BV Lecture 1; 06.10.15

St. phase expansions

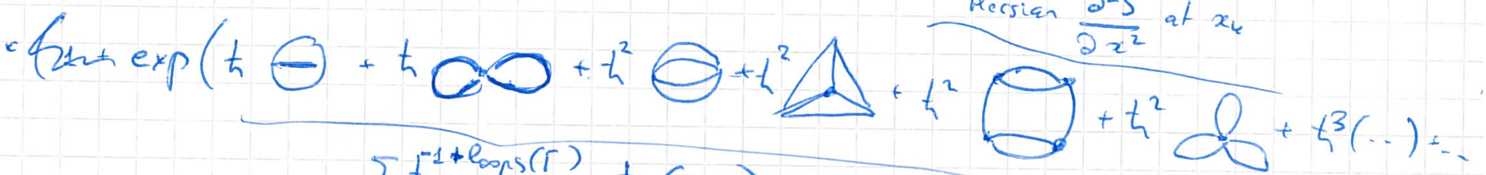
In QFT, one is interested in expressions of form $\int_X e^{\frac{i}{\hbar} S(x)} O_1(x) \dots O_n(x)$ (possibly $n=0$), $\hbar$ - small parameter $\in \mathbb{R}_{>0}$

Stationary phase expansion
(take X to be a smooth, compact mfld)

$$I \underset{\hbar \to 0}{\sim} \sum_{\substack{\text{crit. points} \\ x_k \text{ of } S}} I^{\text{st. phase}}_{x_k}, \quad I^{\text{st. phase}}_{x_k} = e^{\frac{i}{\hbar} S(x_k)} \det{}^{-\frac{1}{2}} \underbrace{B_{x_k}}_{\in \mathrm{End}(T_{x_k} M)} \cdot e^{\frac{\pi i}{4} \cdot \mathrm{sign}\, B_{x_k}} \times$$

take $n=0$

- comes from the Hessian $\frac{\partial^2 S}{\partial x^2}$ at x_k



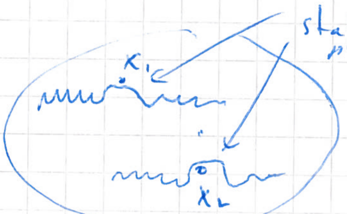
$$\sum_{\substack{\text{connected graphs with valency} \\ \geq 3 \text{ at every vertex } \Gamma}} \hbar^{-1+\mathrm{loops}(\Gamma)} \cdot \underbrace{\Phi_{x_k}(\Gamma)}$$

weight of the graph Γ - function of finite jets of S at x_k

Toy Ex: $\int_{\mathbb{R}} e^{\frac{i}{\hbar}\left(\frac{x^2}{2} + \frac{x^3}{6}\right)} dx = \dots$ sum of 3-valent graphs with trivial $\left(\frac{1}{|\mathrm{Aut}\,\Gamma|}\right)$ weights

$\sim$ Airy function

stationary phase points give contribute "most" to I



- In actual QFTs, there is also a structure of locality w.r.t. some "spacetime" mfld M, and $X = \Gamma(E_M)$ - sections of some bundle (sheaf) on M - "fields"

Gauge systems:

$\mathcal{G} \circlearrowright X$, $S \in C^\infty(X)^{\mathcal{G}}$

Lie group; symmetry

[more generally, infinitesimal gauge sym. - distribution on X]

- Ex: $X = \mathrm{Conn}\left(\begin{smallmatrix} P \\ \downarrow \\ M \end{smallmatrix}\right)$ (G-bundle; Riemannian mfld), $S_{YM} = \|F_A\|^2 = \int_M F \wedge * F$ (Hodge norm)

$\mathcal{G} = \mathrm{Map}(M, G)$ acting by fiberwise rotations on P.

- Toy Ex: $X = \mathfrak{g}^*$, $\mathfrak{g} = \mathrm{Lie}(G)$, $G \circlearrowright_{\text{coad}} X$, $S = f(\underbrace{C_2, C_3, \dots}_{\text{casimirs}}) \in C^\infty(X)^G$

Problem: ~~stationary~~ st. phase points of S are not isolated, since G acts on them!

gauge-fixing in BV formalism: $X \underset{\text{replace}}{\rightsquigarrow} \tilde{X}$ — a $\mathbb{Z}$-graded supermanifold, $X \subset \mathrm{body}(\tilde{X})$. (2)

$S \rightsquigarrow \tilde{S} \in C^\infty(\tilde{X})$ s.t.
- $\tilde{S}|_X = S$
- $\tilde{S}$ "generates" the data of gauge symmetry "$G \circlearrowright X$"
- $\tilde{S}$ satisfies the "QME".

(*) $\int_X e^{\frac{i}{\hbar}S}$ $\underset{\text{replace}}{\rightsquigarrow}$ $\int_{\mathcal{L} \subset \tilde{X}} e^{\frac{i}{\hbar}\tilde{S}} \sim \sum_{\text{crit pts}}$ (sum of Feynman diagrams)

$\int_X e^{\frac{i}{\hbar}S}$ does not have a well-defined st. phase expansion; $\int_{\mathcal{L}} e^{\frac{i}{\hbar}\tilde{S}}$ has a st. phase expansion!

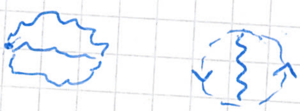
- Here $\mathcal{L}$ – a (Lagrangian) "gauge-fixing subspace" mfd
- result does not depend on (small) deformations of $\mathcal{L}$!

- Remark: lhs in (*) can be viewed as $\mathrm{Vol}(\mathcal{G}) \cdot \int_{X/\mathcal{G}} e^{\frac{i}{\hbar}S}$

~~Here the~~ which may have isolated fixed pts. However, in the local setting $X = \Gamma(\mathcal{E}_M)$, $\mathcal{G} = \Gamma(\mathcal{E}'_M)$ we want to avoid integrating over quotients, and rhs of (*) is the "free resolution" (locally), as $\tilde{X} = \Gamma(\tilde{\mathcal{E}}_M)$ – local object.

Ex: in Yang-Mills, the a special gauge-fixing (Faddeev-Popov) results in appearance of new odd fields $c, \bar{c}$ – Faddeev-Popov ghosts.

Feynman diagrams:



edges: 〰 gluon A (connection); - - > - - ghost

vertices

Structure on $\tilde{X}$:
- $\tilde{X} \circlearrowleft \omega$ – degree -1 odd symplectic structure on $\tilde{X}$
- ~~Symplec~~ $C^\infty(\tilde{X})$ carries $\{,\}$, Δ – odd Poisson bracket, odd Laplacian.
- $\tilde{S}$ satisfies the QME $\frac{1}{2}\{\tilde{S},\tilde{S}\} - i\hbar\Delta\tilde{S} = 0$. $\tilde{S} \in C^\infty(\tilde{X})[[\hbar]]$
- $\mathcal{L} \subset \tilde{X}$ – Lagrangian wrt ω.

Plan of the course

- examples of gauge systems (general structures & class. FTs – YM, CS, PSM, BF, ...)
- Faddeev-Popov construction – supergeom resolution for integrals over group quotients

$$\int_{X/G} e^{i/\hbar S} = \int_{X_{FP}} e^{i/\hbar S_{FP}}$$

- BRST gauge fixing, cohomological vector fields; ghosts as Chevalley-Eilenberg generators

- BV formalism: odd symplectic geometry → classification ~~of odd sympl.~~ thm, classif. of Lagrangians, dubs ~~~~ \& d thm etc.

  half-densities, Δ, integration on supermanifolds

  BV-integrals over Lagrangians

  ~~BV-Stokes theorem~~ ; effective BV actions (pushforward of solutions of QME)

  ⊂ QME, equivalence of solutions

  relation to ∞-algebras and homotopy transfer ($\uparrow$ representations up to homotopy)

  S as a gen. function

  $S = \langle \mathfrak{g}, p_1(x) + \frac{1}{2} p_2(x,x) + \frac{1}{3!} p_3(x,x,x) + \ldots \rangle$

  $\{S,S\} = 0 \rightsquigarrow \infty$-alg. relations

  § BV as gauge-fixing for classical gauge systems
  - solving CME by obstruction theory (BV)
  - existence \& uniqueness result (Felder–Kazhdan '13)

