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BATALIN-VILKOVISKY FORMALISM AND APPLICATIONS IN TOPOLOGICAL QUANTUM FIELD THEORY

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Abstract. Lecture notes for the Fall 2016 topics course in topology, University of Notre Dame.

Contents

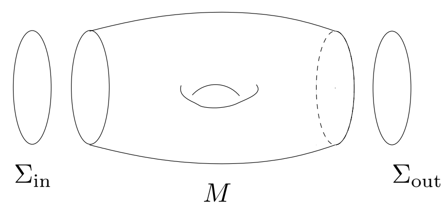
Lecture 1, 08/24/2016.

1. Introduction/motivation

Idea of locality (in the interpretation of Atiyah-Segal): a quantum field theory (QFT) assigns some values ("partition functions") to manifolds. It can be evaluated on manifolds and satisfies a gluing/cutting property. So, a manifold can be chopped into simple (small) pieces, then the QFT can be evaluated on those pieces and then assembled to the value of the QFT on the entire manifold.

1.1. **Atiyah's axioms of topological quantum field theory.** An n-dimensional topological quantum field theory (TQFT) is the following set of data.

- To a **closed** $(n-1)$**-dimensional manifold** Σ, the TQFT assiociates a vector space $\mathcal{H}_\Sigma$ over $\mathbb{C}$ – the "space of states".
- To an n**-manifold** M **with boundary** split into in- and out-parts, $\partial M = \bar\Sigma_{\mathrm{in}} \sqcup \Sigma_{\mathrm{out}}$ (bar refers to reversing the orientation on the in-boundary), the TQFT associates a $\mathbb{C}$-linear map $Z_M:\ \mathcal{H}_{\Sigma_{\mathrm{in}}} \to \mathcal{H}_{\Sigma_{\mathrm{out}}}$ – the "partition function".[1]



[1]Another possible name for Z_M is the "evolution operator".

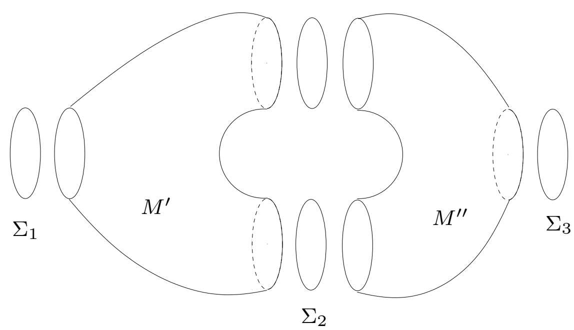
We call such M a *cobordism* between Σ_{in} and Σ_{out}, and we denote

$$\Sigma_{\text{in}} \xrightarrow{M} \Sigma_{\text{out}}$$

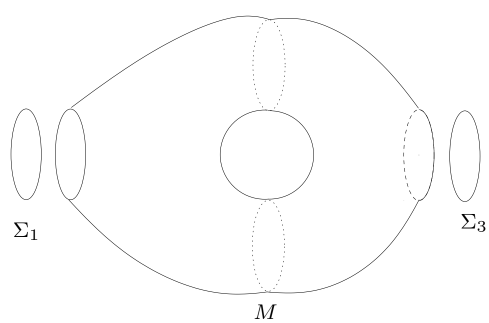
- **Diffeomorphisms of closed** $(n-1)$**-manifolds** act on spaces of states: to $\phi : \Sigma \to \Sigma'$ a diffeomorphism, the TQFT associates an isomorphism $\rho(\phi) : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$ (in the way compatible with composition of diffeomorphisms). For ϕ orientation-preserving, $\rho(\phi)$ is $\mathbb{C}$-linear; for ϕ orientation-reversing, $\rho(\phi)$ is $\mathbb{C}$-anti-linear.

This set of data should satisfy the following axioms:

- **Multiplicativity**: disjoint unions are mapped to tensor products. Explicitly,
$$\mathcal{H}_{\Sigma\sqcup\Sigma'} = \mathcal{H}_\Sigma \otimes \mathcal{H}_{\Sigma'}, \qquad Z_{M\sqcup M'} = Z_M \otimes Z_{M'}$$
- **Gluing**: given two cobordisms $\Sigma_1 \xrightarrow{M'} \Sigma_2$ and $\Sigma_2 \xrightarrow{M''} \Sigma_3$, with out-boundary of the first one coinciding with the in-boundary of the second one,



we can *glue* (or "concatenate") them over Σ_2 to a new cobordism $M := M' \cup_{\Sigma_2} M''$, going as $\Sigma_1 \xrightarrow{M} \Sigma_3$.



Then the partition function for M is the *composition* of partition functions for M' and M'' as linear maps:
$$\boxed{Z_M = Z_{M''} \circ Z_{M'}} : \quad \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_3}$$
- **Normalization**:
  - For $\varnothing$ the empty $(n-1)$-manifold,
  $$\mathcal{H}_\varnothing = \mathbb{C}$$
  - For Σ a closed $(n-1)$-manifold, the partion function for the *cylinder* $\Sigma \xrightarrow{\Sigma\times[0,1]} \Sigma$ is the identity on $\mathcal{H}_\Sigma$.

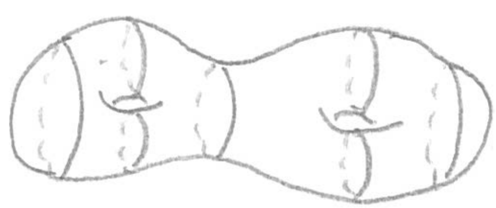
- For $\phi : M \to M'$ a diffeomorphism between two cobordisms, denote $\phi|_{\mathrm{in}}$, $\phi|_{\mathrm{out}}$ the restrictions of ϕ to the in- and out-boundary. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_{\Sigma_{\mathrm{in}}} & \xrightarrow{Z_M} & \mathcal{H}_{\Sigma_{\mathrm{out}}} \\ {\scriptstyle \rho(\phi|_{\mathrm{in}})}\downarrow & & \downarrow{\scriptstyle \rho(\phi|_{\mathrm{out}})} \\ \mathcal{H}_{\Sigma'_{\mathrm{in}}} & \xrightarrow[Z_{M'}]{} & \mathcal{H}_{\Sigma'_{\mathrm{out}}} \end{array}$$

Remark 1.1. Atiyah's TQFT is a functor of symmetric monoidal categories, $\mathrm{Cob}_n \to \mathrm{Vect}_{\mathbb{C}}$, where the structure is as follows:

	Cob_n	$\mathrm{Vect}_{\mathbb{C}}$
objects	closed $(n-1)$-manifolds	vector spaces/$\mathbb{C}$
morphisms	cobordisms $\Sigma_{\mathrm{in}} \xrightarrow{M} \Sigma_{\mathrm{out}}$	linear maps
composition	gluing	composition of maps
identity morphism	cylinder $\Sigma \xrightarrow{\Sigma\times[0,1]} \Sigma$	identity map $\mathrm{id} : V \to V$
monoidal product	disjoint union $\sqcup$	tensor product $\otimes$
monoidal unit	$\varnothing$	$\mathbb{C}$

Remark 1.2. A *closed* n-manifold M can be viewed as a cobordism from $\varnothing$ to $\varnothing$, thus $Z_M : \mathbb{C} \to \mathbb{C}$ is a multiplication by some number $z \in \mathbb{C}$. By abuse of notations, we denote $Z_M := z \in \mathbb{C}$. Thus, with this convention, the partition function for a closed n-manifold is a complex number, invariant under diffeomorphisms and compatible with gluing-cutting. E.g., for $n = 2$, we can cut any closed surface into disks and pairs of pants



Thus, Z for any surface can be calculated from the gluing axiom, provided that Z is known for a disk and for a pair of pants.

