Let $(\mathcal{F},\omega)$ be odd-symplectic, $\mathcal{L}\subset\mathcal{F}$ a Lagrangian submanifold
(max isotropic $\Rightarrow$ isotropic and coisotropic)

μ a Berezinian on $\mathcal{F}$ $\rightsquigarrow$ can construct a Berezinian $\sqrt{\mu}\big|_{\mathcal{L}}$ on $\mathcal{L}$

since $\mathcal{F}\sim T^*[-1]\mathcal{L}$ (tubular nbhd theorem)
locally near
$\mathcal{L}\sim$ zero section of $T^*[-1]\mathcal{L}$

and $\mathrm{Ber}(\mathcal{F})\big|_{\mathcal{L}}=(\mathrm{Ber}\,\mathcal{L})^{\otimes 2}$

(in lin. algebra: $\mathrm{Det}(V\oplus V^*[-1])\cong$
$\cong \mathrm{Det}\,V\otimes(\mathrm{Det}\,V^*)^{-1}\cong(\mathrm{Det}\,V)^{\otimes 2}$)

aside: standard constructions of Lagrangians:
(I) $\mathrm{graph}\,d\Psi\subset T^*[-1]\mathcal{N}$, $\Psi\in C^\infty(\mathcal{N})_{-1}$
(II) $N^*[-1]D$, $D\subset\mathcal{N}$
$T^*[-1]\mathcal{N}$

BV integrals are integrals of the form

$$\int_{\mathcal{L}\subset\mathcal{F}} f\,\sqrt{\mu}\big|_{\mathcal{L}}\quad\text{for }\Delta_\mu f=0.$$

BV-Stokes' theorem (BV, A.S. Schwarz)

(1) $\displaystyle\int_{\mathcal{L}\subset\mathcal{F}}\Delta_\mu f\cdot\sqrt{\mu}\big|_{\mathcal{L}}=0$

Rem Thm (A.S. Schwarz): a supermfd $\mathcal{M}$ with odd-symplectic ω is symplectomorphic to ΠT^*M for ~~some odd~~ $M=\mathrm{body}(\mathcal{M})$

(2) if $\Delta_\mu f=0$, $\mathcal{L}$ and $\mathcal{L}'$ two Lagrangians which can be connected by a smooth family of Lagrangians, then

$$\int_{\mathcal{L}} f\,\sqrt{\mu}\big|_{\mathcal{L}}=\int_{\mathcal{L}'} f\,\sqrt{\mu}\big|_{\mathcal{L}'}$$

Rem (1) locally, in Darboux coords (x,ξ) where $\mathcal{L}$ is given by $\xi=0$, with $\mu=Dx\,D\xi$,
~~looks like~~ ~~is given by~~

$$\int_{\mathcal{L}} Dx\sum\frac{\partial}{\partial x^i}\frac{\partial}{\partial\xi_i}f=0\quad\text{by usual Stokes}$$

(2) small deformation of $\mathcal{L}$ looks like $\mathcal{L}'=\mathrm{graph}\,d\Psi$, $\Psi=\Psi(x)\in C^\infty(\mathcal{L})_{-1}$
$\xi=0$ — flow of $\mathcal{L}$ by Ham v.f. $\{\Psi,\cdot\}$ in time 1
i.e. $\xi_i=\frac{\partial\Psi}{\partial x^i}$

$$\int_{\mathcal{L}'}Dx\,f(x,\xi)=\int Dx\,f(x,0)+\frac{\partial}{\partial\xi_i}f(x,0)\cdot\frac{\partial\Psi}{\partial x^i}+O(\Psi^2)\underset{\text{using }\Delta f=0}{=}$$
$$=\int_{\mathcal{L}}f+\underbrace{\int_{\mathcal{L}}\Delta(f\cdot\Psi)}_{0\text{ by (1)}}+O(\Psi^2)$$

"Odd Fourier transform"

let $\mathcal{F}=T^*[-1]M$, M ordinary mfd

$$C^\infty(\mathcal{F})=\mathcal{V}^\bullet(M)\xrightarrow[\sim]{\;\text{OFT}\;:\;v\mapsto\iota_v\nu\;}\Omega^\bullet(M)$$
ν — degree n map

$\nu\in\Omega^{top}M$

locally:
$$v=v(x^i,\xi_i)\longmapsto\int\underbrace{D\xi}_{\nu_x\in\Lambda^{top}T_x^*M}e^{\langle\xi,\theta\rangle}v(x,\xi)\in C^\infty(T[1]M)=\Omega^\bullet(M)$$
$\theta^i=dx^i$, $\xi_i=\frac{\partial}{\partial x^i}$

$\sum\frac{\partial}{\partial x^i}\frac{\partial}{\partial\xi_i}=\Delta\longleftrightarrow d=\sum\theta^i\frac{\partial}{\partial x^i}$
then Δ on $C^\infty(\mathcal{F})$ $\longleftrightarrow$ d on $\Omega^\bullet(M)$

Lemma:
$$\int_{\mathcal{L}=N^*[-1]D}\sqrt{\mu}\big|_{\mathcal{L}}\,f_v=\int_D\mathrm{OFT}(v)$$
$D\subset M$

Reminder: $N_x^*D=\{\alpha\in T_x^*M\text{ s.t. }\forall\gamma\in T_xD,\ \langle\alpha,\gamma\rangle=0\}=\mathrm{Ann}\,T_xD\subset T_x^*M$

Half-densities

For V a v.sp., $\alpha \in \mathbb{R}$, $\mathrm{Dens}^\alpha V = \{\varphi : \{\text{frames on } V\} \to \mathbb{R} \mid \forall A \in GL(V),\ \varphi(A\cdot e) = |\det A|^\alpha \varphi(e)\}$

$V^\bullet$ a gr. v.sp. $\to \mathrm{Dens}^\alpha V^\bullet = \bigotimes_j (\mathrm{Dens}^\alpha V^j)^{(-1)^j}$

$\to$ can have a vector bundle version (fiber-wise α-density)

