ME231
Measurements Laboratory
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Fourier Series

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If you listen to music you may have noticed that you can tell what instruments are used in a given song or symphony. In some cases, the melody is sequentially played by different instruments. For example, you may hear it played by violins and a little later repeated by flutes. Although the violins and the flutes may play the same exact notes, you can definitely tell when the violins or the flutes are playing. Why? The answer is that, when two different instruments play the same note, the pitch or frequency of the sound waves are the same, but the shape of the sound wave is different. For example, the sound wave of one instrument may have the shape of a sine function while the sound wave of the second one may be a square wave as shown in Fig. 1 for two hypothetical instruments A and B[1].

Looking at time record of the sound wave of different musical instruments may be interesting on its own right but it is strictly a qualitative exercise. The question is, can we quantitatively characterize such waves or, in other words, write them as mathematical expressions. In the case of wave A in Fig. 1 we can write

$$y = A \sin 2\pi f t.$$

In the case of wave B it is harder to write an expression valid for any time. The

[1] Of course, the sound waves of actual musical instruments are much more complex than sine and square waves.

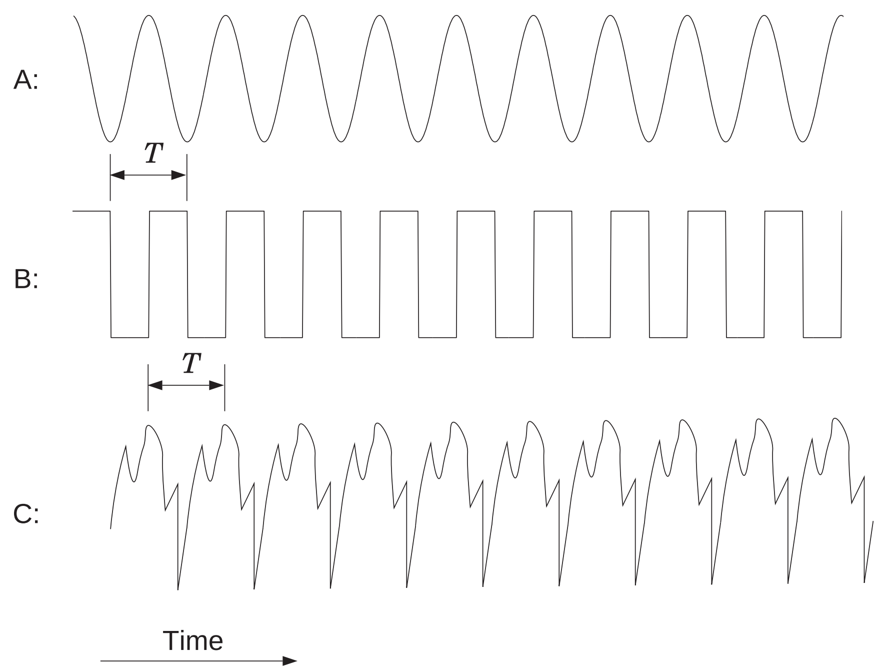

Figure 1: Sound waves of three hypothetical instruments.

best we can do is write an expression for one period as

$$y = \begin{cases} A & : \quad 0 \le t < T/2 \\ -A & : \quad T/2 \le t < T \end{cases}$$

and then announce it is periodic. Obviously, this approach gets to be hopeless as waves get more complex. For example, how would you write an expression for the wave C in Fig. 1?

In 1807, Joseph Fourier proposed the first systematic way to answer the question above. He stated that a completely arbitrary periodic function $f(t)$ could be expressed as a series of the form

$$f(t) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi t}{T} + b_n \sin \frac{2n\pi t}{T} \right) \qquad (1)$$

where n is a positive integer, T is the fundamental period of the function, defined as shown in Fig. 1, and a_o, a_n and b_n are constant coefficients that we will determine later. Imagine the surprise of many with such proposal stating that even discontinuous functions such as square waves could be represented by beautifully smooth sines and cosines! It is said that even Lagrange was so surprised the he categorically denied such possibility. Of course, Fourier was eventually proven right, otherwise we would not be concerned with this now.

The expressions for the coefficients a_o, a_n and b_n are derived in many books[2] and are given by:

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos \frac{2n\pi t}{T} dt, \qquad n = 0, 1, 2, \ldots \qquad (2)$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin \frac{2n\pi t}{T} dt, \qquad n = 1, 2, 3, \ldots \qquad (3)$$

For example, to find the Fourier series for a triangular wave as shown in Fig. 2 we would calculate the coefficients as follows:

[2] See, for example, Boyce and DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 3rd Edition, John Wiley & Sons, 1977.

