in nature. For example, think about day and night, the seasons repeating every year, the waves at the beach, etc. In the engineering world think about the vibration of a motor, the movement of pistons in an internal combustion engine, the turning of wheels, the air conditioning system at your house turning on and off, and so on. Some of these periodic events are complex in nature and hard to study. To get an idea of how measurement systems may respond under periodic inputs, we should study the simplest one, which is the harmonic input. I will also let you know a little secret: many complex, periodic signals are made up of simple sine waves added together. We will look at this a little later.

The steady state[1] solution to (7) with the input (15) is

$$y = MKA\sin(\omega t - \phi) \tag{16}$$

where

$$M = \frac{1}{\sqrt{1 + (\tau\omega)^2}} \tag{17}$$

is called the *magnitude ratio* and

$$\phi = \arctan \tau\omega \tag{18}$$

is called the phase angle.

Note that the frequencies of the input and the output are the same, ω. The magnitude ratio and the phase angle are plotted as functions of frequency in Figure 4. In words, the magnitude ratio tells us how the amplitude of the output of the system depends on the frequency of the input. The phase angle tells us how far the system output is falling behind the input. Note that ϕ is measured in radians. If we want to know the time increment, in seconds, by which the output lags the input, we need to re-write the argument of the sine in

[1] Steady state means that we are looking at the response of the system after the harmonic input has been going for a while. How long? How about longer than five time constants? Do you agree? Why?

(16) as

$$(\omega t - \phi) = \omega(t - \frac{\phi}{\omega}) = \omega(t - \Delta t).$$

Therefore, the time lag, Δt is given by

$$\Delta t = \frac{\phi}{\omega}. \tag{19}$$

For example, in Fig. 5, the input is shown in dashed line and has$A = 1$ and $f = 1$ Hz, which corresponds to $\omega = 2\pi$ rad/s. The output, shown in solid line, has the same frequency as the input but has M=0.5 and $\Delta t = 0.167$ s, or $\phi = 1.05$ rad.

Note, from Fig. 4, that the magnitude ratio remains virtually equal to one if the frequency of the input is low enough so that $\tau\omega$ is below 0.1—recall τ is a constant and is not related to the input. The phase angle also remains rather small in this frequency regime. This means that, for practical purposes, the system behaves in a quasi-static manner if the input has such low frequency. In other words, the response characteristics measured in a static calibration are valid. As the frequency increases, the magnitude ratio decreases and the phase angle increases rapidly, indicating that dynamic effects become important.

Second order systems

Step input response

Now, let us consider the response of a second order system to a step input. In other words we need the solution to (12) with $f(t)$ given by (13). The response of second order systems, as you may imagine, is a bit more complex than that of first order systems. In fact, we need to consider three different cases depending on the value of the parameter ζ in (12), called the *damping ratio*. In the weight scale example we considered above, this relates to how effective the dashpot is in