$\to$ for $\mathcal{F}$ a $\mathbb{Z}$-graded supermfd, $\mathbb{D}\mathrm{ens}^{1/2}\mathcal{F} := \mathrm{Dens}^{1/2}_{\mathcal{F}}T\mathcal{F} \downarrow \mathcal{F}$ - the line bundle of $1/2$-densities

$\mathrm{Dens}^{1/2}\mathcal{F} = \Gamma(\mathbb{D}\mathrm{ens}^{1/2}\mathcal{F})$

Berezinian μ on $\mathcal{F}$ is a 1-density, $\mu \in \mathrm{Dens}^1(\mathcal{F})$

[assuming orientation of odd fibers is fixed]

$\sqrt{\mu} \in \mathrm{Dens}^{1/2}(\mathcal{F})$

Rem: in linear setting, for V oriented (or calculation $\mathcal{O}$)
$\mathrm{Dens}^1 V \cong \Lambda^{top} V^*$ | for $V^\bullet$ a gr. v.sp., $\mathrm{Dens}^1 V^\bullet \cong \mathrm{Det}\, V^\bullet$
$(\varphi : (e_i) \mapsto \pm \langle \alpha, e_1 \ldots e_n \rangle)$ $\bigotimes (\Lambda^{top} V^j)^{-(\ldots)}$
sign depends on or. of $(e_1, \ldots, e_n)$ vs $\mathcal{O}$

There is a canonical BV-Laplacian on $\mathrm{Dens}^{1/2}(\mathcal{F})$, independent of μ (Khudaverdian, Severa)

Locally, $\Delta_{can} : f(x,\xi)\, D^{1/2}x\, D^{1/2}\xi \longmapsto \left(\sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i} f\right) D^{1/2}x\, D^{1/2}\xi$

Rem: for positive densities can take arbitrary powers, e.g. $\mathrm{Dens}^1_+ V^\bullet \to \mathrm{Dens}^{1/2}_+ V^\bullet$

such that

$$\begin{tikzcd} C^\infty(\mathcal{F}) \arrow[r, "\Delta_\mu"] \arrow[d, "\cdot\sqrt{\mu}"'] & C^\infty(\mathcal{F}) \arrow[d, "\cdot\sqrt{\mu}"] \\ \mathrm{Dens}^{1/2}\mathcal{F} \arrow[r, "\Delta_{can}"'] & \mathrm{Dens}^{1/2}\mathcal{F} \end{tikzcd}$$

Note: $\mathrm{Dens}^{1/2}\mathcal{F}, \Delta_{can}$ is a complex but not a BV algebra (cannot multiply)

BV integrals in terms of $1/2$-densities are simply: $\int_{\mathcal{L}\subset\mathcal{F}} \varphi$, $\varphi \in \mathrm{Dens}^{1/2}\mathcal{F}$ s.t. $\Delta_{can}\varphi = 0$

Rem BV integral can be viewed as a pairing between $H^\bullet_\Delta$ and classes $[\mathcal{L}]$ of Lagrangians modulo Lagr. homotopy.
* more generally, one can pair to chains of classes of Lagrangians, $\sum_{[\mathcal{L}]} c_{[\mathcal{L}]}[\mathcal{L}]$

embedding BRST into BV

Given BRST data ~~$(\mathcal{F}_{BRST}, Q_B)$~~ $(\mathcal{F}_B, Q_B, S_{cl}, \mu_B)$, we construct

$\mathcal{F} = T^*[-1]\mathcal{F}_B$ $\quad S = S_{cl} + \overline{Q_B} = $ ~~S_{cl}~~ (locally) $S_{cl}(\Phi) + \sum_i \Phi^+_i Q^i_B(\Phi)$

$\omega = \omega_{can}$ - can. for T^* $\quad$ $\overline{Q_B}$: image of Q under $\mathfrak{X}(\mathcal{F}_B) \subset C^\infty(T^*[-1]\mathcal{F}_B)$ $\quad$ Φ^i coord on $\mathcal{F}_B$, Φ^+_i - coord in fiber of $T^*[-1]\mathcal{F}_B$

$\omega = \sum_i \delta\Phi^+_i \wedge \delta\Phi^i$ $\quad$ $\mathfrak{X}(\mathcal{F}_B) = \Gamma(\mathcal{F}_B, T\mathcal{F}_B)$ $\quad$ $C^\infty(T^*[-1]\mathcal{F}_B) = \Gamma(\mathcal{F}_B, \mathrm{Sym}\, T[1]\mathcal{F}_B)$

$\to Q = \{S, \cdot\} = \left(\begin{array}{c}\text{cotangent lift}\\ \text{of } Q_B\end{array}\right) + dS_{cl}$ (viewed as a vert. v.f. on $T^*[-1]\mathcal{F}_B$) $\underset{\text{locally}}{=} \sum_i Q^i_B(\Phi)\frac{\partial}{\partial\Phi^i} + \sum \Phi^+_i \partial_j Q^i_B(\Phi)\frac{\partial}{\partial\Phi^+_j} + \sum_i \partial_i S_{cl}(\Phi)\frac{\partial}{\partial\Phi^+_i}$

Rem: in ordinary geom., for $\varphi : M \xrightarrow{\sim} M$ diffeo, cot. lift $\tilde\varphi : T^*M \to T^*M$, $(x,\xi) \mapsto (\varphi, (d\varphi)^{-1T}\xi)$ $\to$ infinitesimal version of cot. lift $\mathfrak{X}(M) \to \mathfrak{X}(T^*M)$

$\mu = \mu_B^{\otimes 2} \in \Gamma(\mathrm{Ber}\,\mathcal{F}) = \Gamma((\mathrm{Ber}\,\mathcal{F}_B)^{\otimes 2})$

Lemma $\int_{\mathcal{L}_\psi} e^{\frac{i}{\hbar}S}\sqrt{\mu} = \int_{\mathcal{F}_B} e^{\frac{i}{\hbar}(S_{cl} + Q_B\psi)}\mu_B$ - comparison of BV and BRST integrals