Remark 1.3. In Segal's approach to (not necessarily topological) quantum field theory, one allows manifolds to carry a local geometric structure (of the type depending on the particular QFT): Riemannian metric, conformal structure, complex structure, framing, local system,... Atiyah's axioms above have to be modified slightly to accommodate for the geometric structure.

Example 1.4 (Quantum mechanics)**.** Consider the 1-dimensional Segal's QFT with geometric structure the Riemannian metric on 1-cobordisms. Objects are points with $+$ orientation, assigned a vector space $\mathcal{H}$ and points with $-$ orientation, assigned the dual space $\mathcal{H}^*$. Consider an interval of length $t > 0$ (our partition functions depend on a metric on the interval considered modulo diffeomorphisms, thus only on the length), $I_t = [0, t]$. Denote $Z(t) := Z_{I_t} \in \mathrm{End}(\mathcal{H})$. By the gluing axiom (from considering the gluing $[0, t_1] \cup_{\{t_1\}} [t_1, t_1 + t_2] = [0, t_1 + t_2]$), we have the semi-group law $Z(t_1+t_2)=Z(t_2)\circ Z(t_1)$. It implies in turn that

$$Z(t)=Z(\frac{t}{N})^N \tag{1}$$

for N an arbitrarily large integer. Assume that for τ small, we have $Z(\tau)=\mathrm{id}-\frac{i}{\hbar}\hat{H}\cdot\tau+O(\tau^2)$, for $\hat{H}\in\mathrm{End}(\mathcal{H})$ some operator. Then (1) implies that

$$\boxed{Z(t)=\exp\left(-\frac{i}{\hbar}\hat{H}t\right)}$$

This system is the quantum mechanics, with $Z(t)$ the *evolution operator* in time t and $\hat{H}$ the Schrödinger operator (or *quantum Hamiltonian*), describing the infinitesimal evolution of the system.

E.g. the choice $\mathcal{H}=L^2(X)$ for X a Riemannian manifold and $\hat{H}=-\frac{\hbar^2}{2m}\Delta_X+U(x)\cdot$ would correspond to the quantum particle of mass m moving on the manifold X in the force field with potential U. In this case $Z(t):\psi(x)\mapsto\int_{X\ni y}dy\;Z(t;x,y)\psi(y)$ is the integral operator whose integral kernel $Z(t;x,y)$ is intepreted as the propagation amplitude of the particle from position y to position x in time t.

1.2. The idea of path integral construction of quantum field theory.

1.2.1. *Classical field theory data.* We start by fixing the data of *classical field theory* on an n-manifold:

- A space of fields $F_M=\Gamma(M,\mathbb{F}_M)$ – a space of sections of some sheaf $\mathbb{F}_M$ over M. Typical examples of F_M are:
  - $C^\infty(M)$
  - Space of connections on a principal G-bundle $\mathcal{P}$ over M. (This exaple is typical for some of *gauge theories* e.g. Chern-Simons theory, Yang-Mills theory,. . .)
  - Mapping space $\mathrm{Map}(M,N)$ with N some fixed target manifold. This is typical for so-called *sigma models*.
- The *action functional* $S_M:\;F_M\to\mathbb{R}$ of form

$$S_M(\phi)=\int_M L(\phi,\partial\phi,\partial^2\phi,\ldots)$$

  where L is *the Lagrangian density* – a density on M depending on the value of the field $\phi\in F_M$ and its derivatives (up to fixed finite order) at the point of integration on M. Variational problem of extremization of S (i.e. the critical point equation $\delta S=0$) leads to Euler-Lagrange PDE on ϕ.

Example 1.5 (Free massive scalar field)**.** Let (M,g) be a Riemannian manifold, we set $F_M=C^\infty(M)\ni\phi$ with the action

$$S_M(\phi)=\int_M\left(\frac{1}{2}\langle d\phi,d\phi\rangle_{g^{-1}}+\frac{m^2}{2}\phi^2\right)d\mathrm{vol}$$

Here $m\geq 0$ is a parameter of the theory – the *mass*; $d$vol is the Riemannian volume element on M. The associated Euler-Lagrange equation on ϕ is: $(\Delta+m^2)\phi=0$.

1.2.2. *Idea of path integral quantization.* The idea of quantization is then to construct the partition function for M a closed manifold as

$$Z_M(\hbar) := \text{“} \int_{F_M} \mathcal{D}\phi\, e^{\frac{i}{\hbar} S_M(\phi)} \text{”} \tag{2}$$

Here $\hbar$ is a parameter of the quantization (morally, $\hbar$ measures the "distance to classical theory"); $\mathcal{D}\phi$ is a symbol for a reference measure on the space F_M. Integral (2) is problematic to define directly as a measure-theoretic integral, however it can be defined as an asymptotic series in $\hbar \to 0$, as we will discuss in a moment. So far, r.h.s. of (2) is a heuristic expression which is to be made mathematical sense of.

Consider M with boundary Σ. Denote B_Σ the set of boundary values of fields on M; we have a map of evaluation of the field at the boundary (or pullback by the inclusion $\Sigma \hookrightarrow M$) $F_M \to B_\Sigma$ sending $\phi \mapsto \phi|_\partial$. For the space of states on Σ, we set $\mathcal{H}_\Sigma := \mathrm{Fun}_\mathbb{C}(B_\Sigma)$ – complex-valued functions on B_Σ. For the partition function Z_M, we set

$$Z_M(\phi_\Sigma; \hbar) := \int_{\phi \in F_M \text{ s.t. } \phi|_\partial = \phi_\Sigma} \mathcal{D}\phi\, e^{\frac{i}{\hbar} S_M(\phi)} \tag{3}$$

This path integral gives us a function on $B_\Sigma \ni \phi_\Sigma$ and thus a vector in $Z_M(-;\hbar) \in \mathcal{H}_\Sigma$.

1.2.3. *Heuristic argument for gluing.* Let a closed (for simplicity) n-manifold M be cut by a codimension 1 submanifold Σ into two M' and M'', i.e. $M = M' \cup_\Sigma M''$. Then the integral (2) can be performed in steps:

(i) Fix ϕ_Σ on Σ.
(ii) Integrate over fields on M' with boundary condition ϕ_Σ on Σ.
(iii) Integrate over fields on M'' with boundary condition ϕ_Σ on Σ.
(iv) Integrate over $\phi_\Sigma \in B_\Sigma$.

This yields

$$Z_M = \int_{B_\Sigma \ni \phi_\Sigma} \mathcal{D}\phi_\Sigma\, Z_{M'}(\phi_\Sigma) \cdot Z_{M''}(\phi_\Sigma)$$

One can recognize in this formula the Atiyah-Segal gluing axiom: M' and M'' yield two vectors in $\mathcal{H}_\Sigma$ which are paired in $\mathcal{H}_\Sigma$ to a number – the partition function for the whole manifold.

1.2.4. *How to define path integrals?* Let us first look at finite-dimensional *oscillating* integrals: consider X a compact manifold with μ a fixed volume form and $f \in C^\infty(X)$ a function. The asymptotics, as $\hbar \to 0$, of the integral

$$\int_X \mu\, e^{\frac{i}{\hbar} f(x)}$$

is given by the *stationary phase formula*[2]

$$\int_X \mu\, e^{\frac{i}{\hbar} f(x)} \underset{\hbar \to 0}{\sim} \sum_{x_0 \in \{\text{crit. points of } f\}} e^{\frac{i}{\hbar} f(x_0)} |\det f''(x_0)|^{-\frac{1}{2}} e^{\frac{\pi i}{4} \mathrm{sign} f''(x_0)} (2\pi\hbar)^{\frac{\dim X}{2}}$$

The rough idea here is that the rapid oscillations of the integrand cancel out except in the neighborhood of critical points x_0 of f (i.e. points with $df(x_0) = 0$), which are

[2]See e.g. [14, 27]

the "stationary phase points" for the integrand – points around which oscillations slow down.

