Modifications

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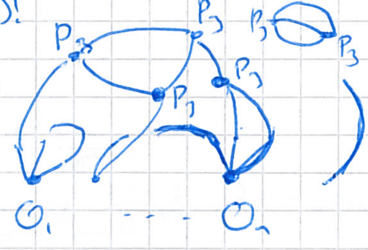
$V_\Gamma \to I$ 6/1

- One can consider graphs with ~~edges~~ vertices marked by elements of a set I. Then one only allows graph automorphisms preserving the marking.

Ex:  has $|\mathrm{Aut}\,\Gamma| = 6$, whereas  has $|\mathrm{Aut}\,\Gamma| = 12$.

Can have colored edges or half-edges, too

- $$\int_V^{pert} d^N x\; e^{-\frac{B(x)}{2}+P(x)} \cdot \frac{1}{(\deg O_1)!} O_1(x) \cdots \frac{1}{(\deg O_n)!} O_n(x) =$$

$$= \det{}^{-\frac12} \beta/_{2\pi} \cdot \sum_{\substack{\text{graphs } \Gamma \text{ with}\\ \text{vertices marked by}\\ \{\underbrace{O_1,\dots,O_n}_{\substack{\text{only one}\\ \text{vertex of each}\\ \text{color}}};\, \underbrace{P_1,\dots,P_r}_{\substack{\text{arbitrary}\\ \text{amount}\\ \text{of vertices}\\ \text{of each color}}}\}}} \frac{1}{|\mathrm{Aut}\,\Gamma|}\; \Phi\Big($$  $$\Big)$$

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- $$\int^{pert} d^N x\; e^{\frac{1}{\hbar}\left(-\frac{B(x)}{2}+P(x)\right)} \overset{x=\sqrt{\hbar}\,\tilde{x}}{=} \hbar^{\frac{N}{2}} \int^{pert} d^N \tilde{x}\; e^{-\frac{B(\tilde{x})}{2} + \sum_{j=3}^{r} \frac{\boxed{\hbar^{\frac{j}{2}-1}}}{j!} P_j(\tilde{x})}$$

$$P(x) = \sum_{j=3}^{r} \frac{1}{j!} P_j(x)$$

no coupling constants

"coupling constants" $\hbar^{\sum_{\text{vertices}}\left(\frac{\mathrm{val}(v)}{2}-1\right)}$

$$= \det{}^{-\frac12}\left(\frac{\beta}{2\pi\hbar}\right) \sum_{\substack{\text{graphs } \Gamma\\ \text{valence} \geq 3}} \frac{1}{|\mathrm{Aut}\,\Gamma|}\; \underbrace{\hbar^{-\#\text{vertices} + \#\text{edges}}}_{\hbar^{-\chi(\Gamma)}}\; \Phi_{B^{-1};\, P_3, P_4 \dots P_r}(\Gamma)$$

$$= \det{}^{-\frac12} \frac{\beta}{2\pi\hbar} \cdot \exp\Big(\hbar\Big(\frac{1}{12}$$  $$+ \frac{1}{8}$$  $$\Big) + \hbar^2(\dots) + \dots\Big) \in \hbar^{\frac{N}{2}}\, \mathbb{R}[[\hbar]]$$

- finitely many graphs contribute to each order in $\hbar$

Fresnel version:

$$\int d^N x\; e^{\frac{i}{\hbar}\left(\frac12 B(x) + P(x)\right)} = \det{}^{-\frac12}\frac{\beta}{2\pi\hbar} \cdot e^{\frac{\pi i}{4}\operatorname{sign}(\beta)} \cdot \sum_{\text{graphs}} \frac{1}{|\mathrm{Aut}\,\Gamma|}\, \hbar^{-\chi(\Gamma)} \cdot i^{\#\text{ver} + \#\text{edges}}\; \Phi(\Gamma)$$

Ex: $$\overset{(*)}{\int} d^N x\; e^{-\frac{B(x)}{2} + \frac{g_2}{2} P_2(x)} = \det{}^{-\frac12}\frac{\beta}{2\pi} \cdot \exp\Big(\frac{g_2}{2\cdot 1}$$  $$+ \frac{g_2^2}{2\cdot 2}$$  $$+ \boxed{\frac{g_2^3}{2\cdot 3}}$$  $$+ \dots\Big) =$$

$$\frac{\cancel{\dots}}{2\cdot n} = |\mathrm{Aut}(\mathrm{wh}_n)|$$

$$= \det{}^{-\frac12}\frac{\beta}{2\pi} \cdot \exp \sum_{n=1}^{\infty} \frac{g_2^n}{2n}\, \mathrm{tr}\,\underbrace{(B_b^{-1} P_2^{\#})}_{\widehat{\mathrm{End}}\, V}{}^n \;\circleddash$$

$$V \xrightarrow{P_2^{\#}} V^* \xrightarrow{B_b^{-1}} V$$

if $P_2(x) = \langle x, p\, x\rangle$, $p : V \to V$, then $(B^{-1} P_2^{\#}) = \beta^{-1} p$

$$\circleddash\; \det{}^{-\frac12}\frac{\beta}{2\pi} \cdot \exp \underbrace{\sum \frac{g_2^n}{2n}\, \mathrm{tr}\,(\beta^{-1} p)^n}_{-\frac12 \cancel{\log}\, \mathrm{tr}\log(1 - g_2 \beta^{-1} p)} = \det{}^{-\frac12}\frac{\beta}{2\pi} \cdot e^{-\frac12 \mathrm{tr}\log(1 - g_2\beta^{-1} p)} = (2\pi)^{+\frac{N}{2}}\left[\det\beta \cdot \det(1 - g_2 \beta^{-1} p)\right]$$

for g_2 small enough

$$= \boxed{(2\pi)^{\frac{N}{2}} \det{}^{-\frac12}(\beta - g_2 p)}$$

= lhs of (*) evaluated as a pure (non-perturbed) Gaussian integral!

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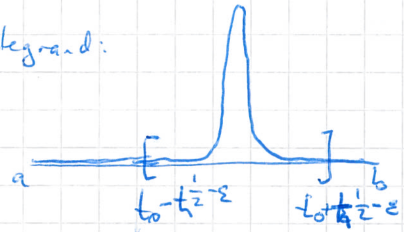
Ex
$$\int_V^{\text{pert}} d^N x \cdot e^{-\frac{B(x,x)}{2} + g_1 \underset{\text{lin. term}}{y(x)}} = (2\pi)^{\frac{N}{2}} \det{}^{-\frac{1}{2}} B \cdot e^{\frac{g_1}{2} \overset{\langle B^{-1}, y\otimes y\rangle}{\bullet\!\!-\!\!\!-\!\!\bullet}}$$
$$= (2\pi)^{\frac{N}{2}} \det{}^{-\frac{1}{2}} B \; e^{\frac{g_1}{2}\langle y, B^{-1} y\rangle}$$
– the actual value of Gaussian $\int$ with source term.

Thm (Laplace) Let $f \in C^\infty([a,b])$, attaining (absolute) a minimum at $t_0 \in (a,b)$; assume that $f''(t_0) > 0$.

Then asymptotically, at $\hbar \to 0$,
$$(*) \qquad \int_a^b e^{-\frac{f(t)}{\hbar}} dt \underset{\hbar\to 0}{\sim} \sqrt{\frac{2\pi\hbar}{f''(t_0)}}\; e^{-\frac{f(t_0)}{\hbar}}$$

Idea: fix $\varepsilon > 0$, $\varepsilon < \frac{1}{2}$ integrand:



step 1: restrict the integral from $I = [a,b]$ to $I_\hbar = [t_0 - \hbar^{\frac{1}{2}-\varepsilon}, t_0 + \hbar^{\frac{1}{2}-\varepsilon}]$

(estimate: complement of $I_\hbar$ contributes negligibly to the integral,
$$e^{-\frac{f}{\hbar}} = e^{-\frac{f(t_0)}{\hbar}} \cdot F, \quad \underset{\text{for } \hbar < \hbar_0(\varepsilon)}{F < C \cdot e^{-C' \hbar^{-2\varepsilon}}} \text{ outside } I_\hbar \;)$$
$$\boxed{F = e^{-\frac{f(t)-f(t_0)}{\hbar}} = e^{-f''(t_0)\cdot\frac{(t-t_0)^2}{2\hbar} + O((t-t_0)^3)}}$$

Step 2: replace the integrand by $\underbrace{e^{-\frac{f(t_0)}{\hbar} - \frac{f''(t_0)}{2\hbar}(t-t_0)^2}}_{\text{first terms of Taylor series for } f \text{ around } t=t_0}$ in $I_\hbar$ – [estimate the error; it is $e^{-\frac{f(t_0)}{\hbar}} \cdot O(\hbar)$]

$$\underbrace{O(\hbar^{3\cdot(\frac{1}{2}-\varepsilon)-1})}_{\text{error in integrand}} \cdot \hbar^{\frac{1}{2}-\varepsilon}$$
$$\text{error in integral} = O(\hbar^{1-4\varepsilon})$$

Step 3 replace $I_\hbar$ by $(-\infty, \infty)$ (estimate the error like in step 1)

– we have a Gaussian integral
$$\int_a^b e^{-\frac{f(t)}{\hbar}} dt \rightsquigarrow \int_{-\infty}^{\infty} e^{-\frac{f(t_0)}{\hbar} - \frac{f''(t_0)}{2\hbar}(t-t_0)^2} dt = \text{rhs of } (*)$$

Rem: modification: for in addition $g \in C^\infty[a,b]$, $g(t_0) \neq 0$, then
$$\int_a^b e^{-\frac{f(t)}{\hbar}} g(t)\, dt \underset{\hbar\to 0}{\sim} \sqrt{\frac{2\pi\hbar}{f''(t_0)}}\; e^{-\frac{f(t_0)}{\hbar}} \cdot g(t_0)$$

covered in lectures

def let $\Phi \in C^\infty((0,\infty))$ and $\varphi_n \in C^\infty((0,\infty))$, $n \in \mathbb{N}$

We say that $\Phi(\hbar) \sim \sum_{n=1}^\infty \varphi_n(\hbar)$ is an asymptotic series for Φ at $\hbar \to 0$

if $\Phi(\hbar) \underset{\hbar\to 0}{\sim} \varphi_1(\hbar)$ (i.e. $\lim_{\hbar\to 0} \frac{\Phi(\hbar)}{\varphi_1(\hbar)} = 1$) and $\forall N \geq 1$, $\Phi(\hbar) - \sum_{n=1}^N \varphi_n(\hbar) \underset{\hbar\to 0}{\sim} \varphi_{N+1}(\hbar)$

(in particular, this implies that $\lim_{\hbar\to 0} \frac{\varphi_{n+1}(\hbar)}{\varphi_n(\hbar)} = 0 \;\; \forall n$)

Examples:
- any Taylor series with nonzero convergence radius
- $\sum n!\, \hbar^n \underset{\hbar\to 0}{\sim} \ldots -\hbar e^{-\hbar} \mathrm{Ei}(-\hbar)$ $\;\leftarrow \mathrm{Ei} \quad \int_{-\hbar}^{\infty} \frac{e^{-t}}{t} dt$
- Stirling formula with corrections: $\Gamma(z) \underset{z\to\infty}{\sim} \sqrt{2\pi}\, z^{z-\frac{1}{2}} e^{-z} \overbrace{\left(1 + \frac{C_1}{z} + \frac{C_2}{z^2} + \cdots\right)}^{\exp \sum_{n=1}^\infty \frac{B_{2n}}{2n(2n-1)z^{2n-1}}}$

  ↗ has zero convergence radius in $\frac{1}{z}$.

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Multidim version of Laplace formula with corrections:

Thm Let M be a compact N-mfld, μ a volume form, $f \in C^\infty(M)$, attaining the (unique) absolute minimum at $p \in M$ [int(M) if M has boundary] ~~assume that~~ Let (x^i) be a coord chart around p, s.t. $x^i(p) = 0$ and s.t. $\mu = \prod_i dx^i$ locally on M and assume that $\det \underbrace{\left.\frac{\partial^2 f}{\partial x^i \partial x^j}\right|_p}_{\text{Hessian at } p.} > 0$. [it cannot be < 0, since p is a minimum]

Then:
$$\int_M e^{-\frac{f(x)}{\hbar}} \mu \underset{\hbar \to 0}{\sim} (2\pi\hbar)^{\frac{N}{2}} \det{}^{-\frac{1}{2}}\left(\left.\frac{\partial^2 f}{\partial x^i \partial x^j}\right|_p\right) e^{-\frac{f(p)}{\hbar}} \cdot \sum_{\substack{\text{graphs } \Gamma \\ \text{val} \geq 3}} \hbar^{-\chi(\Gamma)} \frac{1}{|\mathrm{Aut}\,\Gamma|} \Phi_{(\partial^2_p f)^{-1}; -\partial^3_p f, -\partial^4_p f, \dots}(\Gamma)$$

$$/\!/ \quad \int^{\text{pert}}_{U_p \subset M} e^{\overbrace{-\frac{f(p)}{\hbar}}^{\text{constant}} - \frac{1}{\hbar}\left(\underbrace{\tfrac{1}{2}\partial^2_p f(x,x)}_{\text{Gaussian part}} + \underbrace{\sum_{n \geq 3} \tfrac{1}{n!} \partial^n_p f(x \dots x)}_{\text{perturbation}}\right)} d^N x$$

$$\frac{1}{n!} \sum_{i_1 \dots i_n} \left.\frac{\partial^n f}{\partial x^{i_1} \dots \partial x^{i_n}}\right|_p x^{i_1} \dots x^{i_n}$$

here $\partial^k_p f \in \mathrm{Sym}^k T^*_p M$

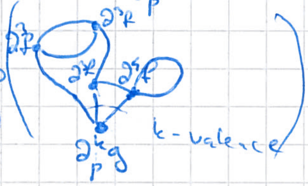
$x \in U \sim T_p M$

$T_p U$

$\varphi: U \to M$ coord chart

polyn. fn. on (nbhd of origin in) $T_p M \sim$ nbhd of p in U (chart)

Rem
$$\int_M e^{-\frac{f(x)}{\hbar}} g(x) \cdot \mu \underset{\hbar \to 0}{\sim} (2\pi\hbar)^{\frac{N}{2}} \det{}^{-\frac{1}{2}}(\partial^2_p f) \cdot e^{-\frac{f(p)}{\hbar}} \cdot \sum_{\substack{\text{graphs} \\ \text{with one marked} \\ \text{vertex}}} \hbar^{1+\chi(\Gamma)} \frac{1}{|\mathrm{Aut}\,\Gamma|} \Phi$$



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Stationary phase formula

1D version:

Thm Let $f \in C^\infty(\mathbb{R})$, $g \in C^\infty_{\text{comp supp}}(\mathbb{R})$. Let $I(\hbar) = \int_{\mathbb{R}} g(t)\, e^{\frac{i}{\hbar} f(t)} dt$.