$\mathcal{L}_\psi = \mathrm{graph}\, d\psi \subset T^*[-1]\mathcal{F}_B$ for $\psi \in C^\infty(\mathcal{F}_B)_{-1}$

because $S|_{\mathcal{L}_\psi} = S(\Phi^i, \Phi^+_i = \frac{\partial\psi}{\partial\Phi^i}) = S(\Phi^i, 0) + \frac{\partial S}{\partial\Phi^+_i}\cdot\frac{\partial\psi(\Phi)}{\partial\Phi^i} = S(\Phi, 0) + \{S, \psi\}|_{\mathcal{F}_B \text{ i.e. } \Phi^+ = 0}$

since S is linear in Φ^+ $\quad = S + \underbrace{Q(S)}_{=Q_B S}|_{\mathcal{F}_B}$

Rem: QME for $S \iff$ $\left.\begin{array}{l} \cdot\ Q_B^2 = 0 \\ \cdot\ Q_B S_{cl} = 0 \end{array}\right\}$ CME
$\cdot\ \mathrm{div}_{\mu_B} Q_B = 0$

BV 14/3

Faddeev-Popov via BV

$G \circlearrowright M$, $S_{cl} \in C^\infty(M)^G$, $\mu_{cl} \in \Omega^{top}(M)^G$

gauge-fixing: $\varphi^{-1}(0) \subset M$ for $\varphi: M \to \mathfrak{g}$

Assoc. BV data: (min. version)

$$\mathcal{F} = T^*[-1]\mathcal{F}^{min}_{BRST} = T^*[-1](M \times \mathfrak{g}[1]) = T^*[-1]M \times \mathfrak{g}[1] \times \mathfrak{g}^*[-2]$$

coordinates:	x^i	x^+_i	c^a	c^+_a
degree ("ghost number")	0	-1	1	-2

$$\omega = \omega_{can} = \sum_i \delta x^+_i \wedge \delta x^i + \sum_a \delta c^+_a \wedge \delta c^a$$

$$S(x, x^+, c, c^+) = S_{cl}(x) + \underbrace{\sum_{i,a} x^+_i\, v^i_a(x)\, c^a}_{Q_{BRST}} + \frac{1}{2}\sum \langle c^+, [c,c]\rangle$$

$= \sum_a \langle x^+, v_a c^a\rangle = d\rho_x \in \mathrm{Hom}(\mathfrak{g}, T_x M)$ understood as a bilinear function on $\mathfrak{g}[1] \times T^*_x[-1]M$

lift to a function on $\mathcal{F}_{BV}$

- S satisfies QME $\qquad \mu = \mu_{cl}^{\otimes 2} \otimes (\mu_{\mathfrak{g}}^{-1})^{\otimes 2}$

- gauge-fixing: $\mathcal{L}_\varphi = N^*[-1]\underbrace{(\varphi^{-1}(0) \times \mathfrak{g}[1])}_{\cap\ \mathcal{F}^{min}_{BRST}} \subset T^*[-1]\mathcal{F}^{min}_{BRST}$ locally given by:

$$\begin{cases} x \text{ s.t. } \varphi^a(x) = 0 \\ c \text{ any} \\ x^+ = d\varphi^T(\bar{c}) \quad \text{i.e. } x^+_i = \frac{\partial \varphi^a(x)}{\partial x^i}\bar{c}_a \\ c^+ = 0 \end{cases}$$

from calculating $\mathrm{Ann}\left(\frac{\ker d_x\varphi}{T_x M}\right) = \mathrm{im}(d_x\varphi)^T$

Thus

$$\int_{\mathcal{L}_\varphi \subset \mathcal{F}} e^{\frac{i}{\hbar}S}\sqrt{\mu} = \int_{\substack{\varphi^{-1}(0) \times \mathfrak{g}[1] \times \mathfrak{g}^*[-1] \\ x \quad c \quad \bar{c}}} \mathcal{D}\ldots\ e^{\frac{i}{\hbar}(S_{cl} + \langle \bar{c}, FP(x)\, c\rangle)}$$

$$\simeq \int_{\substack{M \times \mathfrak{g}^* \times \mathfrak{g}[1] \times \mathfrak{g}^*[-1] \\ x \quad \lambda \quad c \quad \bar{c}}} dx\, d\lambda\, \mathcal{D}c\, \mathcal{D}\bar{c}\ e^{\frac{i}{\hbar}(S_{cl}(x) + \langle \lambda, \varphi(x)\rangle + \langle \bar{c}, FP(x)\, c\rangle)}$$

– FP integral.

Example: Chern-Simons theory. N – closed 3-mfld, $\underline{G}$ – compact gauge group

$$M = \Omega^1(N) \otimes \underline{\mathfrak{g}} \simeq \mathrm{Conn}\begin{pmatrix} N \times \underline{G} \\ \downarrow \\ N \end{pmatrix}, \quad G = \Omega^0(N, \underline{G}) \text{ acting by } A \mapsto A^g = gAg^{-1} + g\,dg^{-1}$$

$$S_{cl} = \int_N \mathrm{tr}\left(\frac{1}{2} A \wedge dA + \frac{1}{6} A \wedge [A,A]\right)$$

$\mathcal{F}^{min}_{BRST} = \Omega^0 \oplus \Omega^1(N, \underline{\mathfrak{g}}) \ni (c, A)$, gh $= 1, 0$

$$\mathcal{F} = \Omega^\bullet(N, \underline{\mathfrak{g}})[1] \overset{\text{(model for.)}}{=} T^*[-1]\mathcal{F}^{min}_{BRST}$$

-1	0	1	2	– degree of component of gr. v.sp.
Ω^0	Ω^1	Ω^2	Ω^3	
c	A	A^+	c^+	coordinate

super-field $\mathcal{A} = c + A + A^+ + c^+ \in \Omega^\bullet(N, \underline{\mathfrak{g}})[1]$

Rem: more precisely, $\mathcal{A}: \mathcal{F} \to \Omega^\bullet(N, \underline{\mathfrak{g}})$ – superfield map, deg $= 1$ – universal coordinate on $\mathcal{F}$ with values in diff. forms.