This formula can be improved to accommodate corrections in powers of $\hbar$:

(4)
$$\int_X \mu\, e^{\frac{i}{\hbar}f(x)} \underset{\hbar\to 0}{\sim} \sum_{x_0\in\{\text{crit. points of } f\}} e^{\frac{i}{\hbar}f(x_0)} |\det f''(x_0)|^{-\frac{1}{2}} e^{\frac{\pi i}{4}\operatorname{sign} f''(x_0)} (2\pi\hbar)^{\frac{\dim X}{2}} \cdot \\ \cdot \sum_\Gamma \hbar^{-\chi(\Gamma)} \Phi_\Gamma$$

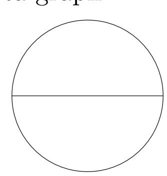
where Γ ranges over graphs with vertices of valence ≥ 3 (possibly disconnected, including $\Gamma = \varnothing$); $\chi(\Gamma) \leq 0$ is the Euler characteristic of the graph. Graphs Γ are called the **Feynman diagrams**. Assume that Γ has E edges and V vertices. We decorate all half-edges of Γ with labels $i_1, \ldots, i_{2E}$ each of which can take values $1, 2, \ldots, p := \dim X$. The weight of the graph Γ, Φ_Γ, is defined as follows.

- We assign to every edge e consisting of half-edges h_1, h_2 the decoration $f''(x_0)^{-1}_{i_{h_1} i_{h_2}}$ – the matrix element of the inverse Hessian given by the labels of the constituent half-edges.
- We assign to every vertex v of valence k with adjacent half-edges $h_1, \ldots, h_k$ the decoration $\partial_{i_{h_1}} \cdots \partial_{i_{h_k}} f(x_0)$ – a k-th partial derivative of f at the critical point.
- We take the product of all the decorations above and sum over all possible values of labels on the half-edges. Φ_Γ is this sum times the factor $\frac{i^{E+V)}}{|\mathrm{Aut}(\Gamma)|}$ with $\mathrm{Aut}(\Gamma)$ the automorphism group of the graph.

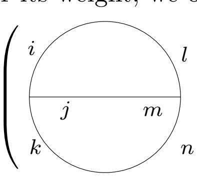
I.e., we have

$$\Phi_\Gamma := \frac{i^{E+V}}{|\mathrm{Aut}(\Gamma)|} \cdot \sum_{i_1,\ldots,i_{2E}\in\{1,\ldots,p\}} \prod_{\text{edges } e=(h_1 h_2)} f''(x_0)^{-1}_{i_{h_1} i_{h_2}} \cdot \prod_{\text{vertices } v} \partial_{i_{h_1}} \cdots \partial_{i_{h_{\mathrm{val}(v)}}} f(x_0)$$

Example 1.6. Consider the "theta graph"



(Note that its Euler characterestic is -1, hence it enters in (4) in the order $\hbar^1$.) For its weight, we obtain

$$\Phi\left(\vphantom{\int}\right.$$


$$\left.\vphantom{\int}\right) = \frac{i^{3+2}}{12} \cdot \sum_{i,j,k,l,m,n\in\{1,\ldots,p\}} f''(x_0)^{-1}_{il} f''(x_0)^{-1}_{jm} f''(x_0)^{-1}_{kn} f'''(x_0)_{ijk} f'''(x_0)_{lmn}$$

Stationary phase formula (4) replaces, in the asymptotics $\hbar \to 0$, a measure-theoretic integral on the l.h.s. with the purely algebraic expression on the r.h.s., involving only values of derivatives of f at the critical points x_0.

The idea then is to define the path integral (2) by formally applying the stationary phase formula, as the r.h.s. of (4), i.e. as a series in $\hbar$ with coefficients given by weights of Feynman diagrams.

We expect that if we started with a classical field theory with S_M invariant under diffeomorphisms of M, the partition functions Z_M coming out of the path integral quantization procedure yield manifold invariants and arrange into a TQFT.

Problem: Stationary phase formula requires critical points of f to be *isolated* (more precisely, we need the Hessian of f at critical points to be non-degenerate). However, diffeomorphism invariant classical field theories are *gauge theories*, i.e. there is a tangential distribution $\mathcal{E}$ on F_M which preserves the action S_M (in some examples, $\mathcal{E}$ corresponds to an action of a group $\mathcal{G}$ – the *gauge group* – on F_M). Thus, critical points of S_M come in $\mathcal{E}$-orbits and therefore are not isolated. Put another way, the Hessian of S_M is degenerate in the direction of $\mathcal{E}$. So, the stationary phase formula cannot be applied to the path integral (2) in the case of a gauge theory.

The cure for this problem comes from using the Batalin-Vilkovisky construction.

1.2.5. *Towards Batalin-Vilkovisky (BV) formalism.* Batalin-Vilkovisky construction replaces the classical field theory package F, S with a new package consisting of:

- A $\mathbb{Z}$-graded supermanifold $\mathcal{F}$ (the "space of BV fields") endowed with odd-symplectic structure ω of internal degree -1.
- A function S_{BV} on $\mathcal{F}$ – the "master action", satisfying the "master equation"
$$\{S_{BV}, S_{BV}\} = 0$$
In particular, this implies that the corresponding Hamiltonian vector field $Q = \{S_{BV}, \bullet\}$ is *cohomological*, i.e. satisfies $Q^2 = 0$. Thus, Q endows $C^\infty(\mathcal{F})$ with the structure of a cochain complex. In other words, $(\mathcal{F}, Q)$ is a *differential graded (dg) manifold*.

The idea is then to replace

$$\int_F e^{\frac{i}{\hbar}S} \to \int_{\mathcal{L}\subset\mathcal{F}} e^{\frac{i}{\hbar}S_{BV}} \tag{5}$$

with $\mathcal{L} \subset \mathcal{F}$ a Lagrangian submanifold w.r.t. the odd-symplectic structure ω.

The integral on the l.h.s. of (5) is ill-defined (by means of stationary phase formula) in the case of a gauge theory whereas the integral on the r.h.s. is well-defined, for a good choice of Lagrangian submanifold $\mathcal{L} \subset \mathcal{F}$ and moreover is invariant under deformations of $\mathcal{L}$.

Remark 1.7. [3] Space $\mathcal{F}$ is constructed, roughly speaking, as Spec of a two-sided resolution of $C^\infty(F)$ construted out of

- Chevalley-Eilenberg resolution for the subspace of gauge-invariant functions of fields $C^\infty(F)^{\mathcal{G}}$ and
- Koszul-Tate resolution for functions on the space of solutions of Euler-Lagrange equations $C^\infty(EL \subset F)$.

So, coordinates on $\mathcal{F}$ of nonzero degree arise as either Chevalley-Eilenberg generators (in positive degree) or Koszul-Tate generators (in negative degree). In particular, this is the reason why $\mathcal{F}$ has to be a supermanifold (since C-E and K-T generators anti-commute).

[3]See [31].

Remark 1.8. In the case of a gauge field theory, one could try to remedy the problem of degenerate critical points in the path integral by passing to the integral over the quotient, $\int_F \to \int_{F/\mathcal{G}}$. The latter may indeed have nondegenerate critical points. But the issue is then that we know how to make sense of Feynman diagrams for the path integral over the space of sections of a sheaf over M, but the quotient $F/\mathcal{G}$ would not be of this type. In this sense, one may think of the r.h.s. of (5) as a resolution of the integral over a quotient $F/\mathcal{G}$ by an integral over a locally free object – the space of sections of a sheaf over M.