"principle of non-stationary phase":
(i) If f has no crit. points on supp(g) then ~~$\int_{\mathbb{R}} g(t) e^{\frac{i}{\hbar} f(t)} dt$~~ $I(\hbar) \underset{\hbar \to 0}{\sim} O(\hbar^\infty)$, i.e. $\forall N \ \lim_{\hbar \to 0} \frac{I(\hbar)}{\hbar^N} = 0$

(ii) Let f has crit points $\{t_{(j)}\}$ on supp(g) which are all isolated and non-degenerate: with $\partial^2 f|_{t_{(j)}} \neq 0$ $/\!/ \ f''(t_{(j)})$

Then
$$I(\hbar) \underset{\hbar \to 0}{\sim} \sum_j \sqrt{2\pi\hbar}\, g(t_{(j)})\, \left|\partial^2 f|_{t_{(j)}}\right|^{-\frac{1}{2}} \cdot e^{\frac{\pi i}{4} \operatorname{sign} \partial^2 f|_{t_{(j)}}} \cdot e^{\frac{i}{\hbar} f(t_{(j)})} + ~~O(\hbar)~~ \ o(\underbrace{\hbar^{\frac{1}{2}}}_{\sqrt{2\pi\hbar}})$$

Proof of (i): can assume supp$(g) \subset [t_0, t_1]$; f is monotonous on $[t_0, t_1]$

make a change of coords s.t. $[t_0, t_1] \longrightarrow [T_0, T_1]$ s.t. $f(t) = \tau$
$t \longmapsto \tau(t)$

$$I(\hbar) = \int_{T_0}^{T_1} \underbrace{\frac{g(f^{-1}(\tau))}{f'(f^{-1}(\tau))}}_{h(\tau)} e^{\frac{i}{\hbar}\tau} d\tau = ~~O(1)~~ \ ~~\text{(bounded) as } \hbar \to 0~~$$

$$0 = \int \frac{\partial}{\partial\tau}\left(h \cdot e^{\frac{i}{\hbar}\tau}\right) d\tau = \int h' e^{\frac{i}{\hbar}\tau} d\tau + \frac{i}{\hbar} \int h e^{\frac{i}{\hbar}\tau} d\tau$$

$$\Rightarrow I(\hbar) = +i\hbar \int h' e^{\frac{i}{\hbar}\tau} d\tau = O(\hbar) \Longrightarrow I(\hbar) = (+i\hbar)^N \int h^{(N)}(\tau) e^{\frac{i}{\hbar}\tau} d\tau = O(\hbar^N),$$

Fedoryuk "Saddle point method"

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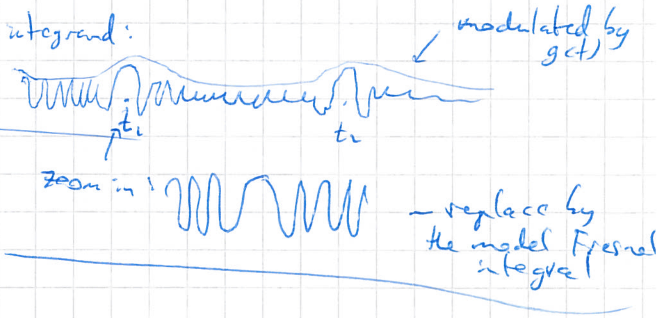
Rough idea for (ii):

take a partition of unity $1 = \sum_j \varphi_j(t) + \psi(t)$, e.g. $\overset{U_j}{\operatorname{supp}\varphi_j}$ = small nbhd of $t_{(j)}$
non-negative
supp ψ supported away from $t_{(j)}$

$$I(\hbar) = \sum_j \underbrace{\int_{U_j} \varphi_j(t)\, g(t)\, e^{\frac{i}{\hbar} f(t)} dt}_{\text{replace by } g(t_j)\, e^{\frac{i}{\hbar}\left(f(t_j) + \frac{1}{2} f''(t_j)(t-t_j)^2\right)}} + \underbrace{\int \psi(t)\, g(t)\, e^{\frac{i}{\hbar} f(t)} dt}_{O(\hbar^\infty)}$$

$\rightarrow$ replace $\int_{U_j} \rightarrow \int_{\mathbb{R}}$



Multidim version (with corrections)

Let M be a mfld, $g \in C_c^\infty(M)$, $f \in C^\infty(M)$
f has only isolated, non-deg (i.e. $\det \partial^2 f \neq 0$) critical points inside supp g, $\{p_j\} \subset M$. Consider $I(\hbar) = \int_M g\, e^{\frac{i}{\hbar} f} \cdot \mu$
Let μ be a volume form

- if $\{p_j\} = \emptyset$, then
$$I(\hbar) \underset{\hbar \to 0}{=} O(\hbar^\infty)$$
- for $\{p_j\} \neq \emptyset$, assume that around p_j we have a local chart $\{x_i^{(j)}\}$ in $\overset{p_j}{\overset{\cap}{U_j}} \subset M$ s.t. $\mu|_{U_j} = \prod_i dx_i^{(j)}$

$$I(\hbar) \underset{\hbar \to 0}{\sim} \sum_j (2\pi\hbar)^{\frac{N}{2}} \det{}^{-\frac{1}{2}} |\partial^2 f|_{p_j}| \cdot e^{\frac{\pi i}{4} \operatorname{sign} \partial^2 f|_{p_j}} \cdot \cancel{g(p_j)}\, e^{\frac{i}{\hbar} f(p_j)} \cdot \sum_{\substack{\text{graphs } \Gamma \\ \text{with 1 marked vertex} \\ \text{and other vertices} \\ \text{of val} \geq 3}} \frac{\hbar^{1-\chi(\Gamma)}}{|\mathrm{Aut}\,\Gamma|} \cdot \Phi_{\Gamma;\, \partial^2 f|_{p_j};\, \{i\partial^k f|_{p_j}\};\, \{\partial^l g|_{p_j}\}_{l \geq 0}}$$

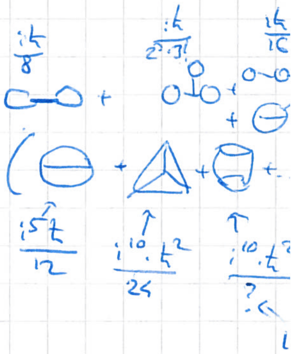
marked (unique) vertex

$$= \sum_j \int^{\text{pert}} e^{\frac{i}{\hbar}\left(f(p_j) + \frac{1}{2}\partial^2 f|_{p_j}(x,x) + \sum_{k \geq 3} \frac{1}{k!} \partial^k f|_{p_j}(\underbrace{x \ldots x}_{k})\right)} \times \left(\sum_{l \geq 0} \frac{1}{l!} \partial^l g|_{p_j}(\underbrace{x \ldots x}_{l})\right) d^N x$$

$x = x^{(j)}$ – loc. coord on U_j

Ex: $\overset{*}{}$ cutoff at $t = 0$

$$\int_{-\infty} e^{\frac{i}{\hbar}\left(\frac{t^2}{2} + \frac{t^3}{6}\right)} dt \underset{\hbar \to +0}{\simeq} \sqrt{2\pi\hbar} \cdot e^{\frac{\pi i}{4}} \exp \sum_{\substack{\text{conn.} \\ \text{3-val} \\ \text{graphs } \Gamma}} \frac{i^{V+E} \cdot \hbar^{-\chi(\Gamma)}}{|\mathrm{Aut}\,\Gamma|} = \sqrt{2\pi\hbar}\, e^{\frac{\pi i}{4}} \exp\Big(\ \ldots$$



$$= \sqrt{2\pi\hbar} \cdot e^{\frac{\pi i}{4}} \sum_{n=0}^\infty \frac{(6n-1)!!}{(2n)!\, 6^{2n}} (i\hbar)^n$$

$V = 2n$
$E = 3n$

counting all matchings for $3 \cdot (2n)$ half-edges

[Rem: this can be expressed via Airy function]

Correction: 2 crit. points $f(t) = \frac{t^2}{2} + \frac{t^3}{6}$, $f'(t) = t \cdot \left(1 + \frac{t}{2}\right)$ $t_1 = 0$, $t_2 = -2$

$f''(t) = 1 + t$
$f'''(t) = 1$

$f''(t_1) = 1$ $f'''(t_1) = 1$ $f(t_1) = 0$
$f''(t_2) = -1$ $f'''(t_2) = 1$ $f(t_2) = 2 - \frac{8}{6} = \frac{2}{3}$

So,
$$\int_{-\infty}^{\infty} e^{\frac{i}{\hbar}\left(\frac{t^2}{2} + \frac{t^3}{6}\right)} dt \underset{\hbar \to +0}{\sim} \sqrt{2\pi\hbar}\, e^{\frac{\pi i}{4}} \exp \sum_{\substack{\text{conn.} \\ \text{3-val. graphs} \\ V = 2n \\ E = 3n}} \frac{(i\hbar)^n}{|\mathrm{Aut}\,\Gamma|} + \sqrt{2\pi\hbar}\, e^{-\frac{\pi i}{4}} \cdot e^{\frac{2}{3}\frac{i}{\hbar}} \exp \sum_{\text{—''—}} \frac{(i\hbar)^n}{|\mathrm{Aut}\,\Gamma|} \cdot \underbrace{(-1)^n}_{(-1)^E}$$

$i^{V+E} \hbar^{E-V}$

$(-1)^E$ – from sign of the propagator

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Berezin integral

V - f.d. vector space "ΠV" - corresp. odd vector space, $C^\infty(\Pi V) := \Lambda^\bullet V^*$

if ~~e~~ $\{e_i\}$ - basis in V, $x^i \in V^*$ - (even) coordinates, $\theta^i \in V^*$ - (odd) coords

& $\mathrm{Fun}_{poly}(V) = \mathbb{R}[x^1 \dots x^N]$ $\quad \underset{\mathrm{Fun}(\Pi V)}{\underset{\shortparallel}{C^\infty(\Pi V)}} = \mathbb{R}\langle\theta^1 \dots \theta^N\rangle / _{\theta^i\theta^j = -\theta^j\theta^i}$

$\frac{\partial}{\partial\theta_i} : \mathrm{Fun}(\Pi V) \to \mathrm{Fun}(\Pi V)$

- odd derivation: $\frac{\partial}{\partial\theta_i}(f\cdot g) = \frac{\partial f}{\partial\theta_i} g + (-1)^{|f|} f \frac{\partial}{\partial\theta_i} g$ s.t. $\frac{\partial\theta_j}{\partial\theta_i} =$

1D Berezin integral

for $V = \mathbb{R}$, $\mathrm{Fun}(\Pi\mathbb{R}) = \mathbb{R}[\theta]/_{\theta^2=0} \ni \underset{a,b\in\mathbb{R}}{a + b\theta} =: f(\theta)$

(*) $\boxed{\int_{\Pi\mathbb{R}} D\theta}\; \underset{\mathrm{Fun}(\Pi\mathbb{R})}{\underset{\cap}{f(\theta)}} \overset{def}{:=} (\text{coeff. of } \theta \text{ in } f) = b$ $\quad$ Rem: $\int_{\Pi\mathbb{R}} D\theta\, f(\theta) = \frac{\partial}{\partial\theta} f(\theta)$

motivation: we want Stokes' f-la: $\int_{\Pi\mathbb{R}} D\theta \frac{\partial}{\partial\theta}(\dots) = 0$ and linearity in f

This implies (*) (up to a constant)

multi-dim version:

$$\int_{\Pi\mathbb{R}^N} D\theta^1 \dots D\theta^N \underset{\mathrm{Fun}(\Pi\mathbb{R}^N)}{\underset{\cap}{\underbrace{f(\theta^1 \dots \theta^N)}}} \overset{def}{:=} \text{coefficient of } \underbrace{\theta^N \dots \theta^1}_{=(-1)^{\frac{N(N-1)}{2}}\theta^1 \dots \theta^N} \text{ in } f = \frac{\partial}{\partial\theta^1} \dots \frac{\partial}{\partial\theta^N} f \in$$

Rem: with this definition, Fubini's thm holds: $\int_{\Pi\mathbb{R}^N} D\theta^1 \dots D\theta^N f = \int_{\Pi\mathbb{R}} D\theta^1 \dots \underbrace{\int_{\Pi\mathbb{R}} D\theta^N f}_{\overset{\cap}{\mathbb{R}[\theta^1 \dots \theta^{N-1}]_{odd}}}$

More abstractly

$\mu \in \Lambda^N V$ - "Berezinian" on ΠV (replacement for the volume form/measure/density for ordinary integration)

Berezin integration

$\boxed{\int_{\Pi V} \mu} \cdot : \mathrm{Fun}(\Pi V) \to \mathbb{R}$

$f \longmapsto \langle \mu, f^{top} \rangle$ ← ~~top~~ component of f in $\Lambda^N V^*$

— pairing between $\Lambda^N V$ and $\Lambda^N V^*$

$\langle \alpha_1 \wedge \dots \wedge \alpha_N, h^N \wedge \dots \wedge h^1 \rangle =$

$= \det(\underset{\substack{\uparrow \\ \text{pairing btw } V \text{ and } V^*}}{\langle \alpha_i, h^j \rangle})_{i,j=1..N}$

Note Constant vol. ~~elements~~ forms on V are elements of $\Lambda^N V^*$

Berezinians on ΠV are elements of $\Lambda^N V$ (without star!)