$$\omega = \frac{1}{2}\int_N \mathrm{tr}\,\delta\mathcal{A} \wedge \delta\mathcal{A} = \int_N \mathrm{tr}(\delta A^+ \wedge \delta A + \delta c^+ \wedge \delta c)$$

$$S = \int_N \mathrm{tr}\Big(\underbrace{\frac{1}{2} A \wedge dA + \frac{1}{6} A \wedge [A,A]}_{S_{cl}} + \underbrace{c^+ \wedge d_A c + \frac{1}{2} c^+ \wedge [c,c]}_{\text{Hamiltonian for } d_{CE}}\Big) = \underbrace{\int_N \frac{1}{2}\mathcal{A} \wedge d\mathcal{A} + \frac{1}{6}\mathcal{A} \wedge [\mathcal{A}, \mathcal{A}]}_{\text{same form. as } S_{cl} \text{ but } A \mapsto \mathcal{A}}$$

superfield

– a more general class of models where similar effect occurs – "AKSZ sigma-models"

one of standard choices of $\mathcal{L}$: $\mathcal{L} = \Omega^\bullet_{coex}(N, \underline{\mathfrak{g}})[1] = \{\mathcal{A} \text{ s.t. } d^*\mathcal{A} = 0\}$ – Lorentz gauge

$d^* = \pm * d *$ for some metric g

actually, for $\mathcal{L}$ to be Lagr. (not isotropic), we have to twist by an acyclic local system (cf. Axelrod-Singer '92)

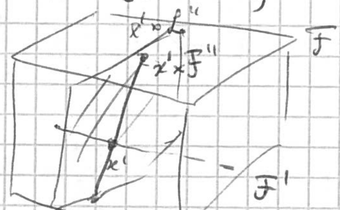
BV pushforward (also: fiber BV integral or effective BV action construction) BV 15/4

Let $(\mathcal{F}', \omega', \mu')$, $(\mathcal{F}'', \omega'', \mu'')$ be two odd-symplectic mfds with compatible Berezinians

then we can construct $\left(\mathcal{F} = \mathcal{F}' \times \mathcal{F}'', \ \omega = \omega' \oplus \omega'', \ \mu = \mu' \otimes \mu''\right)$

for $S \in C^\infty(\mathcal{F})_0[[\hbar]]$, define $S' \in C^\infty(\mathcal{F}')_0[[\hbar]]$ by

$$(*) \quad e^{\frac{i}{\hbar} S'} \sqrt{\mu'} = \int_{\substack{\mathcal{L}'' \subset \mathcal{F}'' \\ \text{Lagr}}} e^{\frac{i}{\hbar} S} \sqrt{\mu}$$



Thm

(1) if S satisfies QME on $\mathcal{F}$

then S' defined by $(*)$ satisfies QME on $\mathcal{F}'$

(2) if $\mathcal{L}''_t \subset \mathcal{F}''$ is a family of Lagrangians, $t \in [0,1]$ (with $\mathcal{L}_{t+\varepsilon}$ given by graph$(\varepsilon d\psi_t)$, $\psi_t \in C^\infty(\mathcal{L}_t)$)

and S'_t the corresp. eff. BV action, then

$$\boxed{\left(e^{\frac{i}{\hbar}S'_1} - e^{\frac{i}{\hbar}S'_0}\right)\sqrt{\mu'} = \Delta_{\mu'} \Psi}, \quad \Psi = \int_0^1 dt \int_{\mathcal{L}''_t} e^{\frac{i}{\hbar}S} \cdot \psi_t \sqrt{\mu''}$$

i.e. S'_0 and S'_1 are related by a "BV canonical transformation"

infinitesimally:
$$\boxed{\frac{\partial}{\partial t} S'_t = \{S'_t, R'_t\}_{\omega'} - i\hbar \Delta_{\mu'} R'_t}$$
with $R'_t = \cancel{\int_{\mathcal{L}_t}} e^{-\frac{i}{\hbar}S'_t} \int_{\mathcal{L}''_t} e^{\frac{i}{\hbar}S} \psi_t \sqrt{\mu''}$

infinitesimal canonical transformation

cf. action of an infinitesimal symplectomorphism on a "$\log$-$\frac{1}{2}$-density".
i.e. symp. v.f. $\{\cdot, R_t\}$

(3) if S_0 and S_1 (sol. of QME) are related by a can. transf. on $\mathcal{F}$

then corresp. S'_0 and S'_1 are related by a can. transf. on $\mathcal{F}'$.

Thus we have a pushforward map

$$\frac{\mathrm{Sol\,QME}(\mathcal{F})}{\text{can. transf.}} \xrightarrow{[\mathcal{L}'']} \frac{\mathrm{Sol\,QME}(\mathcal{F}')}{\text{can. transf.}}$$

dependent on a class of Lagrangians $[\mathcal{L}'']$ modulo Lagr. homotopy.

Corresponding picture on the side of ordinary de Rham theory, via OFT

taking $\mathcal{F} = T^*[-1]M$, $\mathcal{F}' = \dots$, $\mathcal{F}'' = \dots$

$\mathcal{L}'' = N^*[-1]D''$
$\quad\quad\quad\quad \widetilde{M''}$

$N = N' \times N''$
$\quad\quad\quad\quad \cup$
$\quad\quad\quad\quad D''$ cycle

fiber integration: $\Omega^\bullet(N)$
$P_* \quad \downarrow \int_{D''}$
$\cancel{\Omega^\bullet(N')}$
$\Omega^{\bullet - \dim D''}(N')$

P_* sends closed forms to closed forms (1), preserves cohomology classes (3) (exact $\to$ exact)

and change $D'' \to \tilde{D}''$ within same homology class induces a change in P_* by an exact term (2).