Remark 1.9. There are finite-dimensional cases when l.h.s. of (5) exists as a measure-theoretic integral (despite having non-isolated critical points). Then, under certain assumptions, one has a comparison theorem that l.h.s. and r.h.s. of (5) coincide. We will return to this when talking about Faddeev-Popov construction and how it embeds into BV.

Lecture 2, 08/29/2016.

1.3. **Tentative program of the course.**

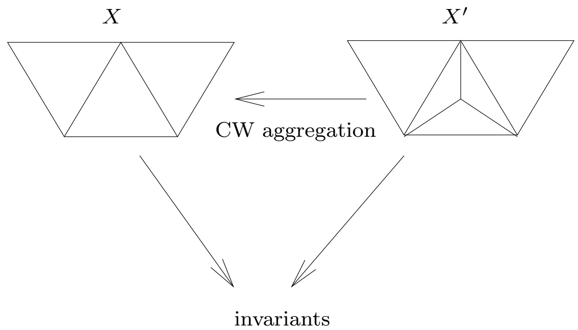
- Classical Chern-Simons theory.
- Feynman diagrams (in the context of finite-dimensional integrals):[4]
  - Stationary phase formula.
  - Wick's lemma for moments of a Gaussian integral. Perturbed Gaussian integral.
  - Berezin integral over an odd vector space.[5] Feynman diagrams for integrals over a super vector space.
- Introduction to BV formalism:
  - ($\mathbb{Z}$-graded) supergeometry: odd-symplectic geometry (after [29]), dg manifolds (partly after [1]), integration on supermanifolds.
  - BV Laplacian, classical and quantum master equation (CME and QME).
  - $\frac{1}{2}$-densities on odd-symplectic manifolds, BV integrals, fiber BV integral as a pushforward of solutions of quantum master equation ([25, 6]).
  - BV as a solution to the problem of gauge-fixing: Faddeev-Popov construction, BRST (as a homological algebra interpretation of Faddeev-Popov), BV (as a "doubling" of BRST).[6]
- AKSZ (Alexandrov-Kontsevich-Schwarz-Zaboronsky) construction [1].

Applications:

(I) A topological quantum field theory (not in Atiyah sense, but in the sense of compatibility with cellular subdivisions/aggregations) on CW complexes X

[4]References: [14, 27].
[5]Reference: [22].
[6]Reference: e.g. [25].

– cellular non-abelian BF theory [25, 12].



Here a CW complex X gets assigned a BV package – a space of fields comprised of cellular cochains and chains twisted by a G-local system E, $\mathcal{F}_X = C^\bullet(X,E) \oplus C_\bullet(X,E^*)$ (with certain homological degree shifts which we omitted here); G is a fixed Lie group – the structure group of the theory. $\mathcal{F}_X$ carries a natural odd-symplectic structure (coming from pairing chains with cochains). The action is given as a sum, over cells $e \subset X$ of all dimensions, of certain universal local building blocks $\bar{S}_e$ depending only on combinatorial type of the cell and on values of fields restricted to the cell.

One calculates certain invariant $\psi(X)$ of X by pushing forward the BV package to the (cellular) cohomology of X, via a *finite-dimensional* fiber BV integral. If X' is a cellular subdivision of X (then we say that X is an "aggregation" of X'), the pushforward of the BV package on X' to X yields back the package on X, and for the invariant one has $\psi(X') = \psi(X)$. More precisely, one gets a simple-homotopy invariant of CW complexes.

We will also discuss here:

- Solutions of the QME vs. infinity algebras (relevant case for this model: unimodular L_∞ algebras). Fiber BV integral as homotopy transfer of infinity algebras. Feynman diagrams from homological perturbation theory.
- Relation to rational homotopy type, to formal geometry (neighborhoods of singularities) of the moduli space $\mathcal{M}_{X,G}$ of local systems on X, to behavior of the R-torsion near the singularities of $\mathcal{M}_{X,G}$.

(II) Perturbative Chern-Simons theory (after Axelrod-Singer [2, 3]). Perturbative invariants of 3-manifolds M given in terms of integrals over Fulton-MacPherson-Axelrod-Singer compactifications of configuration spaces of n distinct points on M.

(III) Kontsevich's deformation quantization of Poisson manifolds (M,π) [20], partly following [5]. Here the problem is to costruct a family (parameterized by $\hbar$) of associative non-commutative deformations of the pointwise product on $C^\infty(M)$, of the form

$$f *_\hbar g(x) = f \cdot g(x) - \frac{i\hbar}{2}\{f,g\}_\pi + \sum_{n\geq 2} (i\hbar)^n B_n(f,g)(x) \tag{6}$$

where B_n are some bi-differential operators (of some order depending on n). The idea of the construction (following [5]) is to write the star-product as as path integral representing certain expectation value for a 2-dimensional topological field theory (the *Poisson sigma model*) on a disk D, with two observables placed on the boundary, at points 0 and 1:

$$f *_\hbar g(x_0) = \int_{X(\infty)=x_0,\eta_{\partial D}=0} \mathcal{D}X\,\mathcal{D}\eta \quad e^{\frac{i}{\hbar}S_{PSM}(X,\eta)} f(X(0)) \cdot g(X(1)) \tag{7}$$

Here the fields X, η are the base and fiber components of a bundle map

$$\begin{array}{ccc} TD & \xrightarrow{\eta} & T^*M \\ \downarrow & & \downarrow \\ D & \xrightarrow[X]{} & M \end{array}$$

and the action is: $S_{PSM} = \int_D \langle \eta, dX\rangle + \frac{1}{2}\langle X^*\pi, \eta\wedge\eta\rangle$. This action possesses a rather complicated gauge symmetry (given by a non-integrable distribution on the space of fields) and one needs BV to make sense of the integral (7). The final result is the explicit construction of operators B_n in (6) in terms of integrals over compactified configuration spaces of points on the 2-disk D.

(IV) BV formalism for field theories on manifolds with boundary, with Atiyah-Segal's gluing/cutting – "BV-BFV formalism" [7, 10] (a very short survey in [11]). Examples:
- Non-abelian BF theory on cobordisms endowed with CW decomposition [12].
- AKSZ theories on manifolds with boundary.

2. Classical Chern-Simons theory

2.1. **Chern-Simons theory on a closed 3-manifold.** Let, for simplicity, $G = SU(2)$ (we will comment on generalization to other Lie groups later) and let M be a closed oriented 3-manifold. Let $\mathcal{P}$ be the trivial G-bundle over M.

2.1.1. *Fields.* We define the space of fields to be the space of principal connections on $\mathcal{P}$. Since $\mathcal{P}$ is trivial, we can use the trivialization to identify connections with $\mathfrak{g}$-valued 1-forms on M (by pulling back the connection 1-form $\mathcal{A} \in \Omega^1(\mathcal{P},\mathfrak{g})$ on the total space of $\mathcal{P}$ to M by the trivializing section $\sigma : M \to \mathcal{P}$). Here $\mathfrak{g}$ is the Lie algebra of G, i.e. in our case $\mathfrak{g} = \mathfrak{su}(2)$. So, we have $F_M = \mathrm{Conn}_{M,G} \simeq \Omega^1(M,\mathfrak{g})$.

2.1.2. *Action.* We define the action functional on F_M as

$$S_{CS}(A) := \int_M \mathrm{tr}\, \frac{1}{2} A\wedge dA + \frac{1}{3} A\wedge A\wedge A$$

with $A \in \Omega^1(M,\mathfrak{g})$ a connection 1-form in fundamental representation of $\mathfrak{su}(2)$.