& in particular, behavior under change of bases: $\theta'^i = \sum_j A_{ij} \theta^j \Rightarrow \underset{D\theta'^1 \dots D\theta'^N}{\underset{\shortparallel}{D\theta'}} = (\det A)^{-1} D\theta$ (**)

as opposed to the transform of vol forms

(Also note: no absolute value in (**) → $\int_{\Pi V}$ depends on orientation)

Gaussian the integral over ΠV

Let $B \in \Lambda^2 V^*$ - a quadratic element in $\mathbb{R}[\theta^1 \dots \theta^N]$, $\underbrace{}_{B(\theta,\theta) = \sum B_{ij}\theta^i\theta^j \text{ odd}}$ & $B_{ij} = -B_{ji}$

$\exp(B) \overset{def}{=} \sum_{k\geq 0} 1 + B + \frac{1}{2} B \wedge B + \dots + \frac{1}{[\frac{N}{2}]!} \underbrace{B \wedge \dots \wedge B}_{[\frac{N}{2}]}$ - higher terms of Taylor series for exp vanish

for $N = 2n$

mma $\int_{\Pi\mathbb{R}^N} D\theta^N \dots D\theta^1 \exp\frac{1}{2}B(\theta,\theta) = \mathrm{Pf}(B_{ij}) := \sum \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \cancel{\mathrm{sign}}\, (-1)^\sigma \prod_{j=1}^{n} B_{\sigma_{2j-1}\sigma_{2j}}$ - Pfaffian of B

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Ex: • $n=1$ $\quad B = B_{12}\theta^1\theta^2 + B_{21}\theta^2\theta^1 = 2B_{12}\theta^1\theta^2$

$$\int D\theta^2 D\theta^1 \exp \underbrace{\tfrac{1}{2}B(\theta,\theta)}_{B_{12}\theta^1\cdot\theta^2} = \int D\theta^2 D\theta^1 (1 + B_{12}\theta^1\theta^2) = B_{12}$$

• $n=2$

$$\int D\theta^4 D\theta^3 D\theta^2 D\theta^1 \cancel{\Sigma} \exp \underbrace{\tfrac{1}{2}B(\theta,\theta)}_{\substack{B_{12}\theta^1\theta^2 + B_{13}\theta^1\theta^3 + B_{14}\theta^1\theta^4 \\ + B_{23}\theta^2\theta^3 + B_{24}\theta^2\theta^4 + B_{34}\theta^3\theta^4}} = \int D\theta^4 \cdots D\theta^1 \tfrac{1}{2}\left(\tfrac{1}{2}B(\theta,\theta)\right)^2 =$$

$$= B_{12}B_{34} - B_{13}B_{24} + B_{14}B_{23} = \mathrm{Pf}(B_{ij})$$

Rem Fact about Pfaffians: $(\mathrm{Pf}\, B_{ij})^2 = \det B_{ij}$

Also: $\mathrm{Pf}(ABA^T) = \det A \cdot \mathrm{Pf}\, B$, $\mathrm{Pf}(A_1 \oplus A_2) = \mathrm{Pf}\, A_1 \cdot \mathrm{Pf}\, A_2$

$\mathrm{Pf}(\lambda \cdot B) = \lambda^n\, \mathrm{Pf}\, B$

E.g. $\mathrm{Pf}\begin{pmatrix} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & a_n \\ & & & -a_n & 0 \end{pmatrix} = a_1 \cdots a_n$

$\det(\text{— " —}) = a_1^2 \cdots a_n^2$

Special case of a Gaussian Berezin integral.

$$(*) \int_{\Pi\mathbb{R}^{2N} = \Pi\mathbb{R}^N \oplus \Pi\mathbb{R}^N} (D\theta^1 D\psi_1) \cdots (D\theta^N D\psi_N) \cdot \exp\Big(\sum_{i,j=1}^{N} C_{ij}\psi_i\theta^j\Big) = \det(C_{ij})$$

$\uparrow$ coords $\theta^1 \ldots \theta^N$ $\quad$ coords $\psi_1 \ldots \psi_N$

$\uparrow$ no condition of anti-symmetry

Proof: lhs $= \frac{1}{N!}\cancel{\Sigma} \int \Big(\sum_{ij} C_{ij}\psi_i\theta^j\Big)^N = \cancel{\Sigma}\frac{1}{N!}\int \sum_{i_1 j_1 \ldots i_N j_N} (C_{i_1 j_1}\psi_{i_1}\theta^{j_1}) \cdots (C_{i_N j_N}\psi_{i_N}\theta^{j_N}) =$

only terms where $i_1 \ldots i_N$ and $j_1 \ldots j_N$ are permutations of $1 \ldots n$ contribute

$$= \frac{1}{N!}\sum_{\sigma,\sigma' \in S_N} (-1)^{\sigma}\cdot(-1)^{\sigma'} C_{\sigma(1)\sigma'(1)} \cdots C_{\sigma(N)\sigma'(N)} = \det C \quad \Box$$

Invariantly ~~Abstractly~~, $(*)$ says:

$$\boxed{\int_{\Pi V \oplus \Pi V^*} \underbrace{\mu_{can}^{\Pi V \oplus \Pi V^*}}_{\underbrace{\mu^{\Pi V}}_{\Lambda^N V} \cdot \underbrace{(\mu^{\Pi V})^*}_{(\Lambda^N V)^* \simeq \Lambda^N V^*}} \cdot \exp \overset{C(\psi,\theta)}{C} = \det C}$$

$$C \in V^* \otimes V \simeq \mathrm{End}\, V$$

$$\cap$$

$$\mathrm{Fun}(\Pi V \oplus \Pi V^*)$$

$$\shortparallel$$

$$\Lambda V^* \otimes \Lambda V$$

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8/1

"Odd Wick's lemma"

$N = 2 \cdot n$

Example: (model correlator)

assume $\frac{1}{2}B(\theta,\theta) = \pm b_1\theta_1\theta_2 \pm b_2\theta_3\theta_4 + \dots \pm b_n\theta_{N-1}\theta_N$

$$\int_{\Pi V} D\theta_N \cdots D\theta_1 \; e^{\frac{1}{2}B(\theta,\theta)} \; \theta_1\theta_2 = \underset{\text{Fubini}}{=} \int_{\Pi\mathbb{R}^{N-2}} D\theta_N \cdots D\theta_3 \cdot e^{\sum_{3\le i<j\le N} B_{ij}\theta_i\theta_j} = \mathrm{Pf}(B_{ij})_{i,j\ge 3} =$$

$$= \underbrace{(\mathrm{Pf}\, B)}_{\int_{\Pi V} D\theta \, e^{\frac{1}{2}B(\theta,\theta)}} \cdot (B_{12})^{-1} = (\mathrm{Pf}\, B)\cdot (-B^{-1})_{12}$$

More generally: • let $\xi_1, \xi_2 \in V^*$

assuming that B is non-degenerate

then
$$\int_{\Pi V} D\theta \; e^{-\frac{1}{2}B(\theta,\theta)} \underbrace{\xi_1 \wedge \xi_2}_{\overset{\cap}{\Lambda^2 V^* \subset \mathrm{Fun}(\Pi V)}} = \underbrace{\mathrm{Pf}(-B)}_{(-1)^n \mathrm{Pf}(B)} \cdot \langle \underset{\overset{\cap}{\Lambda^2 V \subset V^{\otimes 2}}}{B^{-1}}, \underbrace{\xi_1 \otimes \xi_2}_{\overset{\in}{(V^*)^{\otimes 2}}} \rangle$$

Odd Wick's lemma:
for $B \in \Lambda^2 V^*$ non-deg, $\xi_1 \dots \xi_{2m} \in V^* \subset \Lambda^\bullet V^* = \mathrm{Fun}(\Pi V)$ for some m, we have

$$\int_{\Pi V} D\theta \; e^{-\frac{1}{2}B(\theta,\theta)} \underbrace{\xi_1 \wedge \dots \wedge \xi_{2m}}_{\overset{\cap}{\Lambda^{2m} V^*}} = \mathrm{Pf}(-B) \cdot \sum_{\sigma \in S_{2m}/S_m \ltimes \mathbb{Z}_2^m} (-1)^\sigma \langle \sigma \circ (B^{-1})^{\otimes m}, \xi_1 \otimes \dots \otimes \xi_{2m} \rangle$$

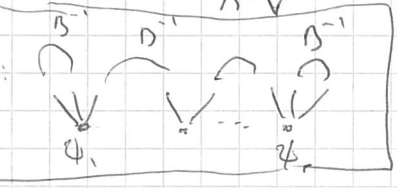
Sign!

• Likewise, for $\psi_j \in \Lambda^{s_j} V^*$, $j = 1 \dots r$, a collection of homogen. polynomials on ΠV,

$2m = \sum s_j$

$$\int_{\Pi V} D\theta \cdot e^{-\frac{1}{2}B(\theta,\theta)} \underbrace{\frac{1}{s_1!}\psi_1 \wedge \dots \wedge \frac{1}{s_r!}\psi_r}_{\overset{\cap}{\Lambda^\bullet V}} = \mathrm{Pf}(-B) \sum_{\sigma \in \prod_j S_{s_j} \backslash S_{2m}/S_m \ltimes \mathbb{Z}_2^m} \frac{(-1)^\sigma}{|\mathrm{Stab}_{[\sigma]}|} \langle \sigma \circ (B^{-1})^{\otimes m}, \underbrace{\tilde{\psi}_1 \otimes \dots \otimes \tilde{\psi}_r}_{\overset{\cap}{(V^*)^{\otimes 2m}}} \rangle$$

Graphically:

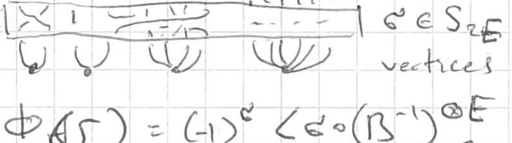


• Perturbed odd Gaussian integral over ΠV:

Thm
$$\int_{\Pi V} D\theta \; e^{-\frac{1}{2}B(\theta,\theta) + \sum_{j=2, \, j \text{ even}}^{r} \frac{g_j}{j!} P_j} = \mathrm{Pf}(-B) \sum_{\substack{\text{graphs } \Gamma \\ (\text{vertices of even valence})}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \Phi_{B^{-1};\{g_j \psi_j\}}(\Gamma) \qquad (*)$$

where $P_j \in \Lambda^j V^*$

$\Gamma =$



$$\Phi(\Gamma) = \underset{\substack{\uparrow \\ \text{Sign!}}}{(-1)^\sigma} \langle \sigma \circ (B^{-1})^{\otimes \underset{\substack{\uparrow \\ \text{no. of edges}}}{E}}, \bigotimes_j \tilde{P}_j^{\otimes \underset{\substack{\uparrow \\ \text{no. of vertices of valence } j \text{ in } \Gamma}}{v_j}} \rangle \cdot \prod_j g_j^{v_j}$$

Rem (*) is not a definition of perturb. evaluation of an integral, but a formula for Berezin int. (which always converges)

Perturbative integral over a $\overset{\text{vector}}{\text{superspace}}$.

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8/2

superspace $\mathbb{V} = (\underset{\text{- pair of vector spaces}}{V^{\text{even}}, V^{\text{odd}}}) = V^{\text{even}} \oplus \Pi V^{\text{odd}}$

$\mathrm{Fun}(\mathbb{V}) = \underline{\mathrm{Sym}^\bullet V^*_{\text{even}}} \otimes \Lambda^\bullet V^*_{\text{odd}}$

or $C^\infty(V_{\text{even}})$

or some other model

$\in \mathrm{Sym}^j V_e^* \otimes \Lambda^k V_o^*$

$$\int^{\text{pert}}_{\mathbb{V}} \underset{\substack{\uparrow\\ \text{coords}\\ \text{on } V_{\text{even}}}}{dx} \; \underset{\substack{\uparrow\\ \text{coords}\\ \text{on } V_{\text{odd}}}}{D\theta} \cdot e^{-\frac12 B_e(x,x) - \frac12 B_o(\theta,\theta) + \sum\limits_{\substack{j,k\\ k \text{ even}}} \frac{g_{j,k}}{j!\,k!} P_{j,k}(\underbrace{x,\dots,x}_{j}, \underbrace{\theta,\dots,\theta}_{k})} =$$

$$= \det{}^{-\frac12}\frac{B_e}{2\pi} \cdot \mathrm{Pf}(-B_o) \cdot \sum_{\substack{\text{graphs } \Gamma\\ \text{with \underline{edges} colored in } \{e,o\}\\ \text{vertices have even}\\ o\text{-valence}}} \frac{1}{|\mathrm{Aut}\,\Gamma|} \cdot \Phi_{\Xi}(\Gamma)$$

Ξ - set of Feynman rules:

$\overset{e}{\text{——}} \rightsquigarrow B_e^{-1}$

$\overset{o}{\text{——}} \rightsquigarrow B_o^{-1}$

edges

j vertices

$\rightsquigarrow g_{j,k} P_{j,k} \in \mathrm{Sym}^j V_e^* \otimes \Lambda^k V_o^* \subset (V_e^*)^{\otimes j} \otimes (V_o^*)^{\otimes k}$

$$\Phi(\Gamma) = \boxed{(-1)^{\sigma_o}} \left\langle \left(\sigma_e \circ (B_e^{-1})^{\otimes E_e}\right) \otimes \left(\sigma_o \circ (B_o^{-1})^{\otimes E_o}\right), \bigotimes_{j,k} (g_{jk}\tilde{P}_{j,k})^{\otimes V_{j,k}} \right\rangle$$

$$\Gamma \longleftrightarrow (\sigma_e, \sigma_o) \in \frac{S_{2E_e} \times S_{2E_o}}{\prod_{j,k} S_{V_{j,k}} \ltimes (S_j \times S_k)^{V_{j,k}}} \Big/ (S_{E_e} \ltimes \mathbb{Z}_2^{E_e}) \times (S_{E_o} \ltimes \mathbb{Z}_2^{E_o})$$

sign of σ_o only.