"Discrete BF theory"

- outlook, for details see arXiv:hep-th/0610326, arXiv: 0809.1160

- let $V^\bullet$ be a fin.dim. $\mathbb{Z}$-graded v.sp. (in examples, it will be non-negatively graded)

define $\mathcal{F} = V^\bullet[1] \oplus V^*[-2]$, i.e. $\mathcal{F}^j = \{ V^{j+1} \oplus (V^{2-j})^*$

$\phantom{define \mathcal{F}} \overset{\shortparallel}{T^*[-1]} V[1]$

ω = canonical symp form of cot. bundle

denote $A: \mathcal{F} \to V[1] \to V$ universal coordinates on $\mathcal{F}$ (superfields), of total degrees 1 and -2

$B: \mathcal{F} \to V^*[-2] \to V^*$

if $\{e_i\}$ - basis in V, $\{e^i\}$ - dual basis in V^*,

then $A = \sum A^i e_i$, $B = \sum B_i e^i$

where $\underbrace{A^i}_{\text{components}} \in C^\infty(\mathcal{F})_{1-|e_i|}$, $B_i \in C^\infty(\mathcal{F})_{-2+|e_i|}$ are the coordinates on $\mathcal{F}$.

$\omega = \langle \delta B, \delta A \rangle = \sum_i \delta B^i \wedge \delta A_i \in \Omega^2(\mathcal{F})_{-1}$

$Fun(\mathcal{F})$ ~~forms~~ has a structure of a BV Algebra $(Fun(\mathcal{F}), \cdot, \{,\}, \Delta)$ $\quad \Delta = \sum \frac{\partial}{\partial A_i}\frac{\partial}{\partial B^i}$

- Consider $S \in Fun(\mathcal{F})$ of the form

$S(A,B) = \sum_{ij} d^i_{\ j} B_i A^j + \frac{1}{2}\sum_{ijk} f^i_{\ jk} B_i A^j A^k$

Lemma: S satisfies QME $\iff$ $(d^i_{\ j}, f^i_{\ jk})$ define a structure of unimodular dgLie algebra on

$\frac{1}{2}\{S,S\} - i\hbar \Delta S = 0$ $\quad$ with $d: V^\bullet \to V^{\bullet+1}$, $[,]: V \otimes V \to V$

$d(e_i) = \sum_j d^j_{\ i} e_j$ $\quad [e_i, e_j] = \sum_k f^k_{\ ij} e_k$

satisfying identities $d^2 = 0$

Leibniz $d[X,Y] = [dX, Y] + (-1)^{|X|}[X, dY]$

Jacobi $[X,[Y,Z]] = [[X,Y],Z] \pm [Y,[X,Z]]$

and $\mathrm{tr}\,[X,\cdot] = 0$

In this case one can write $S = \sum \langle B, dA \rangle + \frac{1}{2}\langle B, [A,A] \rangle$ - a BV theory ("abstract BF") associated to a uDGLA $(V^\bullet, d, [,])$

Lemma': Let $S \in Fun(\mathcal{F})[[\hbar]]$ be of form

$S(A,B) = \sum_{n\geq 0} \frac{1}{n!}\langle B, l_n(\underbrace{A, \ldots, A}_{n}) \rangle + i\hbar \sum_{n\geq 0} \frac{1}{n!} q_n(\underbrace{A \ldots A}_{n})$

with $l_n: \Lambda^n V \to V$, $q_n: \Lambda^n V \to \mathbb{R}$ some multi-linear maps

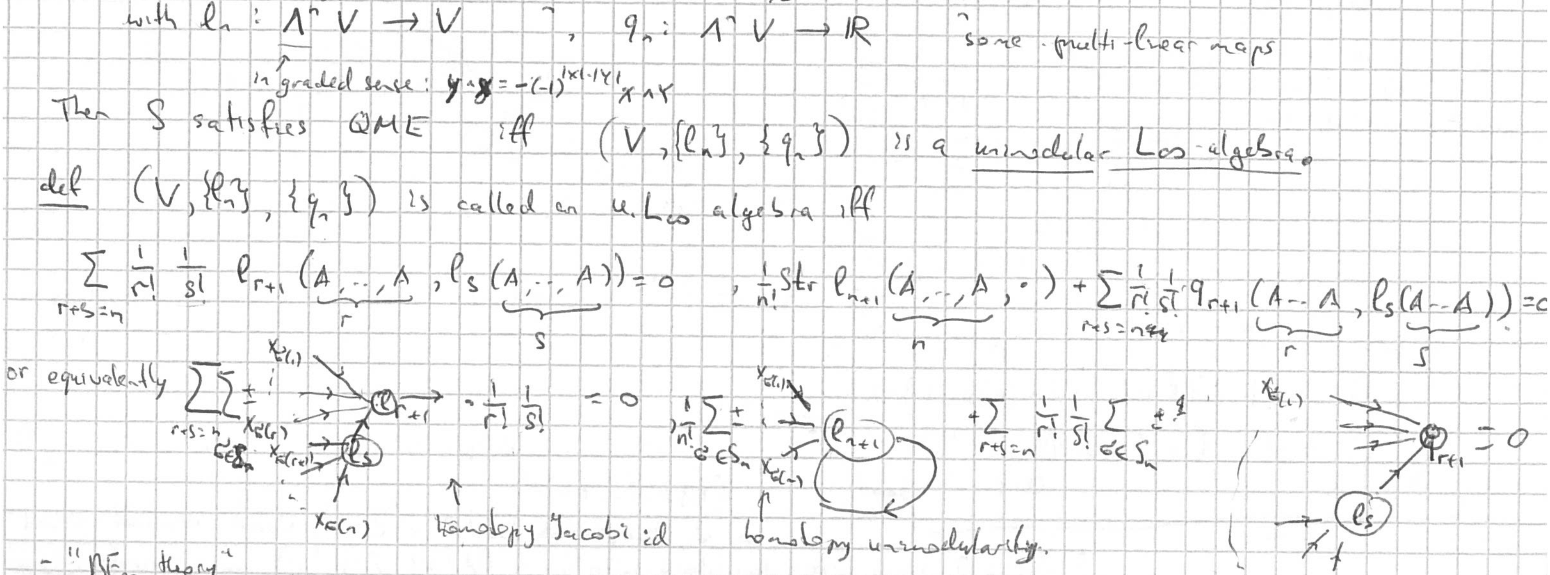
in graded sense: $y \wedge x = -(-1)^{|x|\cdot|y|} x \wedge y$