Remark 2.1. It can be instructive to rewrite the action as $\int_M \mathrm{tr}\, \frac{1}{2}A\wedge dA + \frac{1}{6}A\wedge [A,A]$ where $[,]$ is the (super-)Lie bracket on the differential graded Lie algebra of $\mathfrak{g}$-valued forms, $\Omega^\bullet(M,\mathfrak{g})$; here $[A,A]$ is simply $A\wedge A + A\wedge A$. But this rewriting exhibits denominators $1/2!$, $1/3!$ and suggests that there might be some "homotopy Chern-Simons" action associated to infinity algebras where higher terms would appear, which is indeed correct [6].

2.1.3. *Euler-Lagrange equation.* Let us calculate the variation of the action:

$$\delta S_{CS} = \int_M \mathrm{tr}\, \frac{1}{2}\delta A \wedge dA + \frac{1}{2} A \wedge d\delta A + \delta A \wedge A \wedge A = \int_M \mathrm{tr}\, \delta A \wedge (\underbrace{dA + A \wedge A}_{\text{curvature } F_A})$$

Here in the second equality we used integration by parts to remove d from δA. Note that the coefficient of δA in the final expression is the curvature 2-form of the connection A, $F_A = dA + A \wedge A = A + \frac{1}{2}[A, A] \in \Omega^2(M, \mathfrak{g})$. Thus, the Euler-Lagrange equation $\delta S_{CS} = 0$ (the critical point equation for S_{CS}) reads

$$\boxed{F_A = 0}$$

– flatness condition on the connection.

2.1.4. *Gauge symmetry.* For any group-valued map $g : M \to G$ and a connection $A \in \Omega^1(M, \mathfrak{g})$, we define the *gauge transformation* as mapping

$$A \quad \mapsto \quad \boxed{A^g := g^{-1}Ag + g^{-1}dg} \tag{8}$$

This defines a (right) action of the gauge group $\mathrm{Gauge}_{M,G} = \mathrm{Map}(M, G)$ on $F_M = \mathrm{Conn}_{M,G}$.

One can understand the transformation formula (8) as the effect of a change of trivialization of the principal bundle $\mathcal{P}$: assume that the connection 1-form on total space $\mathcal{A} \in \Omega^1(\mathcal{P}, \mathfrak{g})$ is fixed but we are given two different trivializations $\sigma, \sigma' : M \to \mathcal{P}$ with $\sigma' = \sigma \cdot g$. Then, the corresponding 1-forms on the base, $A_\sigma = \sigma^* \mathcal{A}$ and $A_{\sigma'} = (\sigma')^* \mathcal{A}$ are related by (8).

Alternatively, one can interpret (8) as the action of a bundle automorphism

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow[\simeq]{\cdot g} & \mathcal{P} \\ \downarrow & & \downarrow \\ M & = \!=\! = & M \end{array} \tag{9}$$

on a connection.

Note that A^g is flat iff A is flat.

Chern-Simons action changes under the gauge transformation (8) as

$$S_{CS}(A^g) - S_{CS}(A) = -\frac{1}{6} \int_M \mathrm{tr}\, (g^{-1}dg)^{\wedge 3}$$

where $(g^{-1}dg)^{\wedge 3} = (g^{-1}dg) \wedge (g^{-1}dg) \wedge (g^{-1}dg)$ is a 3-form on M with coefficients in matrices (endomorphisms of the space where $\mathfrak{g}$ is represented).

Recall that for $G \subset U(N)$ a simple compact group, one has the *Cartan 3-form*

$$\theta = -\frac{1}{24\pi^2} \mathrm{tr}\ (g^{-1}dg)^{\wedge 3} \in \Omega^3(G)$$

– a closed G-invariant form on G with integral periods representing the generator of $H^3(G, \mathbb{Z}) \simeq \mathbb{Z}$. In particular, for $G = SU(2)$, θ is the volume form on $SU(2)$ viewed as the 3-sphere, normalized to have total volume 1.

Therefore, (9) implies the following

Lemma 2.2 (Gauge (in)dependence of Chern-Simons action)**.**

$$\frac{1}{4\pi^2} \left(S_{CS}(A^g) - S_{CS}(A)\right) = \int_M g^* \theta = \langle [M], g^*[\theta] \rangle \quad \in \mathbb{Z}$$

Note that, for $G = SU(2)$, the r.h.s. is simply the degree of the map $g : M \to SU(2) \sim S^3$.

Thus, $S_{CS}(A)$ is invariant under infinitesimal gauge transformations; more precisely, it is invariant under $\mathrm{Gauge}^0_{M,G} \subset \mathrm{Gauge}_{M,G}$ – the connected component of trivial transformation $g = 1$ in $\mathrm{Gauge}_{M,G}$. However, under a general gauge transformation $S_{CS}(A)$ can change by an integer multiple of $4\pi^2$.

Lecture 3, 08/31/2016.

Introduce a function

$$\boxed{\psi_k(A) := e^{\frac{ik}{2\pi} S_{CS}(A)}} \tag{10}$$

with $k \in \mathbb{Z}$ a parameter – the "level" of Chern-Simons theory. By Lemma 2.2, ψ_k is a $\mathrm{Gauge}_{M,G}$-invariant function on $F_M = \mathrm{Conn}_{M,G}$. In particular, we can regard ψ_k as a function on the quotient $\mathrm{Conn}_{M,G}/\mathrm{Gauge}_{M,G}$.

2.1.5. *Chern-Simons invariant on the moduli space of flat connections.* Restriction of the function ψ_k to flat connections yields a *locally constant* function on the quotient

$$\mathcal{M}_{M,G} = \mathrm{FlatConn}_{M,G}/\mathrm{Gauge}_{M,G} = \frac{\{A \in \Omega^1(M, \mathfrak{g}) \text{ s.t. } dA + \frac{1}{2}[A, A] = 0\}}{A \sim g^{-1}Ag + g^{-1}dg \qquad \forall g : M \to G}$$

– the *moduli space of flat connections*. The locally constant property of ψ_k on the moduli space follows immediately from the fact that flat connections solve the Euler-Lagrange equation $\delta S_{CS} = 0$.

Recall that $\mathcal{M}_{M,G}$ can be identified[7] with $\mathrm{Hom}(\pi_1(M), G)/G$ – the space of group homomorphisms $\pi_1(M) \to G$, modulo action of G on such homomorphisms by conjugation on the target G.[8]

Moduli space $\mathcal{M}_{M,G}$ is typically disconnected and ψ_k can take different values on different connected components.

Example 2.3. Take $G = SU(2)$ and take M to be a *lens space*:

$$M = L(p, q) := \frac{\{(z_1, z_2) \in \mathbb{C}^2 \text{ s.t. } |z_1|^2 + |z_2|^2 = 1\}}{(z_1, z_2) \sim (\zeta \cdot z_1, \zeta^q \cdot z_2)} \sim S^3/\mathbb{Z}_p$$

where $\zeta = e^{\frac{2\pi i}{p}}$ the p-th root of unity; we assume that (p, q) are coprime (otherwise $L(p, q)$ is not a smooth manifold).