A "Toy QED" integral

Consider a superspace $\mathcal{V} = (V, \underset{\text{odd part}}{U \oplus U^*})$, with V the even part

coords: $x_i \quad \theta_a \quad \bar\theta_a$

<u>data</u>:

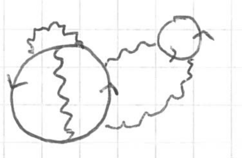
- $B_e(x,x) \in \mathrm{Sym}^2 V^*$ - positive def.
- $B_o(\theta,\bar\theta) = \langle\bar\theta, D\theta\rangle$, $D \in GL(U)$ (non-deg)

$\overset{\Pi}{U^*} \otimes \overset{\Pi}{U} \subset \mathrm{Fun}(\Pi U \oplus \Pi U^*)$

- $P(x,\theta,\bar\theta) \in V^* \otimes U^* \otimes U$ - some element ("photon-electron interaction")

$$\int^{\text{pert}}_{\mathcal{V}} dx\, D\theta\, D\bar\theta\; e^{-\frac12 B_e(x,x) - \langle\bar\theta, D\theta\rangle + g\cdot P(x,\theta,\bar\theta)} = \det{}^{-\frac12}\frac{B_e}{2\pi} \cdot \det(-D) \times$$

$$\times \sum_{\substack{\text{graphs with}\\ \text{edges } \rightsquigarrow\\ \text{and } \rightarrow \text{ (oriented)}\\ \text{vertice}}} \frac{g^{\#\text{vertices}}}{|\mathrm{Aut}\,\Gamma|} \Phi_{B_e^{-1}; B_o^{-1}; P}(\Gamma)$$



← typical Γ.

$P \in V^* \otimes \mathrm{End}(U)$

HE have colors

V ΠU ΠU*

allowed edges

E.g. $\Phi\Big($  $\Big) = g^2 \langle B_e^{-1}, \underbrace{\mathrm{Tr}_U D\circ P\circ D\circ P}_{(\overset{\Pi}{V^*})^{\otimes 2}} \rangle$

Aside: the logic of where we are going / Big Picture

- fin-dim integrals (of stationary phase type) : Feynman graphs give $\hbar \to 0$ asymptotics (to a well-defined integral)

  ↓

  contractions of tensors coming from Taylor-expanding the "action" at a crit pt.

↓

functional integrals $``\int_{\underset{\overset{\uparrow}{\text{bundle}}}{\Gamma(\Sigma,E)}} \mathcal{D}\varphi" \, e^{\frac{i}{\hbar}S[\varphi]}$ - defined (perturbative definition) to be a series in $\hbar$ with coeff. given by Feynman diagrams.

↓

integrals over configurations of points in Σ.

- Functional integrals of interest to physics and mathematics "usually" have local symmetry (gauge)

Yang-Mills

$$\int_{\mathrm{Conn}(\underset{\overset{\uparrow}{\mathbb{R}^{1,3}}}{M},G)} "DA" \, e^{\frac{i}{4g^2}\int_M \mathrm{tr}\, F_A \wedge * F_A}$$

Chern-Simons

$$\int_{\mathrm{Conn}(\underset{\overset{\uparrow}{\text{closed 3-mfd}}}{M},G)} "DA" \, e^{\frac{ik}{2\pi}\int_M \mathrm{tr}\, \frac{1}{2} A\wedge dA + \frac{1}{6} A\wedge[A,A]}$$

Gauge symmetry: action of $\mathrm{Aut}\begin{pmatrix} M\times G \\ \downarrow \\ M\end{pmatrix} \simeq \overset{= \mathcal{G}}{C^\infty(M,G)}$ on connections
$\underset{g}{\rotatebox{90}{$\in$}}$
by $A \longmapsto A^g = gAg^{-1} + gdg^{-1}$.

- instead of crit. points, we have critical G-orbits; to write down Feynman diagram expansion, we need to do something about it - problem of gauge-fixing.

- Methods of gauge-fixing (can be studied for fin-dim integrals)
  - Faddeev-Popov - writing the integral over M/G as an integral over a section.
  - BRST - cohomological formalism for gauge fixing ← allows for reducible gauge-sym
  - BV - symplectic (odd-) cohomological formalism ← works for more general gauge symmetry (e.g. PSM)

- General logic: define $Z = \int_{\Gamma(\Sigma,E)}^{\text{gauge-fixed, perturbative}} \mathcal{D}\varphi \, e^{\frac{i}{\hbar}S[\varphi]}$

  by formal manipulations with PI (e.g. non-linear change of coords, applying a diff. op to integrand, ...)
  we predict a property $\mathcal{P}$ for Z.

  Then $\mathcal{P}$ has to be checked independently for the actual Feynman-diagram expansion.

- Thus, PI is a heuristical tool for constructing (potentially) interesting quantities and predicting their properties.

Some applications

- Chern-Simons theory → invariants of 3-manifolds and knots

$$Z^{\text{pert}}(M,G,k) = \int_{\text{Conn}(M,G)}^{\text{pert g.f.}} DA\; e^{\frac{ik}{2\pi} S_{CS}(A)}$$

– diff. invariant of a (framed, oriented) manifold.

perturbative invariants (Casson etc.)

Kontsevich's integral, Vassiliev's invariants (Bar-Na

Jones polynomial, (Witten)

HOMFLY

$L = K_1 \sqcup \ldots \sqcup K_n \subset M$ a link

$R_1 \ldots R_n$ – representations of G.

$$\int_{\text{Conn}(M,G)}^{\text{pert g.f.}} DA\; e^{\frac{ik}{2\pi} S_{CS}(A)} \cdot \prod_{j=1}^{n} \underbrace{\operatorname{tr}_{R_j} \operatorname{Hol}_{K_j}(A)}_{\text{"Wilson loop"}}$$

- BF theory → Ray-Singer torsion (A.S. Schwarz)
  - 3D: Alexander's polynomial for knots
  - higher-dim knot invariants, cohomology of the space of embeddings
  - homotopy transfer ⇝ rational homotopy type.
  - 3D gravity; $q \to 1$ asymptotics of Turaev-Viro invariant

- Poisson sigma-model (Schaller-Strobl) → Kontsevich's deformation quantization of Poisson manifolds (M,π)

$$C^\infty(M), \cdot \;\overset{\text{deformation}}{\rightsquigarrow}\; C^\infty(M)[[\hbar]], \underset{\text{associative product}}{*_\hbar}$$

$$f *_\hbar g = f\cdot g - \frac{i\hbar}{2}\{f,g\} + \sum_{n\geq 2} \underset{\text{bi-diff. operator}}{B_n(f,g)} \cdot \hbar^n$$

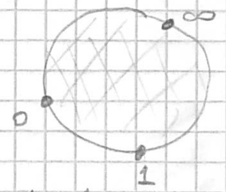
$$f *_\hbar g\,(x) = \text{"}\int DX\,D\eta\text{"}\; e^{\frac{i}{\hbar} S_{PSM}(X,\eta)} \cdot f(X(0)) \cdot g(X(1))$$

$$\begin{tikzcd} TD \arrow[r, "\eta"'] \arrow[d] & T^*M \arrow[d] \\ D \arrow[r, "X"'] & M \end{tikzcd}$$

– bundle maps

s.t. $\eta|_{\partial D} = 0$

$X(\infty) = x$



$$S_{PSM}(X,\eta) = \int_D \langle \eta, dX\rangle + \frac{1}{2}\langle \underbrace{\pi(X)}_{X^*\pi \in \Gamma(D, X^*\wedge^2 TM)}, \underbrace{\eta\wedge\eta}_{\in \Gamma(D, X^*\wedge^2 T^*M)} \rangle$$

$\pi \in \Gamma(M, \wedge^2 TM)$, $[\pi,\pi]_{NS} = 0$

– Poisson bi-vector on M.

- A parallel (which can be made precise).

[Kontsevich-Soibelman… → Losev, P.M.]

BV theories $\xleftrightarrow{\text{[unimod]}}$ cyclic L_∞ algebras

↳ partial integration over fields (passage to an effective action = BV push-forward) $\xleftrightarrow{\text{[unimod]}}$ homotopy transfer for cy. L_∞ to a def. retract

$$\underset{\substack{\text{action}\\ \text{on } \mathcal{F}\\ \shortparallel\\ \Gamma(M,E)}}{S} \rightsquigarrow \underset{\substack{\text{on } \mathcal{F}'\\ \text{related e.g.}\\ \text{to a cell decomp. of } M}}{S_{\text{eff}}} \rightsquigarrow \underset{\substack{\uparrow\\ \text{full integration}}}{Z}$$

so, S_{eff} is an "intermediate quantization"

Faddeev-Papov construction

BV 9/3

Let $\underset{\substack{\uparrow\\ \text{compact Lie group}}}{G} \circlearrowright M$ - a free action, $\gamma: G\times M \to M$ $\qquad m = \dim G$

$$\begin{tikzcd} \mathfrak{g} = \mathrm{Lie}(G) \arrow[r] & \mathfrak{X}(M) \\ \underset{\text{basis vectors}}{T_a} \arrow[r, mapsto] & v_a \end{tikzcd}$$

Let $f \in C^\infty_c(M)^G$ $\quad$ (think of $f = e^{\frac{i}{\hbar}S}\cdot g$; $S \in C^\infty(M)^G$, $g \in C^\infty_c(M)^G$)

$\mu \in \Omega^{\dim}(M)^G$ - G-invariant volume form

$$I := \int_M f\mu = \mathrm{Vol}(G)\int_{M/G} \tilde{f}\cdot\tilde{\mu}$$

where $p: M \twoheadrightarrow M/G$

$f = p^*\tilde{f}$, $\tilde{f} \in C^\infty(M/G)$

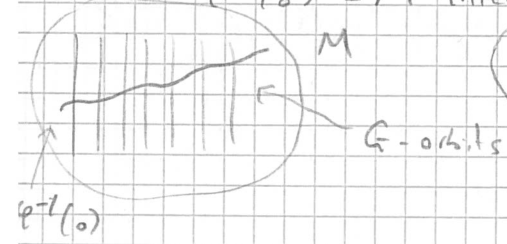
$\iota_{v_1}\cdots\iota_{v_m}\mu = p^*\tilde{\mu}$, $\tilde{\mu} \in \Omega^{top}(M/G)$

- since $\mu = p^*\tilde{\mu}\wedge\chi$,

$\chi \in \Omega^{\dim G}(M)$ s.t. $\chi|_{p^{-1}(\text{point in } M/G)}$ = Haar measure on G, normalized by $\iota_{v_1\wedge\cdots\wedge v_m}\chi = 1$.

Let $\varphi: M \to \mathfrak{g}$ be a function such that

- 0 is a regular value of φ
- $S = \varphi^{-1}(0) \subset M$ intersects every G-orbit transversally and only once*



Rem: there may be topological obstructions for *
(e.g. $G = U(1)$, fibers of $\begin{smallmatrix} M\\ \downarrow\\ M/G\end{smallmatrix}$ = circles; a function φ on S^1 with zero a reg. value cannot have only one zero.)

However this assumption can be relaxed.

Thus

$$I = \mathrm{Vol}(G)\cdot\int_M \underbrace{\delta^{(m)}(\varphi)}\cdot f\cdot\iota_{v_1\wedge\cdots\wedge v_m}\mu = \mathrm{Vol}(G)\cdot\int_{\varphi^{-1}(0)} f\cdot\iota_{v_1\wedge\cdots\wedge v_m}\mu\Big|_{\varphi^{-1}(0)}$$

$$\delta^{(m)}_{\varphi^{-1}(0)} = \underset{\substack{\uparrow\\ \delta\text{-distribution}}}{\delta(\varphi)}\cdot\prod_a d\varphi^a \quad , \quad \delta(\varphi) := \prod_a \delta(\varphi^a(x))$$

Rem: for $C \subset M$ a k-cycle,

$$\delta^{(n-k)}_C : \Omega^k(M) \longrightarrow \mathbb{R}$$

$$\omega \longmapsto \int_C \omega|_C =: \text{"}\int_M \delta^{(n-k)}_C \wedge\omega\text{"}$$

Rem $\delta^{(m)}_{(\varphi)}$ - distributional m-form, invariant e.g. under rescaling of φ

$\delta(\varphi)$ - δ-function - is not invariant

Denote: $\underbrace{\prod_a d\varphi^a}_{=\varphi^*\mu_{\mathfrak{g}}} \wedge \iota_{v_1\wedge\cdots\wedge v_m}\mu = \boxed{J}\cdot\mu$

Then: $J(x) = \det(d\varphi^a \circ v_b)_{a,b} = \det_{\mathfrak{g}}(d_x\varphi\circ d_{(1,x)}\gamma)$ - Faddeev-Popov determinant

$$\mathfrak{g} \longleftarrow T_xM \longleftarrow \mathfrak{g}$$

Thus we have

<u>Thm</u> (Faddeev-Popov):

$$\int_M f\cdot\mu = \mathrm{Vol}(G)\int_M \delta(\varphi(\gamma x))\cdot \underbrace{\mathcal{J}(x)}_{\det_{\mathfrak{g}} FP(x)}\cdot f\,\mu \qquad (*)$$

$$FP(x) = d_x\varphi\circ d_{(1,x)}\gamma \in \mathrm{End}(\mathfrak{g})$$

Integral representations for δ and $\mathcal{J}$:

$$\delta(\varphi(x)) = \int_{\mathfrak{g}^*} d\lambda\, e^{i\langle\lambda,\varphi(x)\rangle}\frac{1}{(2\pi)^m}$$

$$\mathcal{J}(x) = \int_{\Pi\mathfrak{g}\oplus\Pi\mathfrak{g}^*}\Big(\prod_{a=1}^m Dc_a\, D\bar{c}_a\Big)\; e^{\langle\bar{c},FP(x)\,c\rangle}$$

- Berezin integral

$c,\bar{c}$ - "Faddeev-Popov ghosts"

Thus, letting $f = g\cdot e^{\frac{i}{\hbar}S}$ in $(*)$, we get

$$\int_M e^{\frac{i}{\hbar}S} g\,\mu = \mathrm{Vol}(G)\cdot\frac{1}{(2\pi i)^m}\int_{M\times\mathfrak{g}^*\times(\Pi\mathfrak{g}\oplus\Pi\mathfrak{g}^*)}\mu\, d\lambda\, Dc\, D\bar{c}\cdot g(x)\cdot e^{\frac{i}{\hbar}\underbrace{\left(S(x)+\langle\lambda,\varphi(x)\rangle+\langle\bar{c},FP(x)c\rangle\right)}_{S_{FP}(x,\lambda,c,\bar{c})}}$$