  $\int_{\mathcal{L} \subset \tilde{X}}^{} f$ — indep. of $\mathcal{L}$ if $\Delta f = 0$;
  $\mathcal{L}$ Lagr. $\quad e^{\frac{i}{\hbar}\tilde{S}} \cdot \mu^{1/2}_{ref}$

  $\tilde{X} = \tilde{Y} \times \tilde{Z} \supset \mathcal{L}$

  BV pushforward

  $\int_{\mathcal{L}} : \mathrm{Dens}^{1/2}(\tilde{X}) \to \mathrm{Dens}^{1/2}(\tilde{Y})$

  $(\Delta_{\tilde{X}}\text{-closed}) \mapsto (\Delta_{\tilde{Y}}\text{-closed})$

- AKSZ

  - construction of solutions of CME from (super) geom data:

  ~~$\mathcal{X}$~~ $\mathcal{F} = \mathrm{Map}(X, Y)$, $\quad S = S_{source} + S_{target}$, $\quad \omega = \ldots$

  supermanifolds with certain supergeom data:

  X has a coh v.f. Q_X, ~~integration~~ density μ_X of deg $-n$

  typical example: $X = \Pi T\Sigma$, $Q_X = d_\Sigma$, μ_X — can. integration of forms on Σ

  Y has symp form ω of deg $n-1$ and a function Θ of deg n satisfying $\{\Theta, \Theta\} = 0$.

  → e.g. $Y = \mathfrak{g}[1]$ ($\mathfrak{g}$ = Lie(G))

  $\omega = \frac{1}{2}\langle \delta\psi \,\hat{,}\, \delta\psi \rangle$

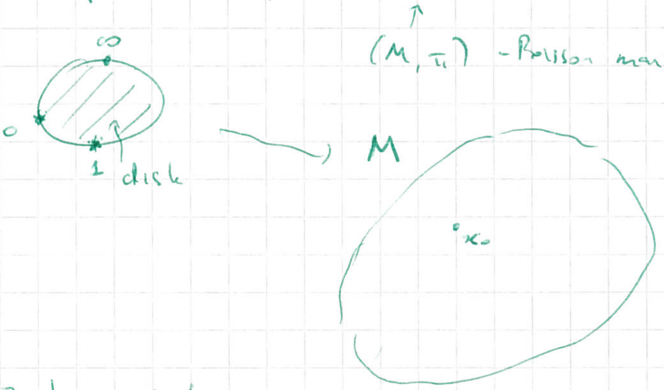
  $\Theta = \frac{1}{6}\langle \psi, [\psi, \psi] \rangle$

  Lots of examples, including Chern-Simons, PSM, BF theory

- Kontsevich's deformation quantization via Poisson sigma model (ref: Cattaneo–Felder '00)

  $\mathcal{F} = \mathrm{Map}(T[1]D, T^*[1]M)$

  (M, π) - Poisson manifold



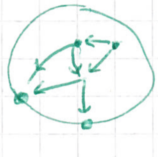
  Class. fields: bundle maps

$$\begin{tikzcd} TD \arrow[r, "\eta"] \arrow[d] & T^*M \arrow[d] \\ D \arrow[r, "X"] & M \end{tikzcd}$$

  $S_{cl} = \int_D \langle \eta, dX \rangle + \frac{1}{2}\langle \eta, \pi(X)\, \eta \otimes \eta \rangle$

[associative] star-product:

$$f \star g\,(x_0) = \text{"}\int_{\substack{\eta|_{\partial D} = 0 \\ X(\infty) = x_0}} e^{\frac{i}{\hbar}S} f(X(0))\, g(X(1)) \text{"} = \text{sum of Feynman diagrams}$$



given by config. space integrals, over configurations of points on D.

comes from formality map

- L_∞ morphism $\quad \mathcal{V}(M), [,]_{NS} \longrightarrow \mathrm{PolyDiff}(M), [,]_{Gerst}, [\bullet,]_{Gerst}$ differential
- deformation of HKR quasi-iso
  - restricted to Maurer-Cartan sets, the map spits out $\star$-products.

- perturbative Chern-Simons theory, invariants of 3-manifolds (& knots)
  (after Axelrod-Singer → Bott-Taubes, Bott-Cattaneo, Cattaneo-Mnev, ? M. Polyak)
  Witten (1-loop) [Kontsevich] (higher loops) (soft propagators) non-acyclic background connections; for surgery

$$Z^{pert}_{\text{Chern-Simons}}(M, G; A_0) = \underbrace{e^{\frac{i}{\hbar}S_{CS}(A_0)} \cdot \tau_{RS}(M, A_0)^{1/2} \cdot e^{\frac{i\pi}{4}\eta_{APS}(M, A_0, g)}}_{e^{Z^{1\text{-loop}}_{CS}}} \cdot \exp\left(\hbar\ \text{[graph]} + \hbar^2\left(\text{[graph]} + \text{[graph]}\right) + \hbar^3 \cdots\right)\cdot e^{c(\hbar)\cdot S_{grav}(g)}$$



$\left[\int_{\mathrm{Conn}(M,E)/\text{Gauge transf}} e^{\frac{i}{\hbar}S_{CS}(A)}\right]_{\text{g.f. around } A_0}$

A_0 — background flat connection

counterterm cancelling dependence on metric g; introduces dependence on framing.

weight of a graph

$$\Phi(\Gamma) = \frac{1}{|\mathrm{Aut}\,\Gamma|}\cdot\int_{\mathrm{Conf}^{V(\Gamma)}(M)} \Big\langle \prod_{\text{edges of }\Gamma} \eta(x_{e_1}, x_{e_2}),\ \prod_{\text{vertices of }\Gamma} \langle \xi^\bullet, [\bullet,\bullet]\rangle_{\text{Killing}} \Big\rangle$$

Γ — graph; $\eta \in \Omega^2(\mathrm{Conf}_2(M), E_{x_{e_1}} \boxtimes E_{x_{e_2}})$; $\Lambda^3\mathfrak{g}^* \simeq \Lambda^3 E^*_{x_i}$; contraction between E and E^*

propagator — parametrix for a chain homotopy $\Omega^\bullet(M,E) \to \Omega^{\bullet-1}(M,E)$

> A_0 is a flat conn. in $P = M\times G \to M$, $E = \mathrm{ad}(P)$, with connection ad_{A_0} — loc. system

Axelrod-Singer:
- Feynman diagrams are finite
- Z^{pert} ~~is independent of~~ depends on g, but the dependence can be cancelled by a local counterterm $\propto S_{grav}(g)$.

Fröhlich-King, Kontsevich, Bar-Natan — inclusion of Wilson lines ⇝ knot invariants

⇝ higher-dim "Wilson surfaces" by Cattaneo-Rossi.

Exact discretization of TQFT via effective BV actions

- non-abelian BF (P.M. '06, '08), version compatible with Segel's gluing (draft in preparation — A.S. Cattaneo, P.M., N. Reshetikhin)
- 1D Chern-Simons (A. Alekseev, P.M. '10)

continuum theory on M (mfld)

Path integral

M, X - cell decomposition

⇝ $\mathcal{F}_X, S_X$ - BV theory with fin. dim. space of fields ($\sim$ cell cochains), "combinatorially-local" action S_X, with QME

fin. dim fiber BV integral

X' - an aggregation of X (i.e. X a subdivision of X')

~~partition function~~, correlators, top. invariants
- partition fun., correlators,
eff. action on cohomology of M

Example: in 1D CS on a circle S^1:

$$\underbrace{e^{\frac{i}{\hbar}S_{eff}(A_0,\psi_0)}}_{\mathrm{Fun}(\Pi\mathfrak{g}\otimes H^\bullet(S^1))} = e^{\frac{i}{2\hbar}\langle\psi_0,[A_0,\psi_0]\rangle}\left(\det_{\mathfrak{g}}\frac{\sinh\frac{\mathrm{ad}_{A_0}}{2}}{\frac{\mathrm{ad}_{A_0}}{2}}\right)^{1/2}$$

⟶ for BF, generates a $\mathrm{u}L_\infty$-algebra structure on $H^\bullet(M,\mathfrak{g})$; in particular, gives Massey operation on $H^\bullet$ and the R-torsion.

For BF, $S_X = \sum_{e\subset X \text{ cells}} S_e$ ← standard local building block, depends only on comb. type of e (... simplex) ← works for X a prismatic complex.