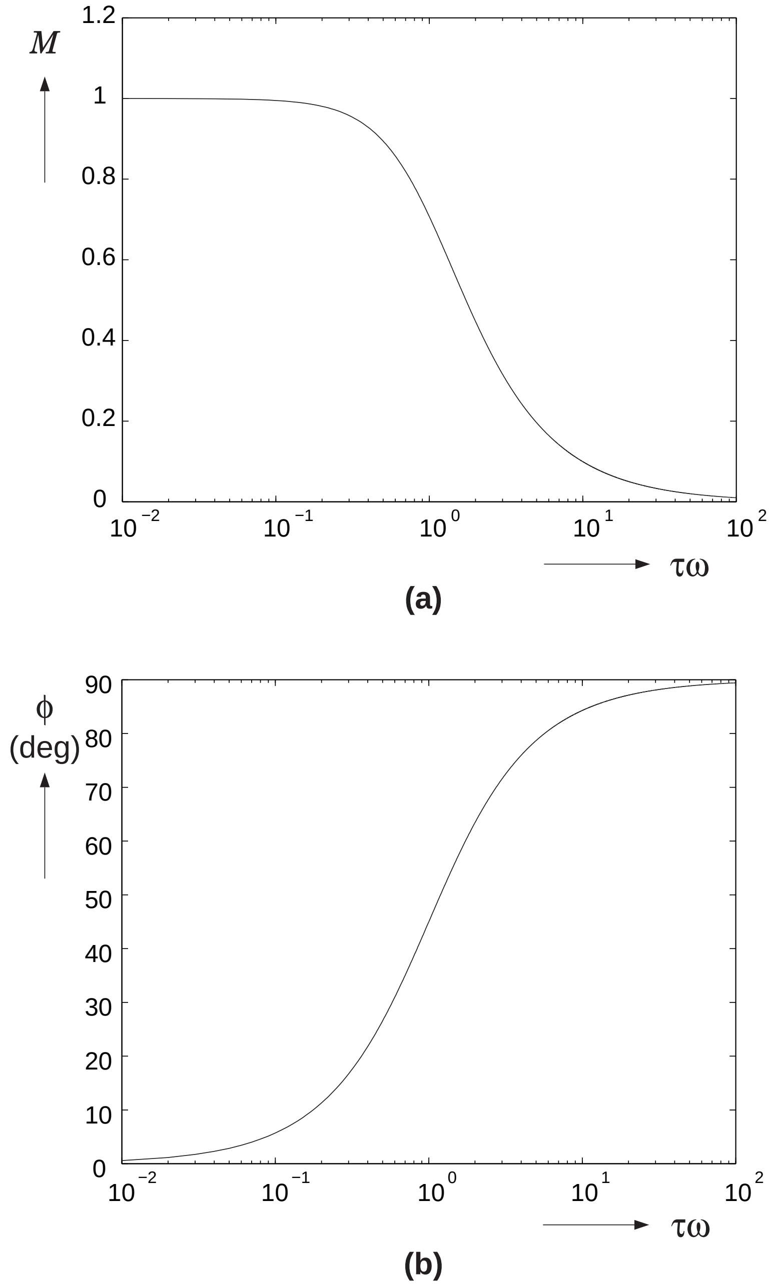

Figure 4: (a) Magnitude ratio for a first order system. (b) Phase angle.

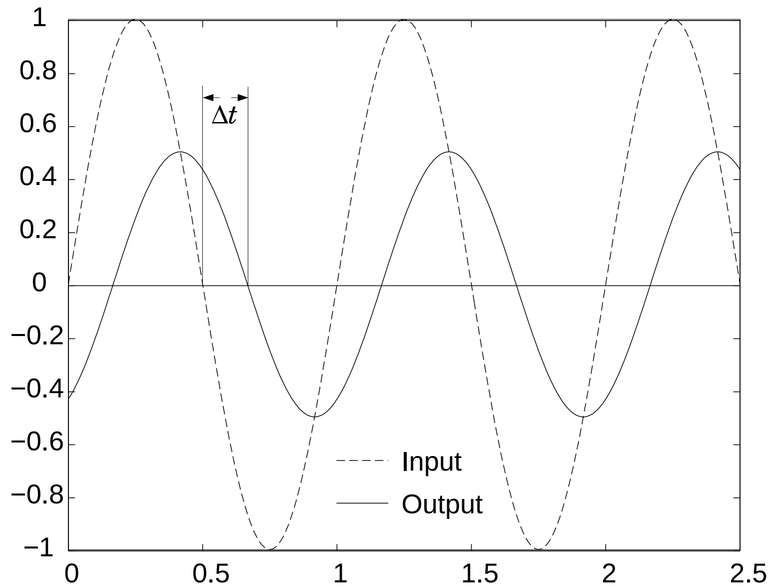

Figure 5: Harmonic input and output for a first order system.

damping the oscillations of the scale or, in other words, on the magnitude of the parameter c. One final note before presenting the three cases: we will assume that the system is at rest with output $y = 0$ when the step input is applied. This also implies that $F_o = 0$ in (13)[2]

Case 1. Underdamped system ($\zeta < 1$). In this case, the solution is

$$y(t) = KF_1 \left[1 - e^{-\zeta\omega_n t} \left(\frac{\zeta}{\sqrt{1-\zeta^2}} \sin \omega_d t + \cos \omega_d t \right) \right]. \tag{20}$$

The sine and cosine terms indicate that the output will oscillate after the step input is applied. The frequency of oscillation is ω_d, which is given by

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}. \tag{21}$$

Note that ω_n is the frequency at which the system would oscillate if no damping were present ($\zeta = 0$). It is called the *natural frequency* and

[2]Those familiar with the solution methods of differential equations will recognize that the conditions of the system just prior to the application of the input are the *initial conditions*. In the current case we have taken $y(0) = 0$ and $\frac{dy(0)}{dt} = 0$.

is a characteristic of the system. The coefficient $e^{-\zeta\omega_n t}$ that multiplies the trigonometric terms decreases as time increases and so dampens the oscillation of the system output. Note that ζ controls how quickly the oscillation is damped out. As $t \to \infty$, $y \to KF_1$, except when $\zeta = 0$. In that case the output will oscillate forever. The response for three cases with $\zeta < 1$ are shown in Fig. 6.

Case 2. Critically damped case ($\zeta = 1$). This is a bit of an unusual case since it requires that ζ be exactly one. In this case, the solution is

$$y(t) = KF_1 \left[1 - (1 + \omega_n t)e^{-\omega_n t}\right]. \tag{22}$$

The claim to fame of this case is that it is the one for which the output approaches the final condition $y = KF_1$ the fastest and without oscillation. The response for this case is also shown in Fig. 6.