$S_{FP}\in C^\infty(M\times\mathfrak{g}^*\times(\Pi\mathfrak{g}\oplus\Pi\mathfrak{g}^*))$ - "Faddeev-Popov action" reminder:

Last time:

$\dim M = n$, $\dim G = m$

BV 10/1

15.12.14

Faddeev-Popov: $\underset{\text{compact}}{G} \overset{\text{free}}{\circlearrowright} M$ $\quad f \in C^\infty_c(M)^G$, $\mu \in \Omega^n(M)^G$

$\gamma: G \times M \to M \qquad f = e^{\frac{i}{\hbar}S} \cdot g$

$\varphi: M \longrightarrow \mathfrak{g}$ such that $\varphi^{-1}(0) \subset M$ intersects each G-orbit transversally, exactly o

Then:

$$I := \int_M f \cdot \mu = \mathrm{Vol}(G) \cdot \int_M \delta(\varphi(x)) \cdot \det_{\mathfrak{g}}\big(\underbrace{d_x\varphi \circ d_{(1,x)}\gamma}_{FP(x)}\big) \cdot f \cdot \mu$$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "d_{(1,x)}\gamma"] \arrow[rr, bend right, dashed, "FP(x)"'] & T_xM \arrow[r, "d\varphi"] & \mathfrak{g} \end{tikzcd}$$

or

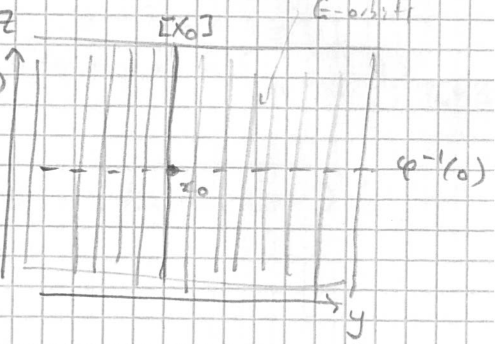
$$\int_M e^{\frac{i}{\hbar}S} \cdot g \cdot \mu = \frac{\mathrm{Vol}(G)}{(2\pi i)^m} \int_{M \times \mathfrak{g}^* \times (\Pi\mathfrak{g} \oplus \Pi\mathfrak{g}^*)} \mu\, d\lambda\, Dc\, D\bar{c} \cdot g(x)\, e^{\frac{i}{\hbar}\big(\underbrace{S(x) + \langle\lambda, \varphi(x)\rangle + \langle\bar{c}, FP(x)\, c\rangle}_{S_{FP}}\big)}$$

Terminology: $\left.\begin{matrix} c \\ \bar{c} \end{matrix}\right\}$ "FP ghosts"
λ - Lagrange multipl

Let x_0 be a crit pt of S, lying on the crit G-orbit $[x_0] \subset M$ and satisfying $\varphi(x_0) = 0$

let $(y_1 \ldots y_{n-m}; z_1 \ldots z_m)$ be an adapted local coord chart on M near x_0

s.t.
- x_0 is $y = z = 0$
- $[x_0]$ is $y = 0$; moreover G-orbits are locally given by $y = \text{const.}$ (*)
- $\varphi: (y, z) \longmapsto z$ corresp. to a point $\sum z_a T_a \in \mathfrak{g}$



S is G-invariant $\Rightarrow S = S(y)$, $\frac{\partial}{\partial z_a} S = 0$

Hessian of S: $\quad \partial^2 S|_{x_0} = \begin{array}{c|cc} & y & z \\ \hline y & \frac{\partial^2 S}{\partial y_i \partial y_j}\Big|_{x_0} & 0 \\ z & 0 & 0 \end{array}$

- We assume that this block is non-degenerate, i.e. all of degeneracy of $\partial^2 S$ comes from G-invariance (in other words, $\mathrm{rank}\, \partial^2 S = n - m$)

$\partial^2 S|_{x_0}$ is degenerate.

However, let us consider

$$\underbrace{\partial^2 (S + \langle\lambda, \underbrace{\varphi(x)}_{z}\rangle)\big|_{(x_0, \lambda=0)}}_{\in\, \mathrm{Sym}^2(T_{x_0}M \oplus \mathfrak{g}^*)^*} = \begin{array}{c|ccc} & y_j & z_b & \lambda_b \\ \hline y_i & \frac{\partial^2 S}{\partial y_i \partial y_j}\Big|_{x_0} & 0 & 0 \\ z_a & 0 & 0 & \delta_{ab} \\ \lambda_a & 0 & \delta_{ab} & 0 \end{array}$$

- non-degenerate!

(*) $\Leftrightarrow$ vector fields defining G-action on M locally have the form $\quad v_a = \sum_b f_{ab}(y, z) \frac{\partial}{\partial z_b}$ with (f_{ab}) a non-degenerate matrix

$\Rightarrow FP(x)_{ab} = f_{ab}(y, z)$

So, FP part of Hessian: $\quad \partial^2 \langle\bar{c}, FP(x)\, c\rangle\big|_{(x_0, 0, 0)} = \begin{array}{c|cc} & c & \bar{c} \\ \hline c & 0 & -f_{ba} \\ \bar{c} & f_{ab} & 0 \end{array}$ - non-deg (with $f_{ab} = f_{ab}|_{x_0}$)

The whole Hessian: $\quad \partial^2 S_{FP}\big|_{(x_0, 0, 0, 0)} = \begin{array}{c|cccc} y & \frac{\partial^2 S}{\partial y_i \partial y_j}\Big|_{x_0} & & & \\ z & & & 1_{\mathfrak{g}^*} & \\ \lambda & & 1_{\mathfrak{g}} & & \end{array}$ = non-degenerate

Stationary phase evaluation of FP integral

(BV 10/2

- Euler-Lagrange equation for $S_{FP}(x,\lambda,c,\bar{c})$:

$$\left.\begin{array}{l}\cdot\ \frac{\partial}{\partial x_i}S(x)+\langle\lambda,\frac{\partial}{\partial x_i}\varphi\rangle=0\\ \cdot\ \varphi(x)=0\end{array}\right\}\Leftrightarrow$$ finding a conditional extremum of S on $\varphi^{-1}(0)$

(in fact, G-invariance of S implies that a conditional extremum is an actual extremum, so $\lambda_{crit}=0$)

- $c=\bar{c}=0$

(without using adapted coords):

$$\partial^2 S\big|_{FP(x_0,0,0,0)}=\begin{array}{c|cccc} & x & \lambda & c & \bar{c}\\ \hline x & \frac{\partial^2 S}{\partial x^2}\Big|_{x_0} & \underset{\mathfrak{g}^*\to T^*_{x_0}M}{(d\varphi|_{x_0})^T} & & \\ \lambda & \underset{T_{x_0}M\to\mathfrak{g}}{d\varphi|_{x_0}} & 0 & & \\ c & & & & \underset{\in End(\mathfrak{g}^*)}{-FP(x_0)^T}\\ \bar{c} & & & \underset{\in End(\mathfrak{g})}{FP(x_0)} & \end{array}\ \rightsquigarrow\ \left(\partial^2 S_{FP}\big|_{x_0}\right)^{-1}=\begin{array}{c|cccc} & x & \lambda & c & \bar{c}\\ \hline x & \mathcal{D} & \beta^T & & \\ \lambda & \beta & 0 & & \\ c & & & & FP(x_0)^{-1}\\ \bar{c} & & & -FP(x_0)^{-1T} & \end{array}$$

where $\mathcal{D}\in Sym^2 T_{x_0}M$

$\beta: T^*_{x_0}M\to\mathfrak{g}^*$, $\beta^T:\mathfrak{g}\to T_{x_0}M$

s.t. $$\begin{cases}\partial^2 S|_{x_0}\circ\mathcal{D}=id_{T^*_{x_0}M}-\underbrace{(d\varphi|_{x_0})^T\circ\beta}_{T^*_{x_0}M\leftarrow\mathfrak{g}^*\leftarrow T^*_{x_0}M}\\ d\varphi|_{x_0}\circ\beta^T=id_{\mathfrak{g}}\\ \underbrace{d\varphi|_{x_0}\circ\mathcal{D}=0}_{\Downarrow\ \mathcal{D}\in im(\Lambda^2 T_{x_0}\varphi^{-1}(0)\to\Lambda^2 T_{x_0}M)}\end{cases}$$

$$\mathfrak{g}\underset{\beta^T}{\overset{d\varphi|_{x_0}}{\rightleftarrows}}T_{x_0}M$$ define splitting $T_{x_0}M\simeq\mathfrak{g}\oplus T_{x_0}\varphi^{-1}(0)$

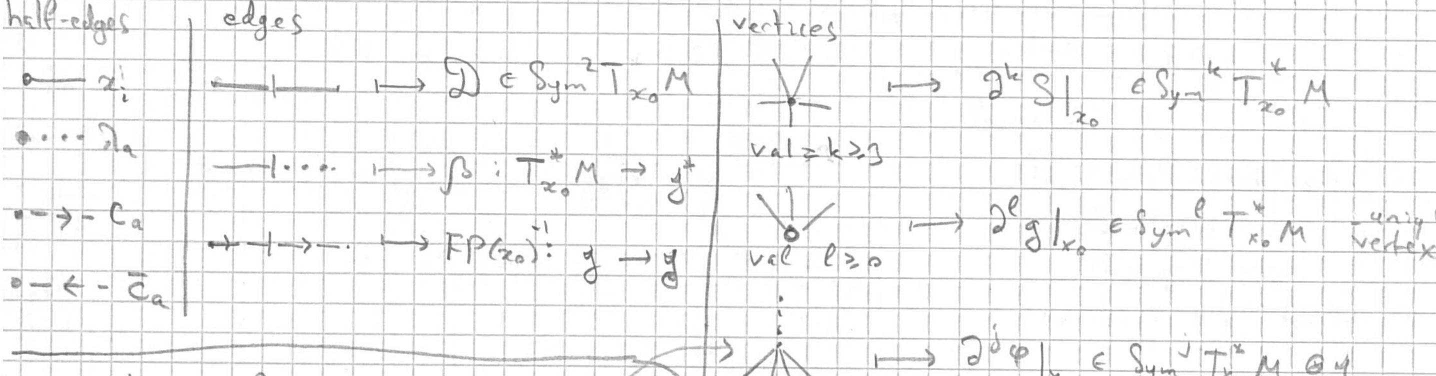
$\mathcal{D}$ is the image of $\tilde{\mathcal{D}}\in\Lambda^2 T_{x_0}\varphi^{-1}(0)$ and $\tilde{\mathcal{D}}$ is the inverse of $\partial^2_{x_0}(S|_{\varphi^{-1}(0)})=\partial^2_{x_0}S|_{T_{x_0}\varphi^{-1}(0)}$

i.e. $\mathcal{D}$ is the propagator / Green's function for $\partial^2 S|_{x_0}$ in the "gauge" $\varphi(x)=0$

Stationary phase formula:

$$\int_M e^{\frac{i}{\hbar}S(x)}g(x)\cdot\mu=\frac{Vol(G)}{(2\pi i)^m}\sum_{\substack{\text{crit } G\text{-orbits of } S,\,[x_0]\\ x_0=[x_0]\cap\varphi^{-1}(0)}}\overbrace{(2\pi\hbar)^{\frac{n+m}{2}}\cdot\left(\frac{i}{\hbar}\right)^m}^{(2\pi i)^m\cdot(2\pi\hbar)^{\overset{\frac{\dim M/G}{2}}{\frac{n-m}{2}}}}e^{\frac{i}{\hbar}S(x_0)}\cdot\det{}^{-\frac{1}{2}}\partial^2_{x_0}S|_{\varphi^{-1}(0)}\cdot\det_{\mathfrak{g}}FP(x_0)\cdot e^{\frac{\pi i}{4}\,\mathrm{sign}\,\partial^2_{x_0}S|_{\varphi^{-1}(0)}}\cdot\sum_{\text{graphs }\Gamma}\frac{\hbar^{1-\chi(\Gamma)}}{|Aut\,\Gamma|}\Phi_{x_0}($$

Feynman rules for calculating $\Phi(\Gamma)$:



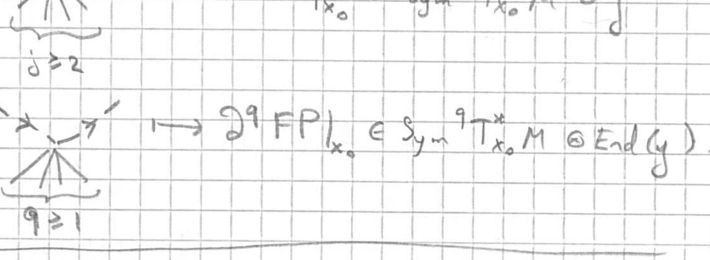
$\longmapsto\mathcal{D}\in Sym^2 T_{x_0}M$

$\longmapsto\beta: T^*_{x_0}M\to\mathfrak{g}^*$

$\longmapsto FP(x_0)^{-1}:\mathfrak{g}\to\mathfrak{g}$

$\longmapsto\partial^k S|_{x_0}\in Sym^k T^*_{x_0}M$

$\longmapsto\partial^\ell g|_{x_0}\in Sym^\ell T^*_{x_0}M$



$\longmapsto\partial^j\varphi|_{x_0}\in Sym^j T^*_{x_0}M\otimes\mathfrak{g}$

$\longmapsto\partial^q FP|_{x_0}\in Sym^q T^*_{x_0}M\otimes End(\mathfrak{g})$

Important special case:

φ is linear in x. Then vertex is absent, hence there are no diagrams involving λ-half-edge and generic diagram looks like.

note: if G-action is by shifts in x (e.g. in local coords) then vertex is also absent

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Rem to define $\det \partial^2_{x_0} S|_{\varphi^{-1}(0)}$ invariantly, we need a volume element on $T_{x_0}\varphi^{-1}(0)$, i.e. element of $\mathrm{Det}\, T^*_{x_0}\varphi^{-1}(0)$.