BV-BFV

- Atiyah-Segal ~~axiomatics~~ of QFT:

closed $(n-1)$-mfld Σ $\longrightarrow$ space of states $\mathcal{H}_\Sigma$

cobordism $\Sigma_{in} \xrightarrow{M} \Sigma_{out}$ $\longrightarrow$ partition function linear operator $Z_M : \mathcal{H}_{\Sigma_{in}} \to \mathcal{H}_{\Sigma_{out}}$

gluing $\Sigma_1 \xrightarrow{M_I} \Sigma_2 \xrightarrow{M_{II}} \Sigma_3$ $\rightsquigarrow$ composition $Z_M = Z_{M_{II}} \circ Z_{M_I}$

disjoint union $\sqcup$ $\to$ $\otimes$

Heuristically:
$Z_M(\varphi_\partial) = \int_{\varphi|_\partial = \varphi_\partial} \mathcal{D}\varphi \; e^{\frac{i}{\hbar} S(\varphi)}$
bdry field configuration
$\rightsquigarrow Z_M \in \mathrm{Fun}(\{\varphi_\partial\}) = \mathcal{H}_\partial$
gluing axiom $\Leftarrow$ locality of $S(\varphi)$ and the measure $\mathcal{D}\varphi$.

in BV-BFV: (hybrid effective action formalism)

$\Sigma \to (\mathcal{H}_\Sigma, \Omega_\Sigma)$ - complex
differential

$M \to \mathcal{F}^T_{back}$, $Z^T_M \in \mathrm{Hom}(\Sigma_{in}, \Sigma_{out}) \hat{\otimes} \mathrm{Fun}(\mathcal{F}^T_{back})$;

space of bulk backgrounds / bulk zero-modes / residual fields...

manifold odd-symp

QME: $\boxed{(\Delta_{back} + \Omega_{\Sigma_{out}} - \Omega^*_{\Sigma_{in}}) Z_M = 0}$

$T \in \mathcal{T}_M$ $\leftarrow$ poset of "realizations"

"realization" of the theory on M

- for $T' < T$, $Z^{T'} = P_*^{T \to T'} Z^T$ $\leftarrow$ "renormalization flow"

fiber BV integral

- gluing: $Z^T_M = P_*^{(T_I \cup T_{II}) \to T} (Z^{T_{II}}_{M_{II}} \circ Z^{T_I}_{M_I})$

"concatenation"

Z_M and Ω_Σ is constructed by a perturbative path integral as a sum of Feynman diagrams

Toy QM on a graph Γ / random walks on Γ (a toy model of a Segal's QFT)

let I = incidence matrix of Γ

I^N_{uv} = #{paths of length N along edges of Γ from v to u}

$\| \; \langle u | I^N | v \rangle$

$\langle u | e^{tI} | v \rangle = \sum_{\text{paths on } \Gamma,\ v \to u} \frac{t^{|p|}}{|p|!}$ $\leftarrow$ "path integral representation"

$Z_t = Z_{[0,t]}$

$\mathcal{H}_{pt} = \mathrm{Fun}_{\mathbb{C}}(\{\text{vertices of } \Gamma\}) \simeq \mathbb{C}^{V(\Gamma)}$

$Z_{[t_2,t_3]} \circ Z_{[t_1,t_2]} = Z_{[t_1,t_3]}$ - gluing axiom holds

$e^{I \cdot (t_3 - t_2)} \cdot e^{I \cdot (t_2 - t_1)} \quad e^{I (t_3 - t_1)}$

BV2 20.10.14

Plan:
- preliminaries: principal bundles, connections, gauge transformations
- (classical) Chern-Simons on a closed 3-manifold
- other examples: p-form electrodynamics, Yang-Mills, BF
[- Lie algebroid symmetry; ~~distr~~ symmetry given by a distribution]

Facts on 3-mflds: TM is trivial, M is cobordant to ∅, G-bundles are trivial for G 1-connected

Let G - a Lie group, $\begin{array}{c} P \circlearrowleft G \\ \pi\downarrow \\ M \end{array}$ a principal bundle Φ_g for $g \in G$, $\Phi_g: P \longrightarrow P \cdot g^{-1}$ ($p\cdot g$)
$\cap$
P

can define the right action of G as $p \mapsto p\cdot g$

G acts freely (& smoothly) on P, $M = P/G$

$\begin{array}{c} P \\ \downarrow \\ M \end{array}$ is in particular a fiber bundle over M with fiber $\pi^{-1}(x)$, i.e. fibers are G-torsors
G acting freely & transitively on the

a connection ∇ on P is a 1-form $\mathcal{A} \in \Omega^1(P) \otimes \mathfrak{g}$, $\mathfrak{g} = \mathrm{Lie}(G)$
such that ~~Φ_{g}~~
- $\Phi_g^* \mathcal{A} = \mathrm{Ad}_g \mathcal{A}$ (equivariance) $\mathrm{Ad}_g R_g^* \mathcal{A} = \mathcal{A}$
- for $\xi \in \mathfrak{g}$ and $X_\xi \in \mathfrak{X}^{vert}(P)$ the corresponding vector field on P, $d_1\Phi(\xi): P \longrightarrow TP \to P$
we have $\boxed{\iota_{X_\xi} \mathcal{A} = \underset{C^\infty(P)}{\mathbb{1}} \otimes \xi}$

A local trivialization of P over an open $U \subset M$ is a local section $s: U \longrightarrow P|_U$.

$A_{U,s} = s^* \mathcal{A} \in \Omega^1(U) \otimes \mathfrak{g}$ - 1-form of the connection in a loc. trivialization (U, s)

change of trivialization
let $s' = s \cdot h^{-1}$ ($s\cdot h$), $h: M \longrightarrow G$; denote $A = \underset{s^*\mathcal{A}}{A_{U,s}}$, $A' = A_{U,s'} = (s')^*\mathcal{A} \in \Omega^1(U) \otimes \mathfrak{g}$

Then: $A' = \underbrace{\mathrm{Ad}_h A}_{hAh^{-1}} + h\, d h^{-1}$

$A' = \mathrm{Ad}_{h^{-1}} A + h^{-1} dh = h^{-1} A h + h^{-1} dh$

Curvature: $F_{\mathcal{A}} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \in \Omega^2(P) \otimes \mathfrak{g}$; $F_{\mathcal{A}}$ - equivariant & horizontal

$\Rightarrow F_{\mathcal{A}} = \pi^* F$, $F \in \Omega^2(M, \mathrm{ad}(P))$, $\mathrm{ad}(P) = P \times_G^{\mathrm{Ad}} \mathfrak{g}$

$F = dA + \frac{1}{2}[A, A]$ locally

$\Rightarrow F' = dA' + \frac{1}{2}[A', A'] = h F h^{-1}$ - transition function in $\mathrm{ad}(P)$
for a diff. triv. of P

DV 2/c

Let $\rho: G \to R$ be a linear representation

let $\rho(\mathcal{P}) := \mathcal{P} \times_G R$ - associated vector bundle

$$\Gamma(M, \rho(\mathcal{P})) \cong C^\infty(\mathcal{P}, R)^{G\text{-equivariant}}$$

$$\Omega^p(M, \rho(\mathcal{P})) \cong \Omega^p(\mathcal{P}, R)^{G\text{-equiv, horizontal}}$$

map $\alpha \longmapsto d\alpha + \rho(A)\wedge\alpha$ preserves ↺

and defines an exterior derivative $\nabla^\rho_A : \Omega^p(M, \rho(\mathcal{P})) \to \Omega^{p+1}(M, \rho(\mathcal{P}))$

E.g. $\mathcal{P}$ trivial, $\mathcal{P} = M\times G \xrightarrow{} M$, $\nabla^\rho_A = d + \rho(A)\wedge : \Omega^p(M,R) \to \Omega^{p+1}(M,R)$

$(\nabla^\rho_A)^2 = F_A \wedge$. A is flat whenever $F_A = 0$.

Chern–Simons

Let $G = SU(2)$, M - oriented compact smooth 3-mfld without boundary

$\mathcal{P} = M \times G \to M$ - triv. G-bundle ; $\mathrm{Conn}(\mathcal{P}) = \{\text{connections on } \mathcal{P}\} \cong \Omega^1(M)\otimes \mathfrak{g}$.