The moduli space $\mathcal{M}_{M,G}$ is the space of elements of order p in $SU(2)$ considered modulo conjugation. Thus, $\mathcal{M}_{M,G}$ consists of $\left[\frac{p+1}{2}\right]$ isolated points corresponding to classes of flat connections $[A]_0, \ldots, [A]_{\left[\frac{p-1}{2}\right]}$ where class $[A]_r$ has the holonomy around the loop γ, representing the generator of $\pi_1(M) = \mathbb{Z}_p$, of the form

$$\mathrm{hol}_\gamma[A]_r = \begin{pmatrix} e^{\frac{2\pi i r}{p}} & 0 \\ 0 & e^{-\frac{2\pi i r}{p}} \end{pmatrix} \in SU(2)$$

[7]The identification goes via mapping a flat connection A to a map associating to based loops γ on M the holonomy of A around γ. Flatness of A implies that this map on loops descends to homotopy classes of loops and implies the group homomorphism property of the map. Final quotient by G corresponds to quotienting out the changes of trivialization of the fiber of $\mathcal{P}$ over the base point.

[8]The identification $\mathcal{M}_{M,G} \simeq \mathrm{Hom}(\pi_1(M), G)/G$ is true for M of arbitrary dimension, if one allows flat connections in all – possibly non-trivial – G-bundles over M. Thus, $\mathcal{M}_{M,G}$ is in fact the moduli space of flat bundles, rather than just flat connections in a trivial bundle.

We consider consider r as defined mod p, and moreover r and $-r$ correspond to conjugate elements in $SU(2)$. Therefore choices $r \in \{0, 1, \ldots, \left[\frac{p-1}{2}\right]\}$ do indeed exhaust all distinct points of $\mathcal{M}_{M,G}$.

The value of the function ψ_k (10) on the point $[A]_r \in \mathcal{M}_{M,G}$ is:

$$\psi_k([A]_r) = e^{\frac{2\pi i k q^* r^2}{p}}$$

(This is the result of a non-trivial calculation.) Here q^* is the residue mod p reciprocal to q, i.e. defined by $q^* q = 1 \bmod p$. In particular, the set of values of ψ_k on $\mathcal{M}_{M,G}$ distinguishes between non-homotopic lens spaces, e.g. distinguishes between $L(5, 1)$ and $L(5, 2)$.

2.1.6. *Remark: more general G.* We can allow G to be any connected, simply-connected, simple, compact Lie group (e.g. $G = SU(N)$) without having to change anything.

We can also allow G to be semi-simple, $G = G_1 \times \cdots \times G_n$ with G_n the simple factors – the corresponding Chern-Simons theory is effectively a collection of n mutually non-interacting Chern-Simons theories for groups $G_1, \ldots, G_n$. In this case we can introduce independent levels $k_1, \ldots, k_n \in \mathbb{Z}$ for different factors.

The assumption that $\pi_0(G)$ and $\pi_1(G)$ are trivial is crucial. By a result of W. Browder, 1961, $\pi_2(G)$ is trivial for any finite-dimensional Lie group (in fact, even for any finite-dimensional H-space). Thus, under our assumptions G is 2-connected and the classifying space BG is 3-connected. Therefore, for M of dimension ≤ 3, $[M, BG] = *$ – all classifying maps are homotopically trivial. Thus *a G-bundle $\mathcal{P}$ over M has to be trivial.* And then we can globally identify connections in $\mathcal{P}$ with $\mathfrak{g}$-valued 1-forms and can make sense of Chern-Simons action. However, if either $\pi_0(G)$ or $\pi_1(G)$ is nontrivial, then there can exist non-trivial G-bundles (and one has to allow connections in all possible G-bundles as valid fields for the theory, if one wants ultimately to construct a field theory compatible with gluing/cutting). In this case special techniques are needed to construct S_{CS} (e.g. by defining the action on patches where the bundle is trivialized and then gluing the patches while taking into account the corrections arising from the change of trivialization on overlaps). In particular, for $G = U(1)$, S_{CS} is constructed in terms of Deligne cohomology.

2.1.7. *Relation to the second Chern class.* We assume again that $G = SU(2)$ (or, more generally, any simply-connected subgroup of $U(N)$).

Fact: any closed oriented 3-manifold M is *null-cobordant*, i.e. there exists a 4-manifold N with boundary $\partial N = M$.

As before, let $\mathcal{P}$ be the trivial G-bundle over M and let $\tilde{\mathcal{P}}$ be the trivial G-bundle over N

Lemma 2.4. Let $A \in \Omega^1(M, \mathfrak{g})$ be a connection in $\mathcal{P}$ and $a \in \Omega^1(N, \mathfrak{g})$ its extension to a connection in $\tilde{\mathcal{P}}$ (i.e. the pullback by the inclusion of the boundary $\iota : M \hookrightarrow N$ is $a|_M := \iota^* a = A$). Then we have

$$S_{CS}(A) = \frac{1}{2} \int_N \mathrm{tr}\, F_a \wedge F_a \tag{11}$$

where $F_a = da + \frac{1}{2}[a, a] \in \Omega^2(N, \mathfrak{g})$ is the curvature of a.

Proof. Indeed, we have

$$d\, \mathrm{tr}\, (\frac{1}{2} a \wedge da + \frac{1}{3} a \wedge a \wedge a) = \mathrm{tr}\, (\frac{1}{2} da \wedge da + da \wedge a \wedge a) \tag{12}$$

and

$$\text{(13)}\quad \operatorname{tr}\frac{1}{2}F_a\wedge F_a = \operatorname{tr}\frac{1}{2}(da+a\wedge a)\wedge(da+a\wedge a) = \operatorname{tr}(\frac{1}{2}da\wedge da+da\wedge a\wedge a+a\wedge a\wedge a\wedge a)$$

Note that the last term on the r.h.s. vanishes under trace: $\operatorname{tr} a^{\wedge 4} = \operatorname{tr} a\wedge a^{\wedge 3} = -\operatorname{tr} a^{\wedge 3}\wedge a = -\operatorname{tr} a^{\wedge 4}$, hence $\operatorname{tr} a^{\wedge 4} = 0$. Thus, (12)=(13) and the statement follows by Stokes' theorem. □

Let N_+, N_- be two copies of N (with N_- carrying the opposite orientation). Let $\bar{N} = N_+ \cup_M N_-$ be the closed 4-manifold obtained by gluing N_+ and N_- along M.

Fix $g : M \to G$ and construct a (generally, non-trivial) G-bundle $\bar{\mathcal{P}}_g$ over $\bar{N}$ which is trivial over N_+ and N_- and has transition function g on the tubular neighborhood of $M \subset \bar{N}$.

Let A be some connection on M; let a_+ be its extension over N_+ and let a_- be an extension of the *gauge transformed* connection $A^g = g^{-1}Ag + g^{-1}dg$ over N_-. The pair (a_+, a_-) defines a connection $\bar{a}$ in $\bar{\mathcal{P}}_g$.

By Lemma 2.4, we have

(14)

$$\frac{1}{8\pi^2}\int_{\bar{N}} \operatorname{tr} F_{\bar{a}}\wedge F_{\bar{a}} = \frac{1}{8\pi^2}\int_{N_+\cup N_-} \operatorname{tr} F_{\bar{a}}\wedge F_{\bar{a}} = \frac{1}{8\pi^2}\left(\int_N \operatorname{tr} F_{a_+}\wedge F_{a_+} - \int_N \operatorname{tr} F_{a_-}\wedge F_{a_-}\right)$$
$$= \frac{1}{4\pi^2}(S_{CS}(A) - S_{CS}(A^g))$$

Input from Chern-Weil theory. Recall that for P a G-bundle over M (with M of arbitrary dimension and with G a subgroup of $U(N)$), for A an arbitrary connection in P, the closed 4-form

$$\text{(15)}\qquad \frac{1}{8\pi^2}\operatorname{tr} F_{\mathsf{A}}\wedge F_{\mathsf{A}} \in \Omega^4(\mathsf{M})_{\text{closed}}$$

represents the image of the second Chern class of P,[9] $c_2(\mathsf{P}) \in H^4(M,\mathbb{Z})$ in de Rham cohomology $H^4(M,\mathbb{R})$. In particular, 4-form (15) has integral periods independent of A.