We use the SES $0\to T_{x_0}\varphi^{-1}(0)\hookrightarrow T_{x_0}M \xrightarrow[d\varphi|_{x_0}]{} \mathfrak{g}\to 0$ which induces a canonical iso

$$\begin{array}{ccc} \mathrm{Det}\,T^*_{x_0}M & \cong & \mathrm{Det}\,\mathfrak{g}^*\otimes \mathrm{Det}\,T^*_{x_0}\varphi^{-1}(0)\\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$}\\ \mu|_{x_0} & & \mu_\mathfrak{g} \end{array}$$

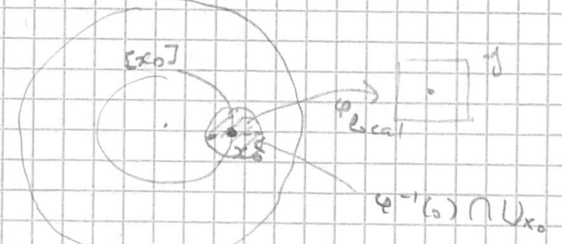
$\mu_\mathfrak{g}$ - restriction of Haar measure on G to $1\in G$

- use this to define $\nu = "\frac{\mu|_{x_0}}{\mu_\mathfrak{g}}" \in \mathrm{Det}\,T^*_{x_0}$

Rem
- can drop the assumption that $\varphi^{-1}(0)$ intersects G-orbits only once
  - then just ~~use~~ choose one intersection for every critical orbit
- instead of choosing φ globally, can pick φ_j in the tubular nbhd of j-th critical orbit

[we may even reverse the logic and choose a representative $[x_0]\ni x_0$ on each crit orbit and then choose locally $\varphi: \underset{\underset{M}{\cap}\,\text{open}}{U_{x_0}} \to \mathfrak{g}$]

- freeness of G-action is required in the nbhd of crit. locus of S, but can be relaxed on the complement in M



Motivating example: Yang-Mills

$$M = \left\{\text{connections in } \begin{array}{c} P=\Sigma\times\underline{G}\\ \downarrow\\ \Sigma\end{array}\right\} \cong \Omega^1(\Sigma)\otimes\underline{\mathfrak{g}}$$

local gauge group

(physical case: $\Sigma=\mathbb{R}^{3,1}$, $\underline{G}=SU(3)$ · QCD)

$$G = \{\text{automorphisms of } P\} \cong C^\infty(\Sigma,\underline{G})$$ acting by $A\longmapsto A^g = gAg^{-1}+g\,dg^{-1}$

$$S(A) = \frac{1}{4g^2}\int_\Sigma \mathrm{tr}\, F_A\wedge *F_A$$

$\mu=$ "Liouville measure on the space of connections" - problematic

Haar measure on G - also problematic.

$\boxed{\varphi(A)=d^*A}$ - Lorentz gauge

$$\varphi: \Omega^1(\Sigma)\otimes\underline{\mathfrak{g}} \to \underset{\underset{\mathrm{Lie}\,G\,=\,\text{gauge}}{\cong}}{\Omega^0(\Sigma)\otimes\underline{\mathfrak{g}}}$$

$FP(A) = d^* d_A$: gauge $\to$ gauge

$c\in\Omega^0(\Sigma)\otimes\Pi\mathfrak{g}$ $\quad\bar{c}\in\Omega^{\dim\Sigma}(\Sigma)\otimes\Pi\mathfrak{g}^*$

Formally:

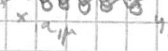
$$Z = "\int \underset{\substack{\text{Conn}\,\oplus\,\text{gauge}^*\\ \oplus\,\Pi\text{gauge}\,\oplus\,\Pi\text{gauge}^*}}{\mathcal{D}A\,\mathcal{D}\lambda\,\mathcal{D}c\,\mathcal{D}\bar{c}}" \, e^{\frac{i}{\hbar}S_{FP}}$$

$$S_{FP} = S_{YM}(A) + \int_\Sigma\langle\lambda, d^*_A A\rangle + \int_\Sigma\langle\bar{c}, d^*d_A c\rangle$$

St. Phase expansion:

edges

gluon field

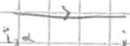


$$\to \int_{\mathbb{R}^{3,1}}\frac{d^4k}{(2\pi)^4}e^{ik(x-y)}\frac{i\delta_{ab}\eta_{\mu\nu}}{k^2+i\varepsilon}$$

ghost field

$$\to \int_{\mathbb{R}^{3,1}}\frac{d^4k}{(2\pi)^4}e^{ik(x-y)}\frac{i\delta_{ab}}{k^2+i\varepsilon}$$

matter (quarks)

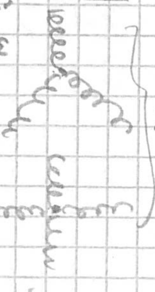


$$\to \int\frac{d^4k}{(2\pi)^4}e^{-ik(x-y)}\left(\frac{i}{\not k-m}\right)_{\alpha\beta}\delta_{ij}$$

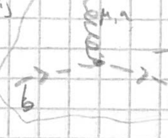
indices of fundam rep of G

spinor indices

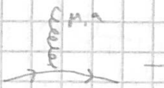
vertices



Coming from cubic and quartic terms in A of $F\wedge *F$



$\to$ ~~$ig\gamma^\mu T_a$~~ $\quad ig f^{abc}\frac{\partial}{\partial x_\mu}(\to)$



$\to ig\gamma^\mu T_a$

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BRST

Plan
- supermanifolds, $\mathbb{Z}$-grading, derivations ($\to$ split case)
- Q-manifolds ($\longleftrightarrow$ L_∞ algebras)
- divergence of a vector field, integration
- BRST integrals, FP $\hookrightarrow$ BRST

def a supermanifold $\mathcal{M}$ $(n|m)$ is the data of
- an ordinary n-mfd M (the "body")
- a sheaf of supercommutative rings of form $\mathcal{O}_{\mathcal{M}}(U) = C^\infty(U) \otimes \Lambda^\bullet V$ (m-dim vector space) for $U \underset{\text{open}}{\subset} M$

- locally, $\mathcal{O}_{\mathcal{M}}(U) \cong C^\infty(U)[\underbrace{\xi_1, \dots, \xi_m}_{\text{odd}}] \ni \sum_k \sum_{\{i_1 \dots i_k\} \subset \{1 \dots m\}} f_{i_1 \dots i_k}(x)\, \xi_{i_1} \cdots \xi_{i_k}$

Ex: ① vector super-space $(V_e, V_o) = \mathcal{V}$

$\mathcal{O}_{\mathcal{V}} = C^\infty(V_e) \otimes \Lambda^\bullet V_o^*$ (can restrict to $U \underset{\text{open}}{\subset} V_e$)

② for a vector bundle $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ rank m, we can construct a $(n|m)$ supermanifold ΠE

s.t. $\text{body}(\Pi E) = M$

$\underline{\mathcal{O}_{\Pi E} = \Gamma(M, \Lambda^\bullet E^*)}$

{ such superman. a called "split"

E.g. we have $(n|n)$ supermanifolds ΠTM, ΠT^*M

def a morphism of supermanifolds $\mathcal{M} \to \mathcal{N}$ is

(1) a map of smooth mfds between bodies $f: M \longrightarrow N$

(2) a morphism of sheaves $\varphi^*_U : \mathcal{O}_{\mathcal{N}}(U) \longrightarrow \mathcal{O}_{\mathcal{M}}(f^{-1}(U))$

$$\begin{tikzcd} \mathcal{O}_{\mathcal{N}}(U) \arrow[r] \arrow[d] & \mathcal{O}_{\mathcal{M}}(f^{-1}(U)) \arrow[d] \\ C^\infty(U) \arrow[r, "f^*"] & C^\infty(f^{-1}(U)) \end{tikzcd}$$

Ex: $(f,\varphi) : \mathbb{R}^{1|2} \longrightarrow \mathbb{R}^{1|2}$

coords (x, ξ_1, ξ_2) $\quad (y, \psi_1, \psi_2)$

defined by $\begin{cases} \varphi^* \psi_i = \xi_i \\ \varphi^* y = x + \xi_1 \xi_2 \end{cases}$

{ Rem/warning: morphisms of bundle in Ex ② induce morphisms of ΠE, but not vice versa.

Ex: $\text{body}(\mathcal{M}) \hookrightarrow \mathcal{M}$ is a canonical morphism, $\mathcal{M} \to \text{body}(\mathcal{M})$ is not.

def a vector field $v \in \mathfrak{X}(\mathcal{M})$ is a derivation of $\mathcal{O}_{\mathcal{M}}$, $v(fg) = v(f) \cdot g + (-1)^{|v| \cdot |f|} f \cdot v(g)$

$|f| = \begin{cases} 0 & f \text{ even} \\ 1 & f \text{ odd} \end{cases}$ $\qquad |v| = 1 \Leftrightarrow v$ "odd"; $\quad |v| = 0 \Leftrightarrow v$ "even"

Similarly: a $\mathbb{Z}$-graded (super-)manifold $\mathcal{M}$ is a sheaf over M of $\mathbb{Z}$-graded super-com rings of form

$\mathcal{O}_{\mathcal{M}}(U) = C^\infty(U) \otimes \widehat{\text{Sym}}_{gr} \mathcal{V}^\bullet$ with $\mathcal{V}^\bullet$ a $\mathbb{Z}$-graded v.sp, $\widehat{\text{Sym}}_{gr} \mathcal{V} = \widehat{\text{Sym}}\, \mathcal{V}^{\text{even}} \otimes \Lambda^\bullet \mathcal{V}^{\text{odd}}$

(allow C^∞ functions of degree 0 monomials)

morphisms = morphisms of $\mathbb{Z}$-graded sc rings (sheaves of)

Ex: ① $E^\bullet = \oplus E^i$ $\begin{matrix} \\ \downarrow \\ M \end{matrix}$ a graded vector bundle induces a graded man. $\mathcal{E}$ with $\mathcal{O}_{\mathcal{E}} = \Gamma(M, \widehat{\text{Sym}}_{gr} E^\bullet)$

② $T[1]M$ has loc coords of degrees 0 and 1, locally: $\mathcal{O}_{T[1]M} \ni \sum_k \sum_{\{i_1 \dots i_k\} \subset \{1 \dots n\}} \underbrace{f_{i_1 \dots i_k}(x)\, \xi_{i_1} \cdots \xi_{i_k}}_{k\text{-form}}$

observe: $\mathcal{O}_{T[1]M} \cong \Omega^\bullet(M)$, $\xi_i \leftrightarrow dx_i$

③ $T^*[-1]M$ has coords of deg 0, -1 ; $\mathcal{O}_{T^*[-1]M} \ni \sum_k \sum_{\{i_1 \dots i_k\} \subset \{1 \dots n\}} g_{i_1 \dots i_k}(x)\, \psi_{i_1} \cdots \psi_{i_k}$

$\mathcal{O}_{T^*[-1]M} \cong \mathcal{V}^{-\bullet}(M)$ - polyvectors ; $\psi_i \leftrightarrow \partial$

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On a graded mfd $\mathcal{M}$, one has the graded Lie algebra of vector fields

$\mathfrak{X}(\mathcal{M})_\bullet$, $v \in \mathfrak{X}(\mathcal{M})_k$ - a derivation of $\mathcal{O}_\mathcal{M}$ of degree k

i.e. $|v(\underset{\overset{\uparrow}{\deg i}}{f})| = k+i$ and $v(fg) = v(f)\cdot g + (-1)^{k\cdot|f|} f\cdot v(g)$

Lie bracket: $[v,w] = \underset{\uparrow}{v \circ w} - (-1)^{|v|\cdot|w|} w \circ v$

as derivations

In particular, there is always the

- Euler vector field $\mathbb{E} \in \mathfrak{X}(\mathcal{M})_0$ defined by $\mathbb{E} f = |f|\cdot f$

locally, if x - coords on body, ξ_α - "formal" coords, $\mathbb{E} = \sum_\alpha |\xi_\alpha| \xi_\alpha \frac{\partial}{\partial \xi_\alpha}$

i.e. $\mathcal{O}_\mathcal{M}(U) = C^\infty(U)[\xi_\alpha]_{gr}$

<u>def</u> a cohomological v.f. on $\mathcal{M}$

is a vector field $Q \in \mathfrak{X}(\mathcal{M})_1$ with the property $[Q,Q] = 0$ $(\Leftrightarrow \underset{\overset{\shortparallel}{Q \circ Q}}{Q^2} = 0)$

<u>Rem</u> Q defines a coboundary operator $Q: \underbrace{C^\infty(\mathcal{M})_k}_{\overset{\shortparallel}{\mathcal{O}_\mathcal{M}}} \to C^\infty(\mathcal{M})_{k+1}$

<u>Ex</u> ① $\mathcal{M} = T[1]M$, $Q = d_M : \underset{\overset{\cong}{\Omega^k(M)}}{C^\infty(\mathcal{M})_k} \to \underset{\overset{\cong}{\Omega^{k+1}(M)}}{C^\infty(\mathcal{M})_{k+1}}$ - de Rham on M viewed as a vector field on $T[1]M$

locally: $Q = \sum \underbrace{\xi_i}_{dx_i} \frac{\partial}{\partial x_i}$ (this formula does define a global object)

② $\mathfrak{g}$ - Lie algebra, $\mathcal{M} = \underset{\overset{\downarrow}{\mathfrak{g}^*}}{\mathfrak{g}[1]}$, $\underset{\overset{\cong}{\mathbb{R}[\underbrace{\psi_1 \ldots \psi_m}_{odd}]}}{C^\infty(\mathcal{M})_\bullet} \cong \Lambda^\bullet \mathfrak{g}^* = C^\bullet_{CE}(\mathfrak{g})$ - Chevalley-Eilenberg cochains of $\mathfrak{g}$

d_{CE} on $C^\bullet_{CE}(\mathfrak{g})$ defines a cohom v.f. on $\mathfrak{g}[1]$, $\Lambda^\bullet \mathfrak{g}^* \to \Lambda^{\bullet+1} \mathfrak{g}^*$

$$\begin{tikzcd} \Lambda^\bullet \mathfrak{g}^* \arrow[r] & \Lambda^{\bullet+1} \mathfrak{g}^* & \\ \mathfrak{g}^* \arrow[r] & \Lambda^2 \mathfrak{g}^* \arrow[u, "\text{extension by Leibnitz}"'] & {[\,,\,]}: \Lambda^2 \mathfrak{g} \to \mathfrak{g} \arrow[l, squiggly, "\text{dualize}"'] \end{tikzcd}$$

in terms of a basis T_a in $\mathfrak{g}$, with $[T_a, T_b] = \sum_c f^c_{ab} T_c$