~~Define~~ For $A \in \mathrm{Conn}(\mathcal{P})$, define action

$$S_{CS}(A) = \int_M \underset{\text{in fund. rep.}}{\mathrm{tr}} \Big(\tfrac12 A\wedge dA + \underbrace{\tfrac16 A\wedge[A,A]}_{\text{or } \frac13 A\wedge A\wedge A} \Big)$$

in fund rep of $su(2)$

$$\underset{\text{variation wrt variation of } A}{\delta S_{CS}(A)} = \int_M \mathrm{tr}\Big(\tfrac12(\delta A\wedge dA + \underbrace{A\wedge d\,\delta A}_{\text{use Stokes'}}) + \tfrac12 \delta A\wedge[A,A]\Big) = \int_M \mathrm{tr}\, \delta A \wedge \underset{dA+\frac12[A,A]}{F_A}$$

<u>Euler-Lagrange equations</u>: $\boxed{F_A = 0}$

(crit. points of S_{CS} on $\mathrm{Conn}(\mathcal{P})$ = <u>flat</u> connections)

<u>Gauge symmetry</u>:

gauge transformations: $A \longmapsto A^g = g^{-1}Ag + g^{-1}dg$ for $g: M \to G$ ~ change of trivialization by g: $s \mapsto s\cdot g$

(<u>Note</u>: A flat $\Leftrightarrow$ A^g flat)

$$S_{CS}(A^g) - S_{CS}(A) = \int \mathrm{tr}\Big(\underbrace{\tfrac12 g^{-1}A\, \overset{-dg\cdot g^{-1}}{g(dg^{-1})} A g - \tfrac12 g^{-1}Ag\,g^{-1}A\,dg + g^{-1}dg\wedge A\wedge A +}$$

$$\underbrace{+ g^{-1}Ag\, \underset{-(g^{-1}dg)^2}{d(g^{-1}dg)} + g^{-1}dg\cdot g^{-1}dg\, g^{-1}Ag}_{} \underbrace{-\tfrac16}_{\frac12(-1)+\frac13}(g^{-1}dg)^3\Big) =$$

$$= -\tfrac16 \int_M \mathrm{tr}(g^{-1}dg)^{\wedge 3}$$

$\Theta = \frac{-1}{24\pi^2}\mathrm{tr}(g^{-1}dg)^{\wedge3} \in \Omega^3_{\text{closed}}(G)$ - the Cartan 3-form on G, left & right-invariant, with integral periods

(actually, a vol. form on $SU(2)\simeq S^3$ of total mass 1) $SU(2)$

$[\Theta]$ - generator of $H^3(G) \cong \mathbb{Z}$

So: $S_{CS}(A^g) - S_{CS}(A) = 4\pi^2 \int_M g^*\Theta$

$= 4\pi^2\langle [M], g^*[\Theta]\rangle = 4\pi^2 \deg(g)$ - degree of $g: M \to S^3$ as a map

BV2/7

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In particular, [denote: $\mathrm{Gauge}_{M,G} = \mathrm{Map}(M,G)$]

S_{CS} is invariant under infinitesimal gauge transf.

and, moreover, $S_{CS}(A^g) = S_{CS}(A)$ for $g \in \mathrm{Gauge}^{\circ}_{M,G}$ ← conn. component of $g \equiv 1$

More generally, S_{CS} is not invariant under "large" gauge transformations (i.e. outside Gauge°)

but $\boxed{e^{\frac{ik}{2\pi} S_{CS}(A)}}$ is invariant, and gives a function on $\mathrm{Conn}_{M,G} / \mathrm{Gauge}_{M,G}$

if $k \in \underline{\mathbb{Z}}$ "level".

Restricting to flat connections,
we obtain a $U(1)$-valued function $e^{\frac{ik}{2\pi} S_{CS}(A)}$: $\underbrace{\dfrac{\mathrm{FlatConn}_{M,G}}{\mathrm{Gauge}_{M,G}}}_{\mathcal{M}_{M,G}} \to U(1)$

$\mathcal{M}_{M,G}$ - moduli space of flat G-connections in P

$\mathcal{M}_{M,G} = \mathrm{Hom}(\pi_1(M), G) \big/ G$ ← acts by conjugation.

- fin. dim. singular (typically) variety.

$e^{\frac{ik}{2\pi} S_{CS}(A)}$ yields a locally-constant (due to $\delta S_{CS} = \ldots$ equation) $U(1)$-valued function on $\mathcal{M}_{M,G}$.

[Ex: lens spaces $M = L(p,q)$, $\pi_1 = \mathbb{Z}_p$, $\underset{\substack{\text{generator}\\ \text{of } \pi_1}}{\gamma} \mapsto \begin{pmatrix} e^{2\pi i r/p} & \\ & e^{-2\pi i r/p} \end{pmatrix} \in SU(2)$

- different points of $\mathcal{M}_{M,G}$ for $r \neq r'$ mod p

values of $e^{\frac{ik}{2\pi} S_{CS}}$ on these points - cf. L. Jeffrey ? "$e^{2\pi i k \frac{q^* r^2}{p}}$", $q \cdot q^* = 1 \bmod p$.

?

Comments

- Everything generalizes to G - any connected, simply-connected, semi-simple, compact Lie group
e.g. $G = SU(N)$

Important fact: Every G-bundle on M is trivial.
if $\dim M \le 3$ and G 1-connected then

because $\pi_2(G) = 0$ always, so G 2-connected $\Rightarrow BG$ 3-connected

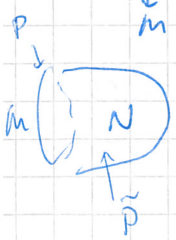
$\left\{ \begin{matrix} \text{iso classes} \\ \text{of } G\text{-bundles} \end{matrix} \right\} \underset{1-1}{\longleftrightarrow} [M, BG] = pt$ (by cellular extension).

- Relation to 2^{nd} Chern class

Fact: every oriented 3-mfd M is null-cobordant, i.e. $\exists\, N$ - a compact 4-mfd with boundary and $\partial N = M$.

if $\begin{matrix} P \circlearrowleft G \\ \downarrow \\ M \end{matrix}$ trivial, can extend it over N, to $\tilde{P}$ (trivially)

and extend connection 1-form A to $\tilde{A} \in \Omega^1(N) \otimes \mathfrak{g}$.



Then $S_{CS}(A) = \int_N \mathrm{tr}\, \frac{1}{2} F_{\tilde{A}} \wedge F_{\tilde{A}}$ (*)

BV 2/4

3/1

Observation on a 4-manifold

$$d\,tr\left(\tfrac12 A\wedge dA+\tfrac13 A\wedge A\wedge A\right) = tr\left(\tfrac12 dA\wedge dA+\tfrac{3}{3} A\wedge A\wedge dA\right)$$

$$tr\, F_A\wedge F_A = tr\{(dA+A\wedge A)\wedge(dA+A\wedge A)\} =$$
$$= tr\big(dA\wedge dA + 2\, A\wedge A\wedge dA + \underbrace{A\wedge A\wedge A\wedge A}\big)$$

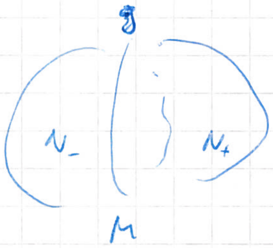
$$tr\, A^4 \overset{*}{=} tr\, A\wedge A^3 = -\,tr\, A^3\wedge A = -\,tr\, A^4$$

$$\Rightarrow \quad d\,tr\left(\tfrac12 A\wedge dA+\tfrac13 A\wedge A\wedge A\right) = \tfrac12 tr\left(F_A\wedge F_A\right)$$

thus $(*)$ holds by Stokes'.