Then S satisfies QME iff $(V, \{l_n\}, \{q_n\})$ is a unimodular L_∞-algebra.

def $(V, \{l_n\}, \{q_n\})$ is called an u.L_∞ algebra iff

$\sum_{r+s=n} \frac{1}{r!}\frac{1}{s!} l_{r+1}(\underbrace{A, \ldots, A}_{r}, l_s(\underbrace{A, \ldots, A}_{s})) = 0$, $\quad \frac{1}{n!}\mathrm{Str}\, l_{n+1}(\underbrace{A, \ldots, A}_{n}, \cdot) + \sum_{r+s=n} \frac{1}{r!}\frac{1}{s!} q_{r+1}(\underbrace{A \ldots A}_{r}, l_s(\underbrace{A \ldots A}_{s})) = 0$

or equivalently



homotopy Jacobi id $\quad$ homotopy unimodularity

- "BF_∞ theory"

• prototypical dgLa: $V^\bullet = \Omega^\bullet(M)\otimes\mathfrak{g}$, M – a mfld, $\dim = n$
d = de Rham on M, $\mathfrak{g}$ – a Lie algebra
$d_M \otimes \mathrm{id}_{\mathfrak{g}}$
$[,]$ = (wedge product of forms) $\otimes [,]_{\mathfrak{g}}$, i.e. $[\alpha\otimes X, \beta\otimes Y] = (\alpha\wedge\beta)\otimes[X,Y]_{\mathfrak{g}}$
$\alpha,\beta\in\Omega^\bullet(M)$ $X,Y\in\mathfrak{g}$

If $\mathfrak{g}$ is unimodular, $\mathrm{Str}_V[\alpha\otimes X,\cdot] = \underbrace{\mathrm{Str}_{\Omega(M)}\alpha\wedge\cdot}_{\infty\ \text{(ill-defined)}} \cdot \underbrace{\mathrm{tr}_{\mathfrak{g}}[X,\cdot]}_{0} \overset{\text{(formally)}}{=} 0$

V is ∞-dimensional,
so unimodularity makes sense only with a regularization of Str.

corresp. BV theory: $S = \int_M \langle B, dA\rangle + \frac{1}{2}\langle B, [A,A]\rangle$, $A\in\Omega^\bullet(M)\otimes\mathfrak{g}$, $B\in\Omega^\bullet(M)\otimes\mathfrak{g}^*$

$\mathcal{F} = \Omega^\bullet(M)\otimes\mathfrak{g}[1]\oplus\Omega^\bullet(M)\otimes\mathfrak{g}^*[n-2]$ — identifying $(\Omega^j(M))^*$ with $\Omega^{n-j}(M)$

$\mathcal{F}_0 = \Omega^1(M,\mathfrak{g})\oplus\Omega^{n-2}(M,\mathfrak{g}^*)$ ← restriction of S here yields the classical BF theory.

$A = \boxed{A^0} + A^1 + \underset{\substack{\shortparallel\\ (B^2_{\ldots})^*}}{A^2 + A^3} + \ldots + A^n$ ← form degrees

ghosts

$B = \boxed{B^0 + \ldots + B^{n-3}} + B^{n-2} + \underset{\substack{\shortparallel\\ (A^1)^+}}{B^{n-1}} + \underset{\substack{\shortparallel\\ c^+}}{B^n}$ anti-field,

tower of higher ghosts — class. fields

– corresp. to the gauge-symmetry tower of BF theory:

$\Omega^0 \to \ldots \to \Omega^{n-4} \to \underbrace{\Omega^0\oplus\Omega^{n-3}}_{\text{gauge}} \to \underbrace{(\Omega^1\oplus\Omega^{n-2})}_{F_{cl}}$

tower of on-shell stabilizers — stabilizer over a point in crit. locus of S ("on-shell")

In the setting of abstr. BF theory assoc to $(V,d,[,])$, consider building an eff. action.

Let $V = \underset{\text{def retract}}{V'} \oplus \underset{\text{acyclic}}{V''}$ – splitting into subcomplexes (does not respect $[,]$)

induces a splitting $\mathcal{F} = \mathcal{F}'\oplus\mathcal{F}''$

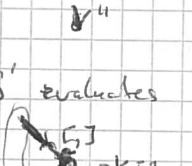
$e^{\frac{i}{\hbar}S'} = \int_{\mathcal{L}\subset\mathcal{F}''} e^{\frac{i}{\hbar}S}$ (let us work modulo ~~the~~ constants, so we do not care about normalization of Lebesgue measures on V, V', V'')

construction of $\mathcal{L}$: take a chain contraction $K: V''^\bullet \to V''^{\bullet-1}$, extended by zero to V' to yield $K: V^\bullet \to V^{\bullet-1}$
satisfying $Kd + dK = P''$, $K^2 = 0$, $KP' = P'K = 0$

then define $\mathcal{L} = N^*[-1](\underset{\cap\ V''}{\mathrm{im}\,K}) \subset \mathcal{F}''$

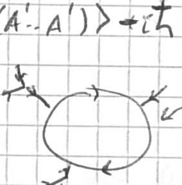
<u>Theorem</u>: the eff. action S' evaluates to $S' = \sum\frac{1}{n!}\langle B', l'_n(A',\ldots,A')\rangle + i\hbar\sum\frac{1}{n!}q'_n(A',\ldots,A')$,

with $l'_n = \sum_{\substack{\text{binary rooted}\\ \text{trees } T}}\frac{1}{|\mathrm{Aut}\,T|}$



put inputs in V' on leaves; $[,]$; $-K[,]$; $-K$; proj to V'