$Q = d_{CE} = \sum_{a,b,c} \frac{1}{2} f^c_{ab} \psi_a \psi_b \frac{\partial}{\partial \psi_c}$

check: $Q^2 = 0$ $(\longleftrightarrow$ Jacobi $)$

<u>Rem</u> $\mathbb{E}$, Q satisfy $[\mathbb{E},\mathbb{E}] = 0 = [Q,Q]$, $[\mathbb{E},Q] = Q$ - an (infinitesimal) action of certain Lie superalgebra: $\underset{\underset{\mathbb{E}}{\shortparallel}}{\text{dilatations}}$ and $\underset{\underset{Q}{\shortparallel}}{\text{affine translations}}$ of the odd line $\mathbb{R}^{0|1}$.

~~~

Classical BRST theory: the data of

- $\mathcal{F}$ - a $\mathbb{Z}$-graded mfd
- Q a cohom v.f. on $\mathcal{F}$
- $S \in C^\infty(\mathcal{F})_0$ satisfying $Q(S) = 0$

<u>Ex</u>: (starting from FP data): $G \circlearrowright M$, $S \in C^\infty(M)^G$

construct $\mathcal{F} = M \times \mathfrak{g}[1]$ - split supermanifold, corresp to triv. bdle $\begin{matrix} M \times \mathfrak{g}[1] \\ \downarrow \\ M \end{matrix}$

$C^\infty(\mathcal{F})_k = C^\infty(M) \otimes \Lambda^k \mathfrak{g}^* = C^k_{CE}(\mathfrak{g}, \underbrace{C^\infty(M)}_{module}) \circlearrowleft d^{module}_{CE}$

in loc. coords

$Q = d \quad = \frac{1}{2} \sum f^c \psi \psi \partial \quad + \sum$

- for $f \in C^\infty(M)$, $Q(f) \in C^\infty(M) \otimes \mathfrak{g}^*$

$Q(f)(x) = d\delta(x)(f)$
~~~

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note

$Q^2 = 0 \iff \begin{cases} \text{Jacobi in } \mathfrak{g} \\ \text{condition that } d_{1*}\gamma : \mathfrak{g} \to \mathfrak{X}(M) \text{ is a Lie alg homomorphism} \end{cases}$

$T_a \mapsto v_a$

$Q(S) = 0 \iff S$ is $\mathfrak{g}$-invariant, i.e. $v_a(S) = 0 \ \forall a$

Integration over a supermanifold

Let $\begin{tikzcd} E \arrow[d, "p"] \\ M \end{tikzcd}$ be a vect. bun., $\mathcal{M} = \Pi E$ - corresp. split super-mfd.

$$\begin{tikzcd} \mathrm{Ber}(\mathcal{M}) := \Lambda^{\dim M} T^*M \otimes \Lambda^{\mathrm{rk} E} E \arrow[d] \\ M \end{tikzcd}$$ - Berezin line bundle (Rem: can also consider the pullback $\begin{tikzcd} p^*\mathrm{Ber} \arrow[d] \\ \mathcal{M} \end{tikzcd}$)

Given a Berezinian $\mu \in \Gamma(M, \mathrm{Ber}(\mathcal{M}))$, we have integration map

$\int_{\mathcal{M}} \mu \, \bullet : C^\infty_c(\mathcal{M}) \to \mathbb{R}$ which is

- $\mathbb{R}$-linear
- for $U \underset{\text{open}}{\subset} M$, $\mu_e \in \Omega^{\dim M}(U)$, $\mu_o \in \Gamma(U, \Lambda^{\mathrm{rk} E} E)$

$$(*) \quad \int_{\Pi E|_U} \mu_e \otimes \mu_o \, f := \int_{U \ni x} \mu_e \underbrace{\int_{\Pi(E|_x)} \mu_o f}_{\overset{\text{def}}{=} \langle \mu_o, f \rangle \in C^\infty(U)}$$ ← definition via Fubini thm

$\langle \mu_o, f \rangle$ - fiberwise Berezin integral

$f \in C^\infty(\Pi E|_U) = \Gamma(U, \Lambda^{\mathrm{rk} E} E^*)$

Rem can consider also Berezinians non-constant in fiber directions:

$$\begin{tikzcd} \widetilde{\mathrm{Ber}}(\mathcal{M}) = \mathrm{Ber}(\mathcal{M}) \otimes \Lambda^{\mathrm{rk} E} E^* \arrow[d] \\ M \end{tikzcd}$$

s.t. $\Gamma(\widetilde{\mathrm{Ber}}(\mathcal{M})) = \Gamma(\mathrm{Ber}(\mathcal{M})) \otimes_{C^\infty(M)} C^\infty(\mathcal{M})$
$=: \text{"}\Gamma(\mathcal{M}, p^*\mathrm{Ber}(\mathcal{M}))\text{"}$

Then integration is

$\int_{\mathcal{M}} : \Gamma(\widetilde{\mathrm{Ber}}(\mathcal{M})) \to \mathbb{R}$

General integral $\int_{\mathcal{M}} \mu \cdot$ can be brought to a sum of integrals of form $(*)$, using partitions of unity $1 = \sum \varphi_\alpha$, $\mathrm{supp}\, \varphi_\alpha \subset U_\alpha$

on U_α, $\mu|_{U_\alpha} = \mu_e^{(\alpha)} \otimes \mu_o^{(\alpha)}$ since $\mathrm{Ber}|_{U_\alpha}$ is trivial..

For $\mathcal{M}^{\mathbb{Z}}$ $\mathbb{Z}$-graded, use same definition of the integral, i.e. using just the structure of a super-mfd.

Divergence of a vector field.

Rem: ΠTM has a canonical Berezinian $\mu_{\Pi TM}$ s.t. $\forall \omega \in \Omega^\bullet(M)$, $\tilde{\omega} \in C^\infty(\Pi TM)$ corresp. function, $\int_{\Pi TM} \tilde{\omega} \cdot \mu_{\Pi TM} = \int_M \overset{\swarrow \text{ as a diff. form}}{\omega}$

for $X \in \mathfrak{X}(\mathcal{M})$, define $\mathrm{div}_\mu(X) \in C^\infty(\mathcal{M})$ for $\mu \in \Gamma(\mathrm{Ber}(\mathcal{M}))$

by $\int_{\mathcal{M}} \mu \, X(f) = - \int_{\mathcal{M}} \mu \cdot \mathrm{div}_\mu X \cdot f$ (we assume $\underset{\underset{\mathrm{body}(\mathcal{M})}{\shortparallel}}{M}$ has no boundary)

- for $\mathcal{M} = M$ an ordinary closed mfd, $\mu \in \Omega^{\mathrm{top}}(M)$,

$$0 \underset{\text{Stokes'}}{=} \int_M \underset{\substack{\uparrow \\ \text{Lie derivative}}}{\mathcal{L}_X} (\mu f) = \int_M \mu \, X(f) + \underbrace{(\mathcal{L}_X \mu)}_{\mu \cdot \mathrm{div}_\mu X} f$$

- in loc. coords x_i, ξ_a, if $\mu = \prod_i dx_i \cdot \prod_a D\xi_a$ locally,

then for $X = \sum_i X_i(x,\xi) \frac{\partial}{\partial x_i} + \sum_a X_a(x,\xi) \frac{\partial}{\partial \xi_a}$, $\boxed{\mathrm{div}_\mu X = \sum_i \frac{\partial}{\partial x_i} X_i - (-1)^{|X|} \sum_a \frac{\partial}{\partial \xi_a} X_a}$

Calculation: $\int \underbrace{dx \, D\xi}_{\mu} \, X(f) = \int dx \, D\xi \left(\sum_i X_i \frac{\partial}{\partial x_i} f + \sum_a X_a \frac{\partial}{\partial \xi_a} f \right) =$

$= \int dx \, D\xi \left(\sum_i - \left(\frac{\partial}{\partial x_i} X_i\right) f + \sum_a - (-1)^{|X_a|} \underbrace{\left(\frac{\partial}{\partial \xi} X_a\right) f}_{} \right) =$ $\quad (-1)^{|X_a|} \left(\frac{\partial}{\partial \xi_a} (X_a f) - \left(\frac{\partial}{\partial \xi_a} X_a\right) \cdot f \right)$ $= - \int dx \, D\xi \, (\mathrm{div}_\mu X) \, f$

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- If μ has density ρ in loc. coords: $\mu = \rho(x,\xi) \underbrace{dx\, D\xi}_{\mu_0}$

then $\boxed{\mathrm{div}_\mu X = \mathrm{div}_{\mu_0} X + \underbrace{\frac{1}{\rho} X(\rho)}_{= X(\log \rho)}}$

or $\mathrm{div}_\mu X = \sum_i \left(\frac{\partial}{\partial x_i} X_i + \frac{1}{\rho} X_i \frac{\partial \rho}{\partial x_i}\right) + \sum_a \left((-1)^{|X_a|} \frac{\partial}{\partial \xi_a} X_a + \frac{1}{\rho} X_a \frac{\partial}{\partial \xi_a}\rho\right)$

$\mathrm{div}_{\mu_0} X$ $\quad \frac{1}{\rho} X(\rho)$

On a $\mathbb{Z}$-graded mfld $\mathcal{M}$ with a Berezinian μ,
$\mathrm{div}_\mu : \mathfrak{X}(\mathcal{M}) \to C^\infty(\mathcal{M})$ is a degree-preserving map.

(Toy) Quantum BRST theory

$\mathcal{F}$ - $\mathbb{Z}$-graded manifold, Q - cohom. v.f. on $\mathcal{F}$, $S \in C^\infty(\mathcal{F})_0$ s.t. $Q(S) = 0$ $\Big\}$ classical data

add the data of μ - a Berezinian on $\mathcal{F}$, require that $\boxed{\mathrm{div}_\mu Q = 0}$

Lemma $\int_{\mathcal{F}} \mu \cdot Q(g) = 0 \quad \forall\, g \in C^\infty(\mathcal{F})$

⌈ follows from def. of div$Q$: $\int_{\mathcal{F}} \mu \cdot Q(g) = -\int_{\mathcal{F}} \mu \cdot \mathrm{div}_\mu Q \cdot g = 0$ ⌋

"BRST integral" is the integral of a Q-cocycle $\int_{\mathcal{F}} \mu \cdot \underbrace{f}_{\text{s.t. } Q(f) = 0}$ which can be shifted by a Q-coboundary $f \mapsto f + Q(g)$

so, we have a map

$$\int_{\mathcal{F}} \mu : H_Q(C^\infty(\mathcal{F})) \longrightarrow \mathbb{R}$$

(or $\mathbb{C}$ if we allow $\mathbb{C}$-valued cocycles)

Prototypical BRST integral:

$$\int_{\mathcal{F}} \mu\, e^{\frac{i}{\hbar} S} = \int_{\mathcal{F}} \mu\, e^{\frac{i}{\hbar}(S + Q(\Psi))} \qquad (*)$$

- can use the freedom to choose Ψ to convert the integral to a form good for st. phase expansion

($\deg S = 0$, $\deg \Psi = -1$)

**

Faddeev-Popov via BRST

FP data: $G \underset{\gamma}{\curvearrowright} M$, $\mu_M \in \Omega^n(M)^G$, $S \in C^\infty(M)$

$\varphi : M \to \mathfrak{g}$ s.t. $\varphi^{-1}(0) \subset M$ - a section of $\begin{matrix} M \\ \downarrow \\ M/G \end{matrix}$

1st attempt: set $\mathcal{F} = M \times \mathfrak{g}[1]$ (as a supermfld it is ΠE for $E = \begin{matrix} M \times \mathfrak{g} \\ \downarrow \\ M \end{matrix}$ - trivial bdle)

$$Q = d_{CE} = \frac{1}{2} \sum_{a,b,c} f^c_{ab} c_a c_b \frac{\partial}{\partial c_c} + \sum_a c_a \underset{\underset{\mathfrak{X}(M)}{\gamma}}{v_a}$$

CE of $\mathfrak{g}$ with coeffs in $C^\infty(M)$

$\mu_{\mathcal{F}} = \mu_M \cdot \underbrace{D\xi}_{\mu_{\Pi\mathfrak{g}} \in \Lambda^m \mathfrak{g}}$ - Berezinian corresp. to the choice Haar measure on G

$$\mathrm{div}\, Q = \sum_a c_a \Big(\underbrace{\sum_b f^b_{ab}}_{\mathrm{tr}_{\mathfrak{g}} [T_a, \bullet]} + \mathrm{div}\, v_a\Big) = 0$$

since
- $\mathfrak{g}$ is unimodular (follows from existence of Haar measure)
- μ_M is G-invariant $\Rightarrow$ action by an element $g \in G$ preserves volumes $\Rightarrow$ vector fields of infinitesimal action are divergence-free

Problem: 1) we have coords x_i of deg $= 0$ and c_a of deg $= +1$, hence $C^\infty(\mathcal{F})$ is non-negatively graded; thus we can't construct Ψ as in $(*)$, with $\deg \Psi = -1$.

2) $\int_{\mathcal{F}} \mu\, e^{\frac{i}{\hbar} S} = 0$ (because of integral over $\mathfrak{g}[1]$) - not an interesting integral.