- let N_+, N_- be two copies of N, $(N_+ = N_-^{\text{op. or.}})$ $\bar N = N_+ \underset{M}{\cup} N_-$

let $g: M\to G$



Construct a G-bundle $\bar P_g$ over $\bar N$, trivial over N_+ and N_- and with transition function g on $M\times[-\varepsilon,+\varepsilon]$

let $A\in \mathrm{Conn}(M)$, $\tilde A_+$ - its extension over N_+

$A^g = gAg^{-1}+g\,dg^{-1}$, $\tilde A_-$ - extension of A^g over N_-

$(\tilde A_+, \tilde A_-)$ define a connection in $\bar P$ $\bar A$

$$\frac{1}{8\pi^2}\int_{\bar N} tr\, F_{\bar A}\wedge F_{\bar A} = \frac{1}{4\pi^2}\int_{N_+\cup N_-} tr\, F_{\bar A}\wedge F_{\bar A} = \frac{1}{4\pi^2}\left(S_{cs}(A^g) - S_{cs}(A)\right)$$

$\shortparallel$ by Chern-Weil

$\langle [\bar N], c_2(\bar P)\rangle \in \mathbb Z$ ← independent of A

$c_2(\bar P) \in H^4(\bar N,\mathbb Z)$ ← 2nd Chern class of $\bar P_g$

So, (in)dependence of $S_{cs}(A)$ on gauge transf. is linked to Chern-Weil theory for G-bundles over 4-manifolds (rk=2 complex vector bundles)

Reminder: Chern-Weil homomorphism:

$$(\mathrm{Sym}\,\mathfrak g^*)^G \longrightarrow H^\bullet(M,\mathbb R)$$
$$p(x) \longmapsto [tr\, p(F_A)] = \gamma^*\underbrace{[p]}_{\in H^\bullet(BG)}, \quad \gamma: M\to BG \text{ - classifying map.}$$

$\cong$

$i_* c_2 \in H^\bullet(BG,\mathbb R)$

$$\to H^\bullet(BG,\mathbb Z) \xrightarrow{i_*} H^\bullet(BG,\mathbb R)\to \dots$$

BV 2/5
3/2

~~Other exampl~~

More examples of gauge systems

- (M, g) - Riemannian closed, oriented mfld (of some $\dim = n$), fix some $1 \le p \le n$

field: $\omega \in \Omega^p(M)$

action: $S(\omega) = \int_M \frac{1}{2} d\omega \wedge * d\omega$ ← Hodge star assoc. to g

Euler-Lagrange equations: $\boxed{d*d\omega = 0}$ (*)

Gauge symmetry: $\omega \longmapsto \omega + d\alpha =: \omega^\alpha$, $\alpha \in \Omega^{p-1}(M)$ (preserves (*))

"Gauge" [for this model]

case $p = 0$ — massless free boson, no gauge symmetry

$(*) \Leftrightarrow \Delta\omega = 0$ — solutions of (*) on M compact - loc. const functions, $EL \simeq H^0(M)$.

case $p = 1$ - class. electrodynamics (without charges, just photons) (Maxwell theory), $d\omega$ = "stress tensor of electromagnetic field"

$EL =$ harmonic 1-forms $\simeq H^1(M)$ (for M non-compact or with bdry, EL becomes ∞-dimensional)

$M = \mathbb{R} \times \Sigma^{n-1}$, $t \in \mathbb{R}$, $\Sigma \xrightarrow{ev_t} \Sigma$, $ev_t^* * d\omega = E_t \xi = \iota_{E_t} dvol_\Sigma$ ← electric field

$ev_t^* * d\omega = B_t = \iota_{B_t} dvol_\Sigma$ ← magnetic field

(pure) Yang-Mills

- (M, g) Riemannian, $P \circlearrowleft G$, $P \to M$ a G-bundle

fields = connections A in P

action: $S_{YM}(A) = \frac{1}{2} \int_M tr (F_A \wedge * F_A)$, $F_A \in \Omega^2(M, ad(P))$, $*F_A \in \Omega^{n-2}(M, ad(P))$, $\propto \frac{1}{2} \int \langle F_A \stackrel{\wedge}{,} * F_A \rangle$ ← Killing form on $\mathfrak{g}$

E-L eq: $\nabla_A * F_A = 0$ (Yang-Mills equation)

Gauge symmetry - same as in C-S: $A \longrightarrow A^g$?

[! here A is on total space, so the interpretation is - action of an automorph. of P on A - active transf. of connection]

Rem $\dim M = 2 \Rightarrow S_{YM}$ depends only on the metric vol form $dvol$ $= \sqrt{g}\, d^2x$,

$F_A = dvol \cdot f_A$, $S_{YM} = \frac{1}{2} \int_M tr f_A^2 \cdot dvol$, $f_A \in \Omega^0(M, ad(P))$

- BF M, $P \circlearrowleft G$ ~~Example~~ semisimple, $P \to M$

YM in 1st order formalism: $A \in Con(P)$, $B \in \Omega^{n-2}(M, adP)$, $S^{1st}_{YM}(A,B) = \int tr\left(B \wedge F_A - \frac{1}{2} B \wedge * B\right)$

EL eq: $F_A = *B$, $\nabla_A B = 0$ $\Leftrightarrow \begin{cases} \nabla_A * F_A = 0 \\ B = * F_A \end{cases}$

fields: $(A, B) \in Conn(P) \times \Omega^{n-2}(M, ad^*(P))$

action: $S(A,B) = \int_M tr\, B \wedge F_A$

more general version: $B \in \Omega^{n-2}(M, ad^*(P))$, $S = \int_M \langle B, F_A \rangle$ - can. pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

EL eq.: $F_A = 0$, $\nabla_A B = 0$

Gauge transf.:

I: $A \longmapsto g A g^{-1} + g d g^{-1}$, $B \longmapsto g B g^{-1}$, $g \in Map(M, G)$

II: $A \longmapsto A$, $B \longmapsto B + \nabla^{ad}_A \tau$, $\tau \in \Omega^{n-3}(M, ad(P))$

Perturbed gaussian integral & Feynman diagrams

BV 3/4

4/0

03.12.14

model integral: $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx = \sqrt{2\pi} \rightsquigarrow \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}\cdot a}dx = \sqrt{\frac{2\pi}{a}}$, $a>0$ — Gaussian integral

Fresnel integral: $\int_{-\infty}^{\infty} e^{\pm\frac{i}{2}x^2 a}dx = \sqrt{\frac{2\pi}{a}}\cdot e^{\pm\frac{i\pi}{4}}$

multidim. version: $\int_{\mathbb{R}^N} e^{-\frac{1}{2}B(x,x)}\prod_{i=1}^N dx_i = \left(\det\frac{\beta}{2\pi}\right)^{-1/2} = \left(\det\left(\frac{B_{ij}}{2\pi}\right)\right)^{-1/2} \cdot (2\pi)^{N/2}$

$B(x,x)$ – positive quadratic form on $\mathbb{R}^N$ — matrix of B

$B: \mathbb{R}^N \otimes \mathbb{R}^N \to \mathbb{R}$ – bilin. form

$\beta: \mathbb{R}^N \to \mathbb{R}^N$ – endomorphism of $\mathbb{R}^N$ corresp. to B in stand. basis on $\mathbb{R}^N$

i.e. $B(x,y) = \langle x, \beta(y)\rangle$

more abstractly: V – vect. space $/\mathbb{R}$

$B: V\otimes V \to \mathbb{R}$, $\mu \in \Lambda^N V^*$

$B^\#: V \xrightarrow{\sim} V^*$

$\operatorname{Det} B^\#: \Lambda^N V \xrightarrow{\sim} \Lambda^N V^*$

"Det B" $= \delta_B \in (\Lambda^N V^*)^{\otimes 2}$ $\rightsquigarrow$ $\boxed{\int_V e^{-\frac{1}{2}B}\mu = (2\pi)^{+\frac{N}{2}}\left(\frac{\delta_B}{\mu^{\otimes 2}}\right)^{-1/2}}$

since $\Lambda^N V$ and $\Lambda^N V^*$ are can. paired by

$(\alpha_1\wedge\cdots\wedge\alpha_N)\otimes(\beta_1^*\wedge\cdots\wedge\beta_N^*) \longmapsto \det\langle\alpha_i,\beta_j^*\rangle \in \mathbb{R}$

Fresnel version: $\int_V e^{-\frac{i}{2}B}\mu = (2\pi)^{\frac{N}{2}}\cdot e^{\frac{\pi i}{4}\cdot \operatorname{sign} B}\cdot\left(\frac{\delta_B}{\mu^{\otimes 2}}\right)^{-1/2}$

phase — $\operatorname{sign} B = \eta(\beta)$ – "APS η-invariant".