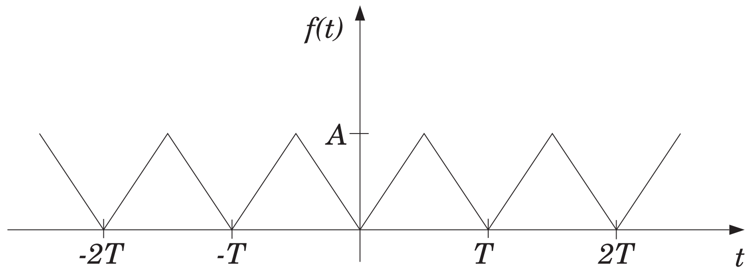

Figure 2: Triangular wave.

- The function $f(t)$ in the range $-T/2 \leq t \leq T/2$ is given by

$$f(t) = \begin{cases} -\frac{2At}{T} & : \quad -T/2 \leq t < 0 \\ \frac{2At}{T} & : \quad 0 \leq t < T/2 \end{cases} .$$

- The coefficient a_o is given by

$$a_o = \frac{4A}{T^2} \left[\int_{-T/2}^{0} -t dt + \int_{0}^{T/2} t dt \right]$$

so

$$a_o = A$$

- The coefficients a_n are given by

$$a_n = \frac{4A}{T^2} \left[\int_{-T/2}^{0} -t \cos \frac{2n\pi t}{T} dt + \int_{0}^{T/2} t \cos \frac{2n\pi t}{T} dt \right]$$

so

$$a_n = \frac{2A}{(n\pi)^2} (\cos n\pi - 1), \qquad n = 1, 2, \ldots$$

- The coefficients b_n are given by

$$b_n = \frac{4A}{T^2} \left[\int_{-T/2}^{0} -t \sin \frac{n\pi t}{T} dt + \int_{0}^{T/2} t \sin \frac{n\pi t}{T} dt \right]$$

so

$$b_n = 0, \qquad n = 1, 2, \ldots$$

- The Fourier series is then

$$f(t) = \frac{A}{2} - \frac{4A}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{2(2n-1)\pi t}{T}.$$

Note that the upper limit of the series is ∞. This says that an infinite number of terms in the series is required to represent the triangular wave. The series does not seem very useful, but we are saved by the fact that it converges rather rapidly. For example, Fig. 3 compares the approximation obtained with truncated series (solid) with the actual triangular wave (dashed line). In this figure, A has been taken as one and T has been taken as one Hertz. N denotes the upper limit of the series. Note that as we take more terms, the approximation becomes better and better. Taking as few as four terms gives a very good approximation to the actual triangular wave.

The most intuitive way to represent a function of time is, perhaps, to plot it with time in the horizontal axis and the value of the function in the vertical axis. This is what we have done in Figs. 2 and 3 and is called the time representation of the function. The Fourier approach, however, suggests an alternative. We can plot the frequency of each trigonometric term in the horizontal axis and the the value of the corresponding coefficient in the vertical axis. This plot is called the frequency representation of the function or the amplitude spectrum. The amplitude spectrum for the first few terms of the triangular wave above is shown in Fig. 4. Note that the amplitude decreases rapidly as the frequency increases. This means that the higher frequencies are not as important as the lower ones. One of the best examples of the Fourier representation of a quantity is a rainbow, which gives us the spectrum of colors contained in white light.