We conclude that the gauge transformation property of Chern-Simons action can be expressed in terms of characteristic classes for G-bundles on 4-manifolds as follows.

Lemma 2.5.

$$\frac{1}{4\pi^2}(S_{CS}(A^g) - S_{CS}(A)) = \langle[\bar{N}], c_2(\bar{P}_g)\rangle \quad \in \mathbb{Z}$$

2.2. **Chern-Simons theory on manifolds with boundary.** Let now M be an oriented 3-manifold with boundary $\partial M = \Sigma$ - a closed surface, or several closed surfaces.

As in the case of M closed, fields are connections on M and the action is unchanged, $S_{CS}(A) = \int_M \operatorname{tr}\frac{1}{2}A\wedge dA + \frac{1}{6}A\wedge[A,A]$.

[9]More precisely, this is the second Chern class of the associated vector bundle $\mathsf{P}\times_G \mathbb{C}^N$.

2.2.1. *Phase space.* We define the *phase space* Φ_Σ associated to the boundary Σ as the space of pullbacks of fields (connections) on M to the boundary. Thus, $\Phi_\Sigma = \mathrm{Conn}_{\Sigma,G}$ – connections on Σ, and we have a natural projection from fields on M to the boundary phase space

$$F_M = \mathrm{Conn}_{M,G} \xrightarrow{\ \pi=\iota^*\ } \Phi_\Sigma = \mathrm{Conn}_{\Sigma,G} \tag{16}$$

– the pullback by the inclusion of the boundary $\iota : \Sigma \hookrightarrow M$.

2.2.2. *δS, Euler-Lagrange equations.* Let us calculate δS. Now we will interpret δ as the exterior derivative on the space of fields, i.e. $\delta S \in \Omega^1(F_M)$ is a 1-form on fields and one can contract it with a tangent vector $v \in T_A F_M \simeq \Omega^1(M,\mathfrak{g})$ to produce a number. This is a (marginally) different interpretation from δ as a variation in variational calculus; the computations are the same but sign conventions are affected as now we treat δ as an odd operator.

Note that now we have two de Rham differentials: d – the "geometric" de Rham operator on M (or Σ) and the "field" de Rham operator δ on F_M (resp. Φ_Σ).

The computation is as follows:

$$\begin{aligned} \delta S &= \int_M \mathrm{tr}\left(-\frac12 \delta A\wedge dA - \frac12 A\wedge d\delta A - \frac12 \delta A \wedge [A,A]\right) \\ &= \underbrace{-\int_M \mathrm{tr}\,\delta A\wedge F_A}_{\text{"bulk term"}} + \underbrace{\int_\Sigma \mathrm{tr}\,\frac12 A|_\Sigma \wedge \delta A|_\Sigma}_{\text{"boundary term"}} \end{aligned} \tag{17}$$

Here we used Stokes' theorem to remove d from δA, and, unlike in the computation for M closed, a boundary term appeared as a result.

Euler-Lagrange equation read off from the first term in the r.h.s. of (17) – the equation that $\langle \delta S, v\rangle = 0$ for a field variation $v \in \Omega(M,\mathfrak{g})$ *supported away from the boundary* – is

$$F_A = 0 \tag{18}$$

– the flatness equation, as for M closed.

2.2.3. *Noether 1-form, symplectic structure on the phase space.* We interpret the boundary term in the r.h.s. of (17) as $\pi^*\alpha_\Sigma$ – the pullback by the projection (16) of the *Noether 1-form* on the phase space $\alpha_\Sigma \in \Omega^1(\Phi_\Sigma)$ defined as

$$\alpha_\Sigma = \int_\Sigma \mathrm{tr}\,\frac12 A_\Sigma \wedge \delta A_\Sigma$$

I.e., for $A_\Sigma \in \mathrm{Conn}_{\Sigma,G}$ a fixed connection on the boundary and for $v \in T_{A_\Sigma}\Phi_\Sigma \simeq \Omega^1(\Sigma,\mathfrak{g})$ a tangent vector ("a variation of boundary field"), we have

$$\iota_v \alpha_\Sigma = -\int_\Sigma \mathrm{tr}\,\frac12 A_\Sigma \wedge v \quad \in \mathbb{R}$$

(symbol ι_v stands for the contraction with a vector or vector field).

The exterior derivative of the 1-form α_Σ yields a 2-form

$$\omega_\Sigma := \delta\alpha_\Sigma = \int_\Sigma \mathrm{tr}\,\frac12 \delta A_\Sigma \wedge \delta A_\Sigma \quad \in \Omega^2(\Phi_\Sigma) \tag{19}$$

In particular, for $u,v\in T_{A_\Sigma}\Phi_\Sigma\simeq \Omega^1(M,\mathfrak{g})$ a pair of tangent vectors, we have

$$\iota_u\iota_v\omega_\Sigma=\int_\Sigma \mathrm{tr}\; u\wedge v\quad \in\mathbb{R}$$

The 2-form ω_Σ is closed by construction. Also, it is weakly non-degenerate (in the sense that the induced sharp-map $\omega^\#: T\Phi_\Sigma\to T^*\Phi_\Sigma$ is *injective*). Thus, ω_Σ defines a symplectic structure on Φ_Σ, viewed as an infinite-dimensional (Fréchet) manifold.

2.2.4. *"Cauchy subspace".* We define the *Cauchy*[10] *subspace* $C_\Sigma\subset\Phi_\Sigma$ as the subspace of fields on the boundary which can be extended to a neighborhood of the boundary, $\Sigma\times[0,\epsilon)\subset M$, as solutions to Euler-Lagrange equations.[11]

For Chern-Simons theory, this means that C_Σ is comprised of connections on Σ which can be extended to flat connections on $\Sigma\times[0,\epsilon)$. Thus, $C_\Sigma=\mathrm{FlatConn}_{\Sigma,G}\subset \mathrm{Conn}_{\Sigma,G}$ is simply the space of all flat connections on Σ.

Lecture 4, 09/05/2016.

Recall that, a vector subspace U of a symplectic vector space (V,ω) is called

- *isotropic* if $U\subset U^\perp$, with $U^\perp=\{w\in V \text{ s.t. } \omega(w,u)=0\ \forall u\in U\}$ – the symplectic orthogonal complement of U (equivalently, $U\subset (V,\omega)$ is isotropic if ω vanishes on pairs of vectors from U);
- *coisotropic* if $U^\perp\subset U$;
- *Lagrangian* if $U=U^\perp$.

Similarly, a submanifold $N\subset(\Phi,\omega)$ of a symplectic manifold is *isotropic/coisotropic/Lagrangian* if, for any point $x\in N$, the tangent space T_xN is a isotropic/coisotropic/Lagrangian subspace in $(T_x\Phi,\omega_x)$.

Recall that, for $C\subset(\Phi,\omega)$ a coisotropic submanifold, the *characteristic distribution* is defined as $(TC)^\perp\subset TC$ – a subbundle of the tangent bundle of C assigning to $x\in C$ a subspace $(T_xC)^\perp$ in T_xC. This distribution is integrable (by Frobenius theorem and $d\omega=0$) and thus induces a foliation of C by the leaves of charactersitic foliation. We denote $\underline{C}$ the corresponding space of leaves (the "coisotropic reduction" of C). The reduction $\underline{C}$ inherits a symplectic structure $\underline{\omega}$ characterized by $p^*\underline{\omega}=\omega|_C$ where $p:C\to\underline{C}$ is the quotient map.[12]

Lemma 2.6. (i) The submanifold $C_\Sigma\subset\Phi_\Sigma$ is coisotropic.
(ii) The characteristic distribution $(TC_\Sigma)^\perp$ on C_Σ is given by infinitesimal gauge transformations.