(14)

Gaussian correlators:

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}B(x,x)} x_{i_1}\cdots x_{i_n}\underbrace{\prod_1^N dx_j}_{d^N x} = \left(\int_{\mathbb{R}^N} e^{-\frac{1}{2}B(x,x)}d^N x\right)\cdot\begin{cases} 0 & \text{if } n \text{ odd}\\ (B^{-1})_{i_1 i_2}, & n=2\\ \sum (B^{-1})_{i_{r_1} i_{s_1}}\cdots(B^{-1})_{i_{r_m} i_{s_m}}, & n=2m \end{cases}$$

$\sum$ over ways to split into pairs $\{1\ldots n\} = \{r_1,s_1\}\cup\cdots\cup\{r_m,s_m\}$

$(2m-1)!!$ terms in the sum

$$* \int_{\mathbb{R}^N} e^{-\frac{1}{2}B(x,x)+\langle y,x\rangle}d^N x = (2\pi)^{\frac{N}{2}}\det{}^{-\frac{1}{2}}(B_{ij})\cdot e^{\frac{1}{2}B^{-1}(y,y)}$$

$y \in V^*$, $B^{-1}: V^*\otimes V^* \to \mathbb{R}$

$$\Rightarrow \langle\langle x_{i_1}\cdots x_{i_n}\rangle\rangle_B = \frac{\partial}{\partial y_{i_1}}\cdots\frac{\partial}{\partial y_{i_n}} e^{\frac{1}{2}B^{-1}(y,y)}\Big|_{y=0} \quad **$$

** $n=2m$: $\overset{*}{=} \frac{\partial}{\partial y_{i_1}}\cdots\frac{\partial}{\partial y_{i_{2m}}}\frac{1}{2^m m!}\underbrace{B^{-1}(y,y)\cdots B^{-1}(y,y)}$ $= \frac{1}{2^m m!}\sum_{\sigma\in S_{2m}}(B^{-1})_{i_{\sigma(1)}i_{\sigma(2)}}\cdots(B^{-1})_{i_{\sigma(2m-1)}i_{\sigma(2m)}}$ $= \sum_{[\sigma]\in S_{2m}/S_m\ltimes\mathbb{Z}_2^m}$ ---- □

generally, for $p\in \operatorname{Sym} V^*$ "$\operatorname{Fun}_{\text{poly}}(V)$", $\langle\langle p\rangle\rangle_B = p\left(\frac{\partial}{\partial y}\right)\circ e^{\frac{1}{2}B^{-1}(y,y)}\Big|_{y=0}$

diff. operator on $C^\infty(V^*)$

$V^* \longrightarrow \mathcal{D}(V^*)$

$x_i \longmapsto \frac{\partial}{\partial y_i}$

extend by to an algebra hom.

$\operatorname{Sym} V^* \longrightarrow \mathcal{D}(V^*)$

$y^\# \in V$ — $\langle,\rangle: V\otimes V\to\mathbb{R}$, $V\to V^*$

$$* \int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle x,\beta(x)\rangle+\langle x,y^\#\rangle}d^N x$$

$$= \int_{\mathbb{R}^N} e^{-\frac{1}{2}\langle x-\beta^{-1}y',\beta(x-\beta^{-1}y')\rangle+\frac{1}{2}\langle\beta^{-1}y',y'\rangle}$$

BV 3/4 4/1

$\langle\langle \varphi_1 \varphi_2 \rangle\rangle = \langle B^{-1}, \varphi_1 \otimes \varphi_2 \rangle$ "propagator"

$\in Sym^2 V^*$, $\varphi_1, \varphi_2 \in V^*$; $B^{-1} \in Sym\, V \otimes V$

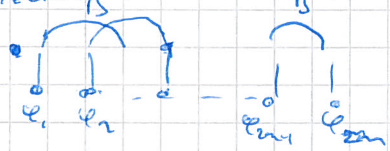
$P_\sigma : V^{\otimes n} \to V^{\otimes n}$ – perm. of n copies of V

$\langle\langle \underbrace{\varphi_1 \cdots \varphi_{2m}}_{\in Sym^{2m} V^*} \rangle\rangle = \langle \sum P_\sigma (\underbrace{B^{-1} \otimes \cdots \otimes B^{-1}}_{m}), \varphi_1 \otimes \cdots \otimes \varphi_{2m} \rangle$

$\sigma \in$ permutations of $(1 \ldots 2m)$
mod transpositions of $(2j, 2j+1)$
and permutations of pairs $(2j, 2j+1)$

$\underbrace{\sigma \in S_{2m} / S_m \ltimes \mathbb{Z}_2^m}_{\text{"matchings" of } 2m \text{ half-edges}}$ – stabilizer of the element $(B^{-1})^{\otimes m} \in V^{\otimes 2m}$

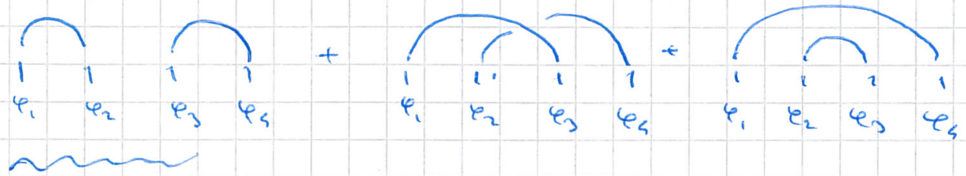
visualization



B^{-1}, φ_1, φ_2, φ_{2m-1}, φ_{2m}

Ex: $m = 2$, $n = 4$:

$\langle\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle\rangle = B^{-1}(\varphi_1, \varphi_2) \cdot B^{-1}(\varphi_3, \varphi_4) + B^{-1}(\varphi_1, \varphi_3) \cdot B^{-1}(\varphi_2, \varphi_4) + B^{-1}(\varphi_1, \varphi_4) \cdot B^{-1}(\varphi_2, \varphi_3)$



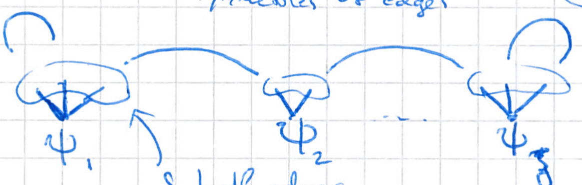
Let ~~$\psi_1 \ldots \psi_r$~~ $\psi_j \in Sym^{s_j} V^*$, $j = 1 \ldots r$, $\sum s_j = 2m$

$\langle\langle \frac{1}{s_1!}\psi_1, \ldots, \frac{1}{s_r!}\psi_r \rangle\rangle = \sum \underbrace{\frac{1}{s_1! \cdots s_r!}}_{|\prod S_{s_j}|} \sum_{\sigma \in S_{2m}/S_m \ltimes \mathbb{Z}_2^m} \langle \sigma \circ (B^{-1})^{\otimes m}, \psi_1 \otimes \cdots \otimes \psi_r \rangle =$

$= \sum_{[\sigma] \in \prod_j S_{s_j} \backslash S_{2m} / S_m \ltimes \mathbb{Z}_2^m} \frac{1}{|Stab_{[\sigma]}|} \langle \sigma \circ (B^{-1})^{\otimes m}, \psi_1 \otimes \cdots \otimes \psi_r \rangle$, $Stab_{[\sigma]} \subset \prod_j S_{s_j}$

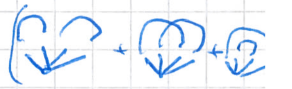
$\frac{|\text{orbit of } \sigma \text{ under action } \prod_j S_{s_j} \curvearrowright S_{2m}/S_m \ltimes \mathbb{Z}_2^m|}{|\prod_j S_{s_j}|}$

symmetries of vertices in pictures; "symmetries of edges"



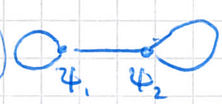
edges
← vertices
ψ_1, ψ_2, …, ψ_r
s_1 half-edges

Example: ① ∃ $\psi \in Sym^4 V^*$

$\langle\langle \frac{1}{4!}\psi \rangle\rangle = \frac{1}{4!} \sum_{\sigma \in S_4 / S_2 \ltimes \mathbb{Z}_2^2} \langle \sigma \circ (B^{-1})^{\otimes m}, \psi \rangle = \frac{1}{4!}$ 

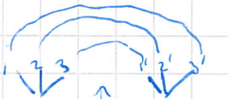
$= \frac{3}{24}$  $= \frac{1}{8} \langle (B^{-1})^{\otimes 2}, \psi \rangle$; $\frac{3}{24} = \frac{1}{8}$