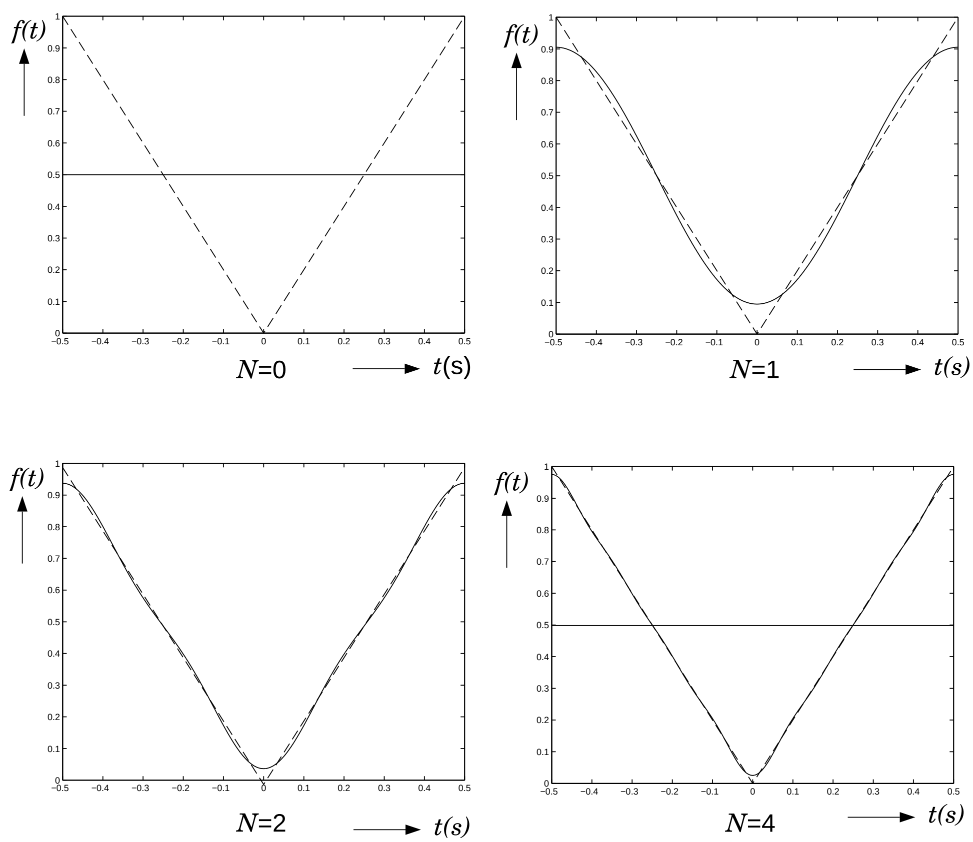

Figure 3: Fourier representation of a triangular wave when the series is truncated at the Nth term.

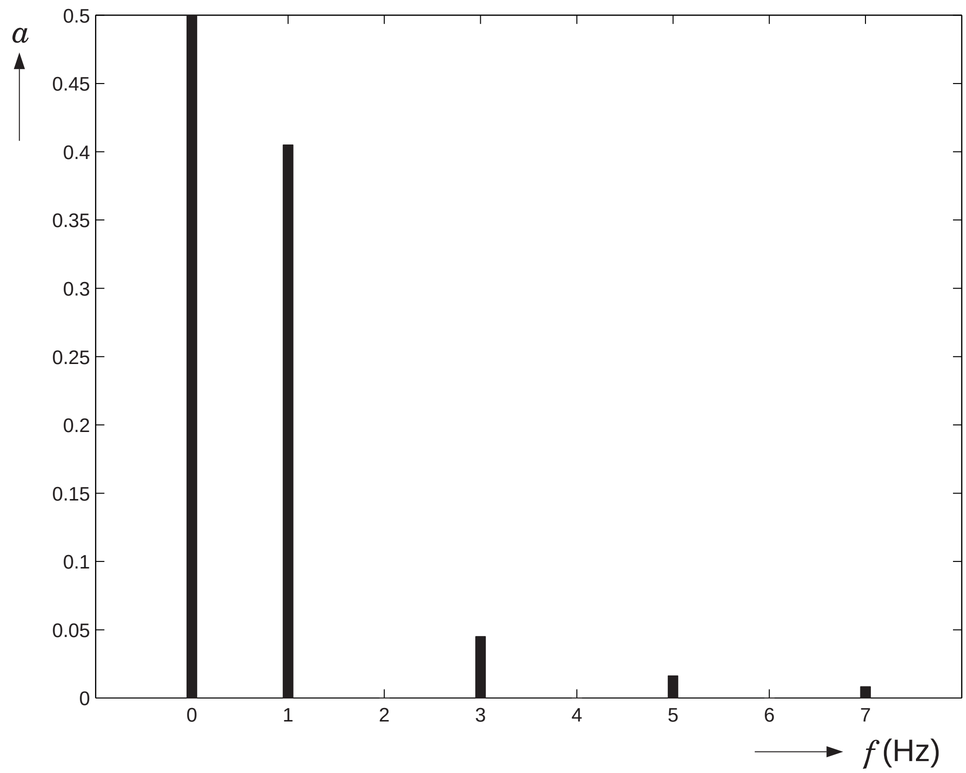

Figure 4: Amplitude spectrum of the triangular wave discussed.

Exercises

1. Derive the Fourier series for a square wave with period T, amplitude A and zero mean. Generate plots similar to those in Figs. 3 and 4 for $A = 5$ and $T = 0.3$ s.