Case 3. Overdamped case ($\zeta > 1$). In this case, the solution is

$$y(t) = KF_1 \left[1 - e^{-\zeta\omega_n t}\left(\frac{\zeta}{\sqrt{\zeta^2 - 1}} \sinh \omega_n \sqrt{\zeta^2 - 1}t + \cosh \omega_n \sqrt{\zeta^2 - 1}t\right)\right]. \tag{23}$$

Therefore, overdamped systems do not oscillate when subjected to a step input. Instead, they approach the final value $y = KF_1$ slowly and monotonically. The speed at which y approaches its final value depends on the value of ζ. The higher ζ is, the slower y changes.

Harmonic input response

The steady state response of a second order system to a harmonic input as given by (15) is given by

$$y(t) = MKA \sin(\omega t - \phi) \tag{24}$$

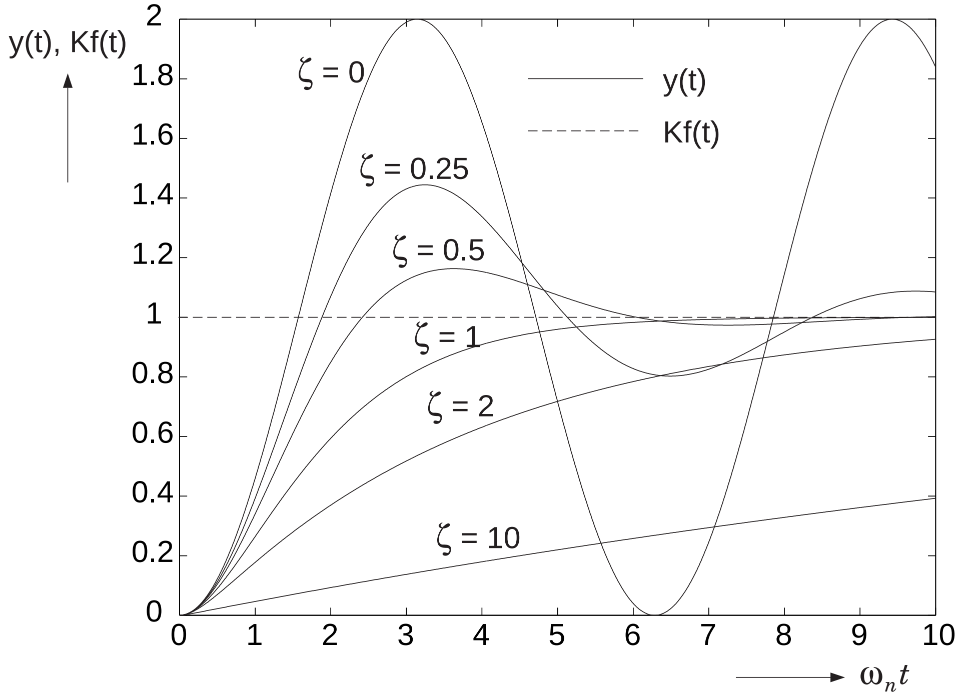

Figure 6: Step input responses of second order systems.

where the magnitude ratio is

$$M = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}}, \tag{25}$$

and the phase angle is given by

$$\phi = \arctan\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2} \tag{26}$$

Like in the first order systems, the frequencies of the input and output are the same. Unlike in the first order system response shown in Fig. 4, where we obtained one curve for the magnitude ratio and the phase angle, the response of the second order system depends on the value of the damping ratio ζ, as shown in Fig. 7. Note that for damping ratios between zero and one, the magnitude ratio is virtually equal to one if the frequency ratio ω/ω_n is smaller than 0.1. The phase angle is also small. This indicates that inputs in this frequency regime would appear to be quasi-static to a second order system.

As the frequency ratio increases, however, dynamic effects become important. For damping ratios smaller than 0.707, the magnitude ratio becomes greater than one in the vicinity of $\omega/\omega_n = 1$. This is called *resonance*. Also note that the phase angle changes rather rapidly around resonance. Have you experienced resonance? I think so. For example, when helping someone in a swing we push him/her once per cycle, at the same frequency as motion of the swing. In addition to the fact that it is convenient (we can stand in one place and do the work there) this is a condition of resonance which makes the swing motion amplitude increase. In another example, some of you may have experienced driving a car with an unbalanced wheel. This can make the car shake. Generally, a speed exists when the shaking is worst. Going faster or slower tends to reduce the shaking. This is because at that particular speed the wheel is rotating near the natural frequency of the suspension of the car, so the vibration is amplified. If, on top of that, the shock absorbers are bad, so that the damping ratio is low, then the car will really be shaking!