② ∃ $\psi_1, \psi_2 \in Sym^3 V^*$

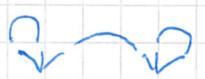
$\langle\langle \frac{1}{3!}\psi_1, \frac{1}{3!}\psi_2 \rangle\rangle = \frac{1}{3!3!} \sum_{\sigma \in S_6 / S_3 \ltimes \mathbb{Z}_2^3} \langle \sigma \circ (B^{-1})^{\otimes 3}, \psi_1 \otimes \psi_2 \rangle = \frac{6}{3!3!}$  $+ \frac{9}{3!3!}$ 

$\frac{6}{3!3!} = \frac{1}{6}$, $\frac{9}{3!3!} = \frac{1}{4}$

$[\sigma] = \begin{bmatrix} (1\,2\,3\,1'2'3') \\ (1'2'3'\,1\,2\,3) \end{bmatrix}$, $[\sigma] = [id]$



6 permutations $(1\,2\,3) \mapsto \pi(1'2'3')$

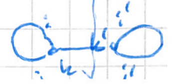


$3 \cdot 3$ ~~choose~~ matchings like that

Explicitly: ∃ $\psi_{1,2} = \sum_{i,j,k} \psi_{1,2}^{ijk} x_i x_j x_k$

$\langle\langle \frac{1}{3!}\psi_1, \frac{1}{3!}\psi_2 \rangle\rangle = \frac{1}{6} \psi_1^{ijk} \psi_2^{i'j'k'} (B^{-1})_{ii'} (B^{-1})_{jj'} (B^{-1})_{kk'} +$

$+ \frac{1}{4} \psi_1^{ijk} \psi_2^{i'j'k'} (B^{-1})_{ij} (B^{-1})_{i'j'} (B^{-1})_{kk'}$



Graphs - reminder

data: V - set of vertices $\qquad i: HE \to V$ (incidence) $\quad$ | for $v \in V$, $i^{-1}(v)$ = star / corolla of v

HE - set of half-edges $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ | $\# i^{-1}(v) = \mathrm{val}_v$ - valence

E = partition of HE into unordered pairs - edges

(alternatively: $\Gamma: HE \to HE$ involution with no fixed points; then $E = \{\text{orbits of } \Gamma\}$)

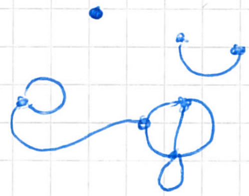
---

graph automorphism:

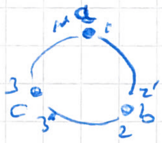
$(\sigma_{HE}, \sigma_V) \in S_{HE} \times S_V$ pair of permutations

commuting with i and preserving partition E (commuting with Γ).

Example:



Example:



$V = \{a, b, c\}$ $\qquad i: \begin{matrix}1\\1'\end{matrix} \to a, \quad \begin{matrix}2\\2'\end{matrix} \to b, \quad \begin{matrix}3\\3'\end{matrix} \to c$

$HE = \{1, 2, 3, 1', 2', 3'\}$

$E = \{\{1,2'\}, \{2,3'\}, \{3,1'\}\}$ (Γ)

$$\sigma_{HE} = \begin{pmatrix} 1 & 2 & 3 & 1' & 2' & 3' \\ 2' & 1' & 3' & 2 & 1 & 3 \end{pmatrix}$$

edges

$\{1\ 2'\} \mapsto \{2'\ 1\}$

$\{2\ 3'\} \mapsto \{1'\ 3\}$

$\{3\ 1'\} \mapsto \{3'\ 2\}$

$$\sigma_V = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}$$

---

automorphism group of a graph $\quad \mathrm{Aut}\,\Gamma \subset \prod_{j=0}^{\text{max valence}} S_{v_j} \ltimes (S_j)^{\times v_j}$, $\quad v_j = \#\{\text{vertices of valence } j\}$

j - valence

S_{v_j} - perm. of vertices of val j

$(S_j)^{\times v_j}$ - permutations of half edges in individual vertices of val $= j$

Ex:

(1) $\Gamma =$

n vertices, $n > 2$

$\mathrm{Aut}\,\Gamma \simeq \mathbb{Z}_2 \ltimes \mathbb{Z}_n$, $\quad |\mathrm{Aut}\,\Gamma| = 2n$

$\mathbb{Z}_2$ - reversing orientation, $\mathbb{Z}_n$ - cyclic shifts

(2) $\Gamma =$

$\mathrm{Aut}\,\Gamma \simeq \mathbb{Z}_2 \ltimes S_3$ $\qquad$ (or $\mathbb{Z}_2 \times S_3$!) $\qquad |\mathrm{Aut}| = 12 = 2 \cdot 3!$

$\mathbb{Z}_2$ - perm. of 2 vertices; S_3 - perm. of edges

(3) $\Gamma =$

$\mathrm{Aut}\,\Gamma = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$

$\mathbb{Z}_2$ - perm. of edges; $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ - rev. orient. of edges (perm. ½-edges within edges)

Perturbed Gaussian integral.

Consider integral of the form

$$\int_V e^{-\frac{1}{2}B(x,x)+P(x)}\,d^n x$$

$$P(x)=\sum_{j=1}^{d}\frac{g_j}{j!}P_j(x) \quad \text{- polynomial perturbation}$$

g_j – "coupling constants"

Define $$\int_V^{\text{pert}} e^{-\frac{1}{2}B(x,x)+P(x)}\,d^n x := \underbrace{\int_V e^{-\frac{1}{2}B(x,x)}d^n x}_{\det^{-1/2}\frac{B}{2\pi}}\cdot \underbrace{\langle\langle e^{P}\rangle\rangle_B}_{\in\mathbb{R}[[g_0,\dots,g_d]]}$$

Here $e^P\in\mathbb{R}[[g_0,\dots,g_d]]\otimes \operatorname{Sym}V^*$

(*) $$\text{''}\sum_{v_1\dots v_d=0}^{\infty}\prod_{j=1}^{d}\left[\left(\frac{g_j}{j!}\right)^{v_j}\cdot\frac{1}{v_j!}\,P_j(x)^{v_j}\right]\text{''}$$

coeff. of $g_1^{v_1}\cdots g_d^{v_d}$ is a finite-degree polynomial!

$\langle\langle e^P\rangle\rangle$ is $\langle\langle\dots\rangle\rangle:\operatorname{Sym}V^*\to\mathbb{R}$ extended by linearity and applied to e^P, i.e. we apply $\langle\langle\dots\rangle\rangle$ to (*) termwise.

Calculation: $$\langle\langle e^P\rangle\rangle=\sum_{v_1\dots v_d=0}^{\infty}\prod_{j=1}^{d}\frac{g_j^{v_j}}{v_j!}\cdot\langle\langle\prod_{j=1}^{d}\frac{1}{j!}P_j^{v_j}\rangle\rangle=$$

$$=\sum_{\substack{v_1\dots v_d=0\\ \sum v_j=2m,\ m\in\mathbb{Z}}}^{\infty}\prod_{j=1}^{d}\frac{g_j^{v_j}}{j!^{v_j}v_j!}\cdot\sum_{\sigma\in S_{2m}/S_m\ltimes\mathbb{Z}_2^m}\langle\sigma\circ(B^{-1})^{\otimes m},\underbrace{\bigotimes_{j=1}^{d}P_j^{\otimes v_j}}_{\in\otimes V^{*\otimes 2m}}\rangle=$$

$$=\sum_{\substack{v_1\dots v_d=0\\ \sum v_j=2m}}^{\infty}g_1^{v_1}\cdots g_d^{v_d}\frac{1}{\left|\prod_{j=1}^{d}S_{v_j}\ltimes S_j^{\times v_j}\right|}\sum_{[\sigma]\in\prod S_{v_j}\ltimes S_j^{\times v_j}\backslash S_{2m}/S_m\ltimes\mathbb{Z}_2^m}\left|\text{orbit of }\sigma\in S_{2m}/S_m\ltimes\mathbb{Z}_2^m\text{ under action of }\prod S_{v_j}\ltimes S_j^{v_j}\right|\langle\sigma\circ(B^{-1})^{\otimes m},\bigotimes_{j=1}^{d}P_j^{\otimes v_j}\rangle=$$