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Solution call the set of data from Attempt 1 $(\mathcal{F}_{min}, Q_{min}, \mu_{\mathcal{F}_{min}})$

set $\mathcal{F} = \mathcal{F}_{min} \times \underbrace{(\mathfrak{g}^*[-1] \oplus \mathfrak{g}^*)}_{= T[1]\,\mathfrak{g}^*[-1] = \mathcal{F}_{auxiliary}}$ with coords

x^i	on M	$deg = 0$
c^a	on $\mathfrak{g}[1]$	$deg = +1$
$\bar{c}_a$	on $\mathfrak{g}^*[-1]$	$deg = -1$
λ_a	on $\mathfrak{g}^*$	$deg = 0$

$$Q = Q_{min} + \underbrace{\sum_a \lambda_a \frac{\partial}{\partial \bar{c}_a}}_{\text{"de Rham vector field" on } T[1]\,\mathfrak{g}^*[-1]} , \quad \mu_{\mathcal{F}} = \mu_M\, Dc\, D\bar{c}\, D\lambda$$

$\int_{\mathcal{F}} \mu \cdot e^{\frac{i}{\hbar} S}$ "contains $0 \cdot \infty$ indeterminacy"

0 ← from int over $c, \bar{c}$

∞ ← from int. over λ

however, we can construct $\Psi_\varphi = \langle \bar{c}, \varphi(x) \rangle \in C^\infty(\mathcal{F})_1$

$\Rightarrow Q(\Psi_\varphi) = \langle \lambda, \varphi(x) \rangle + \langle \bar{c}, FP(x)\, c \rangle$

Thus $\int_{\mathcal{F}} \mu\, e^{\frac{i}{\hbar}(S + Q\Psi_\varphi)}$

- exists (converges), and is $\frac{(2\pi i)^n}{Vol(G)} \int_M \mu_M\, e^{\frac{i}{\hbar} S}$
- is independent under changes of Ψ_φ, in particular under changes of φ.

- can use different choices of Ψ,

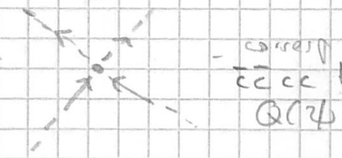
e.g. $\Psi = \langle \bar{c}, \varphi(x) \rangle + \varkappa (\bar{c}, \lambda)_{\mathfrak{g}^*}$ ⇝ $Q(\Psi) = Q(\Psi_\varphi) + \varkappa \cdot (\lambda, \lambda)_{\mathfrak{g}^*}$

a pairing on $\mathfrak{g}^*$

(one can then evaluate the int. over λ and get back an int. of st. phase form)

or Ψ a polynomial of higher degree in $c, \bar{c}$

- this will produce new vertices in st. phase expansion, like



- corresp to $\bar{c}\bar{c}cc$ term in $Q(\Psi)$

Rem $H^0_Q \xleftrightarrow{\sim} C^\infty(M)^G = C^\infty(M/G)$, $H^{\neq 0}_Q = 0$ due to freeness of G-action.

$H^0_Q \cong H^0_{Q_{min}}$

in this sense $(\mathcal{F}, Q)$ is a "resolution" of the quotient M/G.

Remarks

- BRST can accomodate group actions with stabilizers $G' \hookrightarrow G \circlearrowright M$

or $G^N \hookrightarrow \cdots \hookrightarrow G'' \hookrightarrow G' \hookrightarrow G \circlearrowright M$

- Lie algebroid actions (foliations)

$$\begin{tikzcd} E \arrow[rr, hook] \arrow[dr] & & TM \arrow[dl] \\ & M & \end{tikzcd}$$

- There is a similar construction resolving the integral over a submanifold given as zero-locus of a map

⇝ Mathai-Quillen representative of the Thom class of a vector bundle

~ one has to introduce odd "anti-ghosts" [ref: arXiv: hep-th/9511210 survey of 2D Y-M]

- Duistermaat-Heckman localization

(M, ω) - symplectic, H - hamiltonian for a $U(1)$ action, $U(1) \circlearrowright M$

$D = d + \iota_v$ - equivar differential

$$\int_M \frac{\omega^n}{n!} e^{\alpha H} = \alpha^{-n} \int_M e^{\alpha(H+\omega)} = \alpha^{-n} \int_{\Pi TM} \mu_{\Pi TM}\, e^{\Theta + Q\Psi_t}$$

$Q = D$

$\Theta = \alpha(H + \omega)$, $\Psi = (\iota_v g, \cdot)$ ~~$g(v, \cdot)$~~ $t\lambda$

with g equivar. metric, $\lambda = -g(v$

$\rightarrow Q\Psi = D\Psi = t(d\lambda - g(v,v))$

integral does not depend on t,

$t \to \infty$ → localization to fixed points of $U(1)$-action

Batalin-Vilkovisky formalism

- a BV algebra is a $\mathbb{Z}$-graded supercommutative ring $\mathcal{A}^\bullet, \cdot$ endowed with a degree 1 Poisson bracket $\{,\} : \mathcal{A}^j \otimes \mathcal{A}^k \to \mathcal{A}^{j+k+1}$

  s.t. $\{,\}$
  - is skew-symmetric $\{Y,X\} = -(-1)^{(|X|+1)(|Y|+1)}\{X,Y\}$

  [mnemonic rule for signs: comma in $\{\underset{\uparrow}{,}\}$ is odd, of degree +1]
  - is a (bi-)derivation: $\{fg,h\} = f\{g,h\} + \underbrace{(-1)^{(|h|+1)|g|}\{f,h\}g}_{=(-1)^{|f|\cdot|g|}g\{f,h\}}$ $\left\{\begin{array}{l}\text{i.e. } \{\cdot,h\} \text{ is a degree } |h|+1 \text{ right-derivation} \\ \{f,\cdot\} \text{ is a degree } |f|+1 \text{ left-derivation}\end{array}\right.$
  - satisfies Jacobi $\{f,\{g,h\}\} = \{\{f,g\},h\} + (-1)^{(|f|+1)(|g|+1)}\{g,\{f,h\}\}$

  [the structure defined so far $(\mathcal{A}^\bullet, \cdot, \{,\})$ is called a Gerstenhaber algebra]

$\mathcal{A}$ is also endowed with an $\overset{\text{degree 1}}{\vphantom{x}}$ operator $\Delta : \mathcal{A}^j \to \mathcal{A}^{j+1}$ satisfying
- $\Delta^2 = 0$
- Δ is a "2nd order derivation":

(*) $$\Delta(fgh) \pm \Delta(fg)\cdot h \pm \Delta(fh)\cdot g \pm \Delta(gh)\cdot f \pm \Delta f\cdot gh \pm \Delta g\cdot fh \pm \Delta h\cdot fg = 0$$
(7-term relation, or 2nd order Leibniz identity)

- defect of 1st order Leibniz id. for Δ yields $\{,\}$:

(**) $$\Delta(fg) = \Delta f\cdot g + (-1)^{|f|} f\,\Delta g + (-1)^{|f|}\{f,g\}$$

Rem
- it follows from axioms that $\Delta(1) = 0$ (if $\mathcal{A}^\bullet$ is unital)
- a BV algebra $(\mathcal{A}^\bullet, \cdot, \{,\}, \Delta)$ is an "enrichment" of a Gerstenhaber algebra $(\mathcal{A}^\bullet, \cdot, \{,\})$
- axioms above are inter-dependent. E.g. (*) follows from (**) and bi-derivation property of $\{,\}$.

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Example 1. Let $V^\bullet$ be a $\mathbb{Z}$-graded v.sp. with an odd, degree $=-1$ symplectic form ω,

$\omega : V^j \otimes V^{-1-j} \to \mathbb{R}$ skew-symmetry sign convention: $\omega(Y,X) = \underbrace{(-1)^{(|X|+1)(|Y|+1)}}_{+ \text{ for } |X|+|Y|=-1}\omega(Y,X)$

Set $\mathcal{A}^\bullet = \mathrm{Sym}_{gr} V^* = \mathrm{Sym}^\bullet (V^{even})^* \otimes \Lambda^\bullet (V^{odd})^*$
$= \bigotimes_{j \text{ even}} \mathrm{Sym}^\bullet (V^j)^* \otimes \bigotimes_{j \text{ odd}} \Lambda^\bullet (V^j)^*$

$C^\infty(V) \xleftarrow[\text{completion}]{} \text{"}\mathrm{Fun}(V)\text{"}$ with grading inherited from $(V^*)^j = (V^{-j})^*$.

For $f \in C^\infty(V) = \mathcal{A}^\bullet$, we can define a "Hamiltonian vector field" $X_f \in \mathfrak{X}(V) \simeq V^\bullet \otimes C^\infty(V$
by $\iota_{X_f}\omega = \delta f$ (or $\omega(X_f, v) = \pm v(f)$ $\forall v \in \mathfrak{X}(V)$)

$|X_f| = |f|+1$

Then define $\{f,g\} = X_f(g)$ - Poisson bracket

BV Laplacian $\Delta f = \frac{1}{2}\,\underset{\uparrow}{\mathrm{div}}\, X_f$
w.r.t. canonical (coordinate) Berezinian $\mu \in \Lambda^{top}(V^{even})^* \otimes \Lambda^{top} V^{odd}$

In terms of a $\overset{\text{Darboux}}{\text{basis}}$ in V,

$\omega = \sum_i \delta x^i \wedge \delta y_i$, $\{f,g\} = \sum_i f\left(\frac{\overleftarrow{\partial}}{\partial x^i}\frac{\overrightarrow{\partial}}{\partial y_i} - \frac{\overleftarrow{\partial}}{\partial y_i}\frac{\overrightarrow{\partial}}{\partial x^i}\right)g$ $\Delta f = \overbrace{\left(\sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial y_i}\right)}^{\text{2nd order diff op}} f$

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Example 2 M - an ordinary mfd, $\nu \in \Omega^{top}(M)$ a volume form

$\mathcal{A}^\bullet = \mathcal{V}^{-\bullet}(M) = \Gamma(M, \Lambda^{-\bullet} TM)$ - polyvectors (with reverse grading)

i.e. $\mathcal{A}^{-j} = \Gamma(M, \Lambda^j TM)$, $0 \leq j \leq \dim M$

$\{X, Y\} = [X, Y]_{NS}$ - Schouten-Nijenhuis bracket of polyvectors:

$[f,g]_{NS} = 0$, $[v,w]_{NS} = [v,w]_{Lie}$, $[v,f] = v(f)$, $[u, v\wedge w] = [u,v]\wedge w + [u,w]\wedge$

$f, g \in C^\infty(M)$ $\quad v, w \in \mathfrak{X}(M)$ $\quad \vdots$

$\Delta X = \mathrm{div}_\nu X \qquad \mathrm{div}_\nu : \mathcal{V}^j(M) \longrightarrow \mathcal{V}^{j-1}(M)$

$\mathrm{div}\, f = 0$

$\mathrm{div}\, v = \mathrm{div}\, v$

$\mathrm{div}(v\wedge w) = (\mathrm{div}\, v)\, w - v\, \mathrm{div}\, w - [v,w]$

. . . .

Example 3 (Main example) $\mathcal{F}$ - a $\mathbb{Z}$-graded supermanifold

$\omega \in \Omega^2(\mathcal{F})$ - non-degenerate,

i.e. defining an iso $\omega^\#: T\mathcal{F} \longrightarrow T^*\mathcal{F}$

μ - a Berezinian on $\mathcal{F}$.

Set $\mathcal{A}^\bullet = C^\infty(\mathcal{F})$

$\{f, g\} = X_f(g)$ where X_f is defined by $\iota_{X_f}\omega = df$

$\Delta f = \frac{1}{2} \mathrm{div}_\mu X_f$. $\qquad \Delta^2 = 0$ is not automatic but imposes a compatibility condition between μ and ω.

Rem: p-forms on a split supermfd ΠE

$\Omega^p(\Pi E) = \bigoplus_j \Gamma(M, \Lambda^\bullet E^* \otimes \Lambda^j T^*M \otimes S^{p-j}_{\ldots}$

$\Omega^\bullet(\mathcal{M}) = C^\infty(T[1]\mathcal{M})$

Locally: $\Omega^\bullet(\mathcal{M})_U = \mathcal{O}_\mathcal{M}(U) \otimes \mathbb{R}[dx^i]$

$|dx^i| = |x^i| + 1$

〃 completion of $\mathbb{R}[x^i, dx^i]$

Rem Example 1 = Ex. 3 for $\mathcal{F}$ a gr. v. sp. with coord-independent ω and μ (which is unique up to normalization)

Ex 2 = Ex 3 for $\mathcal{F} = T^*[-1]M$ with ω_{can} of a cotangent bdle,

$\mu = \nu^{\otimes 2}$ (Note: $\mathrm{Ber}\,\mathcal{F} = (\Lambda^{top} T^*M)^{\otimes 2}$ is the tensor square of the bundle of volume forms, so $\mu = \nu^{\otimes 2}$ makes sense)

$$\begin{tikzcd} \mathrm{Ber}\,\mathcal{F} \arrow[d] \\ M \end{tikzcd}$$

Master equation

for $(\mathcal{A}^\bullet, \cdot, \{,\})$ a Gerstenhaber algebra, one says that

$S \in \mathcal{A}^0$ solves the classical master equation if $\{S, S\} = 0$

for $(\mathcal{A}^\bullet, \cdot, \{,\}, \Delta)$ a BV algebra, $S \in \mathcal{A}^0[[\hbar]]$ solves quantum ME if

$\frac{1}{2}\{S,S\} - i\hbar \Delta S = 0 \quad \Leftrightarrow \quad \Delta e^{\frac{i}{\hbar}S} = 0$.

If $S = S^{(0)} + S^{(1)}\hbar + S^{(2)}\hbar^2 + \ldots$ solves QME $\Leftrightarrow$

$\{S^0, S^0\} = 0$

$\{S^0, S^1\} = i\Delta S^0$

$\{S^0, S^2\} = -\frac{1}{2}\{S^1, S^1\} + i\Delta S^1$

. . . .

$Q = \{S^0, \cdot\} \in \mathrm{Der}_1 \mathcal{A}^\bullet$, $Q^2 = 0$.

Rem: obstructions for extension of sol. of CME to sol. of QME are $[\Delta S^0] \in H^1_Q$, $[-\frac{1}{2}\{S^1, S^1\} + i\Delta S^1] \in H^1_Q$ etc.

"Classical BV data": $(\mathcal{F}, \underset{\text{deg } -1 \text{ symp}}{\omega}, \underset{\text{sol. of CME}}{S})$ $\qquad Q = \{S, \cdot\}$ is a coh. v. f. on $\mathcal{F}$

"quantum BV data": $(\mathcal{F}, \omega, \underset{\text{Berezinian}}{\mu}, \underset{\text{sol. of QME}}{S^\hbar})$ $\qquad \delta_{BV} = \{S^\hbar, \cdot\} - i\hbar\Delta$ is a coboundary op. on $C^\infty($ but not a derivation.