$q'_n = \sum_{\substack{\text{1-loop graphs } \Gamma\\ \text{3-valent with leaves}}}\frac{1}{|\mathrm{Aut}\,\Gamma|}$



cut the loop and calculate $\mathrm{Str}_{V'}$ of the resulting endomorphism

$+\sum_{\substack{\text{binary rooted}\\ \text{trees}}}\frac{1}{|\mathrm{Aut}\,T|}$

in particular, $S' = \langle B', dA'\rangle + \frac{1}{2}\langle B', \underline{[A',A']}\rangle - \frac{1}{2}\langle B', [A', K[A',A']]\rangle + \ldots$
$[,]$ in $\underline{V}$

$-i\hbar\left(-\mathrm{Str}\,K[A',\cdot] + \frac{1}{2}\mathrm{Str}\,K[A', K[A',\cdot]] + \frac{1}{2}\mathrm{Str}\,K[[A',A'],\cdot] + \ldots\right)$

<u>Rem</u> if we start with S corresp to a uLoo alg. $(V, \{l_n\}, \{q_n\})$
similar formula holds for S' – which corresponds to a uLoo structure $(V', \{l'_n\}, \{q'_n\})$; we allow higher valences in graphs & graphs like

So we have

BV 15/3

u. dgLa $(V, d, [\,,\,])$ $\longleftrightarrow$ abstr. BF theory on $\mathcal{F} = V[1] \oplus V^*[-2]$

$\Big\downarrow$ homotopy transfer of algebraic structure to a subcomplex $\qquad\qquad \Big\downarrow$ eff BV action

u L_∞ alg. $(V', \{l_1'\}, \{q_n'\})$ $\longleftrightarrow$ BF_∞ theory on $\mathcal{F}' = V'[1] \oplus V'^*[-2]$

- BF_∞ theories / uL_∞ algebras are stable under BV pushforwards / homotopy transfer.

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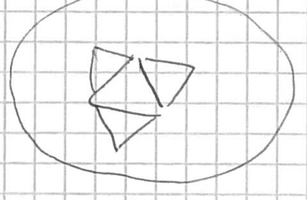
Geometric situation

$V^\bullet = \Omega^\bullet(M) \otimes \mathfrak{g}$, d, $[\,,\,]$
u. Lie alg.



~~We~~ fix a triangulation T of M,
set $V' = C^\bullet(T) \otimes \mathfrak{g}$ - cell cochains of T with coeff in $\mathfrak{g}$.

$\sum_{\sigma \in T} A^\sigma_{\mathfrak{g}}\, e_\sigma$, $A^\sigma_{\mathfrak{g}} \in \mathfrak{g}$ - coefficients
simplices, e_σ basis cochain assoc to σ

superfield $A_T = \sum_\sigma A^\sigma_{\mathfrak{g}}\, e_\sigma$, $\{A^\sigma \in \mathfrak{g}$, $|A_\sigma| = 1 - \dim \sigma$
$B_T = \sum_\sigma B_\sigma e^\sigma$, $B_\sigma \in \mathfrak{g}^*$, $|B_\sigma| = -2 + \dim \sigma\}$
basis chain

$\mathcal{F}' = C^\bullet(T) \otimes \mathfrak{g}[1] \oplus C_\bullet(T) \otimes \mathfrak{g}^*[-2]$
cochains on T ; chains on T

we realise cochains by Whitney elementary forms $\quad e_\sigma \longmapsto \dim\sigma!\, \sum_i (-1)^i t_i\, dt_0 \wedge \dots \widehat{dt_i} \dots \wedge dt_{\dim\sigma}$ $\in \Omega^{\dim\sigma}(\sigma)$

This extends to a chain inclusion $C^\bullet(T) \hookrightarrow \Omega^\bullet(M)$. $\qquad (t_0, \dots, t_{\dim\sigma})$ - barycentric coordinates on $\sigma \subset M$,
Projection $\Omega^\bullet(M) \longrightarrow C^\bullet(T)$ $\qquad \sum_0^{\dim} t_i = 1$ - constraint.
is the integration over ~~faces~~: $\alpha \longmapsto \sum_\sigma e_\sigma \int_\sigma \alpha$.

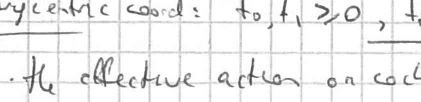
This defines a splitting $\Omega^\bullet(M) = \Omega^{\bullet\prime}(M,T) \oplus \Omega^{\bullet\prime\prime}(M,T)$ $\rightsquigarrow$ $V = V' \oplus V''$ (tensoring by $\mathfrak{g}$)
Whitney forms $\cong C^\bullet(T)$; forms with vanishing integrals over simplices ; $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$.

chain homotopy ~~is ...~~ is assembled from a standard chain homotopy operator $K_{\Delta^n} : \Omega^\bullet_{poly}(\Delta^n) \longrightarrow \Omega^{\bullet-1}_{poly}(\Delta^n)$ due to J. Dupont
stand. simplex

Example for $\Delta^1 \cong [0,1]$:
Whitney forms are: $\chi_0 = 1 - t$, $\chi_1 = t$, $\chi_{01} = dt$, $\quad K : f(t) + g(t)\, dt \mapsto \int_0^t g(\tau)\, d\tau - t \int_0^1 g(\tau)\, d\tau$
$t = t_1$ (barycentric coord: $t_0, t_1 \geq 0$, $t_0 + t_1 = 1$).