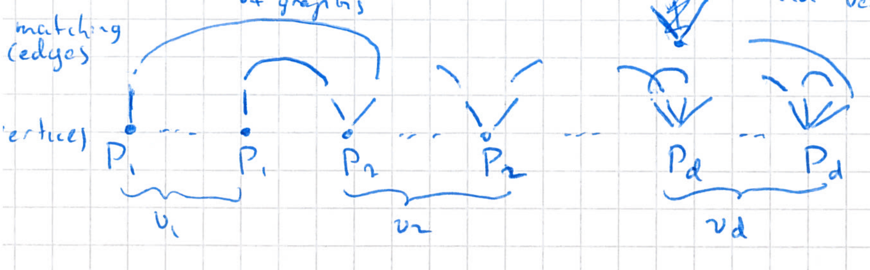
$$=\sum_{v_1\dots v_d=0}^{\infty}g_1^{v_1}\cdots g_d^{v_d}\sum_{[\sigma]\in\prod S_{v_j}\ltimes S_j^{v_j}\backslash S_{2m}/S_m\ltimes\mathbb{Z}_2^m}\frac{1}{|\mathrm{Stab}_{[\sigma]}|}\langle\sigma\circ(B^{-1})^{\otimes m},\bigotimes_{j=1}^{d}P_j^{\otimes v_j}\rangle=$$

$\mathrm{Stab}_{[\sigma]}\subset\prod_j S_{v_j}\ltimes S_j^{v_j}$

$$=\sum_{v_1\dots v_d=0}^{\infty}g_1^{v_1}\cdots g_d^{v_d}\sum_{\text{graphs }\Gamma}\frac{1}{|\mathrm{Aut}\,\Gamma|}\underbrace{\Phi_{B^{-1};\{P_j\}_{j=1}^{d}}(\Gamma)}$$

graphs Γ with v_j vertices of valence j, $j=1\dots d$ (mod isomorphism of graphs)

"value" of the graph Γ with "propagator" B^{-1} and "vertex functions" $\{P_j\}$ (tensors) – "Feynman rules"



$$\Phi(\Gamma_\sigma)=\langle\sigma(B^{-1})^{\otimes m},\bigotimes_{j=1}^{d}P_j^{\otimes v_j}\rangle$$

Γ_σ – graph corresponding to $[\sigma]\in\prod S_{v_j}\ltimes S_j^{v_j}\backslash S_{2m}/S_m\ltimes\mathbb{Z}$

In terms of a basis in V:

$$P_j(x) = \sum_{i_1 \dots i_j = 1}^{\dim V} \underbrace{P_j^{i_1 \dots i_j}}_{\text{coefficients}} x_{i_1} \cdots x_{i_j}$$

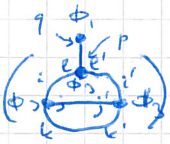
then $\underbrace{\Phi(\Gamma)}_{\text{to calculate}}$: order half-edges somehow and label them with "dummy indices" $i_1 \dots i_{2m}$

For $\forall$ vertex write $P_{[\text{val. of vertex}]}^{[\text{indices of incident half-edges}]}$

For $\forall$ edge write $(B^{-1})_{[\text{indices of constituent half-edges}]}$; take product and sum over values of dummy indices in $i_1 \dots i_{2m} \in \{1 \dots 2m\}$

Exp

$$\Phi_{B^{-1}; \phi_1, \phi_3}\left(\right) = \sum_{i,j,k,i',j',k',\ell,\ell',p,q=1}^{\dim V} \phi_1^{q} \, \phi_3^{p \ell \ell'} \, \phi_3^{ijk} \, \phi_3^{i'j'k'} \, B^{-1}_{qp} \, B^{-1}_{\ell\ell'} B^{-1}_{ii'} B^{-1}_{jj'} B^{-1}_{kk'}$$



So, we have proven:

Thm:
$$\int_V^{\text{pert}} e^{-\frac{1}{2}B(x,x) + P(x)} d^n x = \det{}^{-\frac{1}{2}}\left(\tfrac{B}{2\pi}\right) \cdot \sum_{\text{graphs } \Gamma} \frac{1}{|\mathrm{Aut}\,\Gamma|} \Phi_{B^{-1}; g_1 P_1 \dots g_d P_d}(\Gamma) \qquad (*)$$

$$P(x) = \sum_{j=1}^{d} \frac{g_j}{j!} P_j(x)$$

Ex:
$$\int_{\mathbb{R}}^{\text{pert}} e^{-\frac{1}{2}x^2 + \frac{\lambda}{4!}x^4} dx = \sqrt{2\pi} \cdot \sum_{\substack{4\text{-valent} \\ \text{graphs } \Gamma}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \lambda^{\#\text{vertices}} = \sqrt{2\pi}\left(1 + \frac{1}{8}\lambda + \frac{1}{2 \cdot 8^2}\lambda^2 + \frac{1}{2 \cdot 4!}\lambda^2 + \dots + O(\lambda^3)\right)$$

$(\lambda = g_4)$ $\quad \Gamma = \emptyset$, $\Gamma = \infty$, $\Gamma = \infty\ \infty$

(Rem:
$$\int_{\mathbb{R}} e^{-\frac{1}{2}x^2 - \frac{\nu}{4!}x^4} dx \underset{\nu > 0}{=} \sqrt{\frac{3}{\nu}} \cdot e^{\frac{3}{4\nu}} K_{\frac{1}{4}}\left(\frac{3}{4\nu}\right)$$
– convergent integral; hyperbolic Bessel's function; rhs = asympt series for lhs)

Rem:
$$\text{rhs} = \sqrt{2\pi} \sum_{n \geq 0} \left(\frac{\lambda}{4!}\right)^n \frac{1}{n!} \underbrace{(4n-1)!!}_{\substack{\text{no. of matchings} \\ \text{for } 4n \text{ vertices} \\ \text{(half-edges)}}} = \sqrt{2\pi} \sum_{n \geq 0} \lambda^n \frac{(4n)!}{2^{2n}(2n)!\, n!\, 4!^n}$$

coefficients grow super-exponentially $\Rightarrow$ convergence radius in λ is 0!

Note: $\Phi(\Gamma_1 \sqcup \Gamma_2) = \Phi(\Gamma_1) \cdot \Phi(\Gamma_2)$

Thm Rhs of $(*)$
$$= \det{}^{-\frac{1}{2}} \frac{B}{2\pi} \exp\left(\sum_{\substack{\text{connected graphs} \\ \Gamma}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \Phi(\Gamma) \right)$$

as an exp of a formal power series in $g_1 \dots g_d$

Proof
$$\frac{\text{Rhs}(*)}{\det^{-\frac{1}{2}} B/2\pi} = \sum_{\text{graphs } \Gamma} \frac{1}{|\mathrm{Aut}\,\Gamma|} \Phi(\Gamma)$$

$$\forall\, \Gamma = \underbrace{\Gamma_1 \sqcup \dots \sqcup \Gamma_{s(\Gamma)}}_{\substack{\text{connected} \\ \text{subgraphs}}} = \underbrace{\Gamma_{(1)}^{\sqcup q_1}}_{q_1 \text{ copies}} \sqcup \dots \sqcup \underbrace{\Gamma_{(r)}^{\sqcup q_r}}_{q_r \text{ copies}}, \quad \Gamma_{(1)} \dots \Gamma_{(r)} \text{ – connected, pairwise non-isomorphic}$$

Observation:
$$|\mathrm{Aut}\,\Gamma| = \prod_{k=1}^{r} \underbrace{S_{q_k}}_{\text{permutations of copies of } \Gamma_{(k)}} \ltimes (\mathrm{Aut}\,\Gamma_{(k)})^{\times q_k}$$

$$\exp \sum_{\substack{\text{conn. graphs} \\ \gamma}} \frac{1}{|\mathrm{Aut}\,\gamma|} \Phi(\gamma) = \prod_{\gamma} \sum_{q=0}^{\infty} \frac{1}{q!\, |\mathrm{Aut}\,\gamma|^q} \Phi(\gamma)^q \underset{\text{expand } \prod_\gamma}{=} \sum_{\gamma_1 \dots \gamma_r} \sum_{q_1 \dots q_r} \frac{1}{\prod_{k=1}^{r} q_k! (\mathrm{Aut}\,\gamma_k)^{q}} \Phi(\gamma_1)^{q_1} \cdots \Phi(\gamma_r)^{q_r} \equiv$$

$$\equiv \sum_{\text{all}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \Phi(\Gamma) \qquad \frac{1}{|\mathrm{Aut}\,\Gamma|}\Phi(\Gamma), \quad \Gamma = \coprod \gamma_k^{\sqcup q_k}$$