For frequency ratios above resonance, the magnitude ratio decreases rather rapidly and the phase angle approaches 180°. Systems with damping ratios greater than 0.707 do not exhibit resonance. Instead, the magnitude ratio decreases monotonically as the frequency ratio increases. The phase angle changes more gradually from zero to 180°.

The 'best' damping ratio for a measuring instrument is $\zeta = 0.707$ because the magnitude ratio stays close to one for the largest range of frequency. Therefore manufacturers of measurement systems are careful design their products so that they are 'tuned' in terms of natural frequency and damping ratio

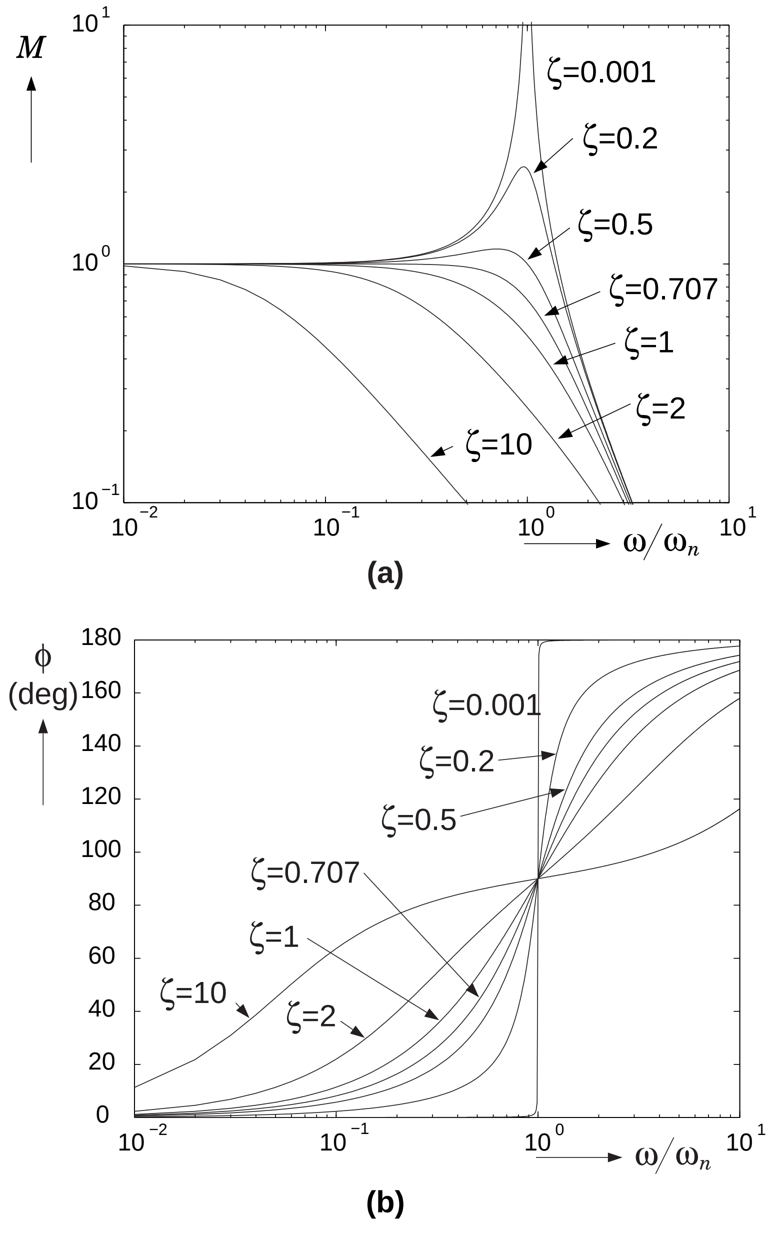

Figure 7: (a) Magnitude ratio for second order systems. (b) Phase angle.

for maximum performance.

Remarks

- Always keep in mind that the zeroth, first and second order systems presented in this handout are mathematical idealizations of system response. An actual instrument may be modeled differently under different conditions. For example, a given instrument may be considered to be zeroth order if the input quantity varies very slowly. The same instrument, when used in an application when the input changes rapidly may have to be considered to be a first, second or maybe even higher order system. The idea is to use the simplest model that will describe the behavior of an instrument within the accuracy required for a given application.

- Also note that the differential equations (7) and (12) are linear. This means that superposition can be used. In other words, if the input to the system can be written as

$$f(t) = f_1(t) + f_2(t) \tag{27}$$

the output will be given by

$$y(t) = y_1(t) + y_2(t) \tag{28}$$

where $y_1(t)$ is the solution when the input is $f_1(t)$ alone and $y_2(t)$ is the solution when the input is $f_2(t)$ alone.