———————————————

With this splitting, we can calculate S. the effective action on cochains $S' = S_T(A_T, B_T)$
as given by a sum of Feynman graphs



, ... it splits into local contributions of simplices:

$S_T(A_T, B_T) = \sum_{\sigma \in T} \overline{S}_\sigma(A_\sigma, B_\sigma)$
universal local building block, depends just on $\dim \sigma$; can be calculated for stand simplices of ~~different~~ dimensions $\geq$

Examples

- $\bar{S}_{\Delta^0=[0]} = \frac{1}{2}\langle B_0, [A_0, A_0]\rangle$

  Δ^0: a point; $\mathfrak{g}^*[-2]$

Coeffs: $\frac{B_{2k}}{(2k)!}$ – Bernoulli numbers

- $\bar{S}_{\Delta^1=[0,1]} = \sum \langle B_{01}, A_1 - A_0 + [A_{01}, \frac{A_0+A_1}{2}] + \frac{1}{12}[A_{01},[A_{01}, A_1 - A_0]] - \frac{1}{720}[A_{01},[A_{01},[A_{01},[A_{01}, A_1 - A_0]]]] + \ldots$

  $-i\hbar\left(\frac{1}{24} tr_{\mathfrak{g}}\, ad^2_{A_{01}} - \frac{1}{4\cdot 720} tr_{\mathfrak{g}}\, ad^4_{A_{01}} + \ldots\right) =$

  $\frac{B_{2k}}{2k\cdot(2k)!}$

  $= \langle B_{01}, [A_{01}, \frac{A_0+A_1}{2}] + \frac{ad_{A_{01}}}{2}\cot\frac{ad_{A_{01}}}{2}\rangle -$

  $-i\hbar\, \underbrace{tr_{\mathfrak{g}} \log}_{=\log\det_{\mathfrak{g}}} \frac{\sinh\frac{ad_{A_{01}}}{2}}{\frac{ad_{A_{01}}}{2}}$

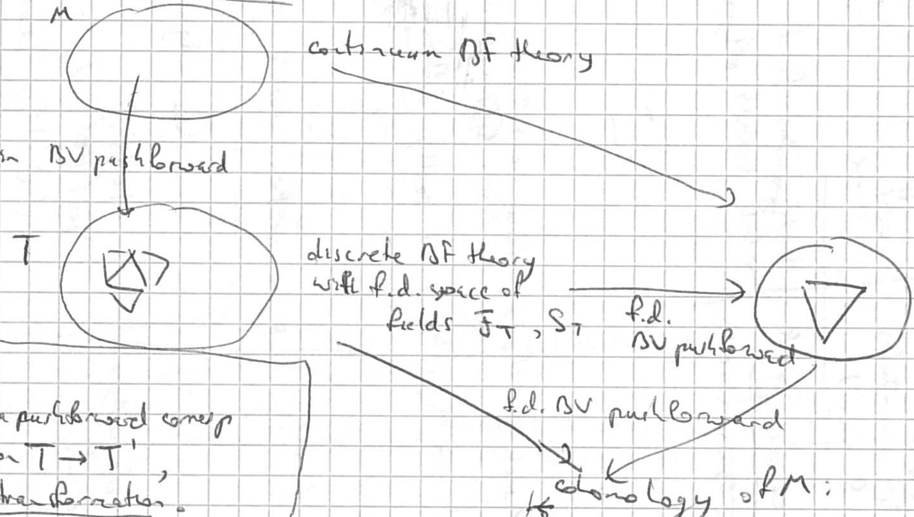
  – corresp to an explicit uL_∞ algebra on cochains of an interval.

- $\bar{S}_{\Delta^m} = \sum_{\substack{\sigma_1 - \text{faces}\\ \text{of codim } 1}} \pm\langle B_{\Delta^m}, A_{\sigma_1}\rangle + \frac{1}{2}\sum_{\substack{\sigma_1,\sigma_2 - \text{faces}\\ \text{s.t. } \sigma_1\cup\sigma_2 = \Delta^m\\ \sigma_1\cap\sigma_2 = pt}} \pm \langle B_{\Delta^m}, [A_{\sigma_1}, A_{\sigma_2}]\rangle\cdot\frac{|\sigma_1|!|\sigma_2|!}{(m+1)!} + \frac{1}{2}\sum \pm\langle B_{\Delta^m}, [A_{\sigma_1},[A_{\sigma_2}, A_{\sigma_3}]]\rangle\cdot\frac{|\sigma_1|!|\sigma_2|\ldots}{(|\sigma_2|+|\sigma_3|+1)\ldots(m+\ldots)} + \ldots$

  $-i\hbar\left(\frac{1}{2}\frac{\pm 1}{(m+1)^2(m+2)}\sum_{\substack{\sigma_1\subset\Delta^m\\ \text{of dim } 1}} tr\,(ad_{A_{\sigma_1}})^2 + \ldots\right)$

  – perturbative result.

So, we have



T' – aggregation of T (i.e. T is a subdivision of T')

$\mathcal{F}_{T'}, S_{T'}$

$\mathcal{F}_{H^\cdot} = H^\cdot(M,\mathfrak{g})[1] \oplus H_\cdot(M,\mathfrak{g}^*)[-2]$

<u>Thm</u>: action S_T induces $S_{T'}$ by a pushforward corresp to an aggregation $T\to T'$, <u>modulo a can. transformation</u>.

On the level of cohomology, we obtain an uL_∞ structure on $H^\cdot(M,\mathfrak{g})$ which encodes the <u>Massey operations</u> on $H^\cdot(M)$ (which in turn know the <u>rational homotopy type</u> of M)

Resulting invariant of M is combinatorially computable (if $\bar{S}_{\Delta^m}$ are known) and is <u>stronger</u> than just the RHT.

Example: S^1 and Klein Bottle: $H^\cdot(M,\mathbb{R})$ are isomorphic as rings and C_∞-algebras, but not as uL_∞

+ behavior of the <u>Reidemeister torsion</u> of M in the neighborhood of 0 on the moduli space of flat local systems (Milnor's torsion)

circle S^1

$S_{H^\cdot} = \langle B_1, [A_0, A_1]\rangle - i\hbar\, tr_{\mathfrak{g}} \log \frac{\sinh\frac{ad_{A_1}}{2}}{\frac{ad_{A_1}}{2}}$

↑ corresp to generator of H^0 and H^1

Klein bottle

$S_{H^\cdot} = \langle B_1, [A_0, A_1]\rangle - i\hbar\, tr_{\mathfrak{g}} \log\left(\frac{ad_{A_1}}{2}\coth\frac{ad_{A_1}}{2}\right)^{-1}$

distinguished by quantum operations on cohomology.