Proof. Fix $A_\Sigma\in C_\Sigma$ a flat connection on Σ. The tangent space $T_{A_\Sigma}C_\Sigma$ is the space of first order deformations of A_Σ as a flat connection. For the curvature of a generic small deformation of A_Σ, we have $F_{A_\Sigma+\epsilon\cdot\alpha}=\epsilon\cdot\underbrace{d_{A_\Sigma}\alpha}_{=:\,d\alpha+[A_\Sigma,\alpha]}+O(\epsilon^2)$ for a deformation $\alpha\in\Omega^1(\Sigma,\mathfrak{g})$ and $\epsilon\to 0$ a small deformation parameter. Hence,

$$T_{A_\Sigma}C_\Sigma=\{\alpha\in\Omega^1(\Sigma,\mathfrak{g})\text{ s.t. } d_{A_\Sigma}\alpha=0\}=\Omega^1(\Sigma,\mathfrak{g})_{d_{A_\Sigma}-\mathrm{closed}}$$

[10]Or "constraint" or "coisotropic" (see below).

[11]Thus, a "Cauchy subspace" – space of valid (in the sense of guaranteeing existence of a solution) initial data on $\Sigma\times\{0\}$ for the Cauchy problem for Euler-Lagrange equations on $\Sigma\times[0,\epsilon)$.

[12]Put another way, forgetting about the ambient symplectic manifold, $(C,\omega|_C)$ is itself a *pre-symplectic manifold*, i.e. one equipped with a *pre-symplectic structure* – a closed 2-form which can be degenerate but its kernel is required to be a subbundle of the tangent bundle TC (in particular, is required to have constant rank). From this point of view, $\underline{C}$ is the space of leaves of the kernel of pre-symplectic structure $\ker\omega|_C\subset TC$.

Let us calculate the symplectic orthogonal:

$$(T_{A_\Sigma}C_\Sigma)^\perp = \{\beta \in \Omega^1(\Sigma,\mathfrak{g}) \text{ s.t. } \int_\Sigma \operatorname{tr}\alpha\wedge\beta = 0 \quad \forall \alpha \in \Omega^1(\Sigma,\mathfrak{g})_{d_{A_\Sigma}-\text{closed}}\} \tag{20}$$

Let us put a metric on Σ and let $*$ be the corresponding Hodge star operator. We then continue (20) making a change $\beta = *\gamma$:

$$(T_{A_\Sigma}C_\Sigma)^\perp = *\{\gamma \in \Omega^1(\Sigma,\mathfrak{g}) \text{ s.t. } (\alpha,\gamma) = 0 \quad \forall \alpha \in \Omega^1(\Sigma,\mathfrak{g})_{d_{A_\Sigma}-\text{closed}}\} \tag{21}$$

where $(\alpha,\gamma) = \int_\Sigma \operatorname{tr}\alpha\wedge\gamma$ is the positive definite Hodge inner product on $\Omega^\bullet(\Sigma,\mathfrak{g})$. By Hodge decomposition theorem, we have

$$\Omega^\bullet(\Sigma,\mathfrak{g}) = \underbrace{\Omega^\bullet(\Sigma,\mathfrak{g})_{d_{A_\Sigma}-\text{exact}} \oplus \Omega^\bullet(\Sigma,\mathfrak{g})_{\text{harmonic}}}_{\Omega^\bullet(\Sigma,\mathfrak{g})_{d_{A_\Sigma}-\text{closed}}} \oplus \Omega^\bullet(\Sigma,\mathfrak{g})_{d^*_{A_\Sigma}-\text{exact}}$$

Thus, the orthogonal complement of $\Omega^1(\Sigma,\mathfrak{g})_{d_{A_\Sigma}-\text{closed}}$ w.r.t. Hodge inner product is $\Omega^1(\Sigma,\mathfrak{g})_{d^*_{A_\Sigma}-\text{exact}}$. Therefore,

$$(T_{A_\Sigma}C_\Sigma)^\perp = *\left(\Omega^\bullet(\Sigma,\mathfrak{g})_{d^*_{A_\Sigma}-\text{exact}}\right) = \Omega^1(\Sigma,\mathfrak{g})_{d_{A_\Sigma}-\text{exact}} \tag{22}$$

Since exact forms are a subspace of closed forms, we have $(T_{A_\Sigma}C_\Sigma)^\perp \subset T_{A_\Sigma}C_\Sigma$ which proves item (i) – coisotropicity of C_Σ.

Infinitesimal gauge transformations are the action of the Lie algebra $\text{gauge}_\Sigma = \text{Lie}(\text{Gauge}_\Sigma) \simeq \text{Map}(\Sigma,\mathfrak{g})$ by vector fields on Conn_Σ; this infinitesimal action arises from considering the action of a path of gauge transformations, $g_t \in \text{Gauge}_\Sigma$ with $t \in [0,\epsilon)$, starting at $g_{t=0} = 1$, on a connection A_Σ and taking the derivative at $t = 0$. Thus the gauge transformation formula

$$g \in \text{Gauge}_\Sigma \quad \mapsto \quad (A_\Sigma \mapsto A_\Sigma^g = g^{-1}A_\Sigma g + g^{-1}dg) \in \text{Diff}(\text{Conn}_\Sigma)$$

implies that infinitesimal gauge transformations are given by

$$\gamma \in \text{gauge}_\Sigma \quad \mapsto \quad (A_\Sigma \mapsto \underbrace{d_{A_\Sigma}\gamma}_{\in T_{A_\Sigma}\text{Conn}_\Sigma}) \in \mathfrak{X}(\text{Conn}_\Sigma) \tag{23}$$

Note that, fixing A_Σ and varying γ in (23), we obtain the subspace

$$\{d_{A_\Sigma}\gamma \mid \gamma \in \Omega^0(\Sigma,\mathfrak{g})\} \subset T_{A_\Sigma}\text{Conn}_\Sigma$$

which coincides with the value (22) of the characteristic distribution on C_Σ at $A_\Sigma \in C_\Sigma$. This proves item (ii). □

2.2.5. $L_{M,\Sigma}$. Let $EL_M = \text{FlatConn}_M$ be the space of solutions of Euler-Lagrange equation on M – the space of flat connections, and let $L_{M,\Sigma} := \pi(EL_M) \subset \Phi_\Sigma$ be the set of boundary values of flat connections on M. Since a solution of E-L equation on M is in particular a solution of E-L equation on the neighborhood of Σ, we have

$$L_{M,\Sigma} \subset C_\Sigma \subset \Phi_\Sigma$$

Remark 2.7 (**Aside on the evolution relation in classical mechanics.**). Consider a classical mechanical system in Hamiltonian formalism as a 1-dimensional field theory on an interval. It assigns to a point with + orientation a phase space Φ (a symplectic manifold (Φ,ω)) and to a point with − orientation the same space with the opposite sign of symplectic structure, $\bar{\Phi}$ (i.e. $(\Phi,-\omega)$). To an interval $[t_0,t_1]$ it assigns $L = \pi(EL_{[t_0,t_1]}) \subset \bar{\Phi}\times\Phi$; L consists of pairs of (initial state, final state) related by time evolution of the system from time t_0 to time t_1. In the case of a non-degenerate classical system, any point in Φ_{t_0} defines a unique solution for the Cauchy problem for E-L equation and evaluating it at $t = t_1$ we obtain an evolution map $U_{[t_0,t_1]} : \Phi_{t_0} \to \Phi_{t_1}$ which is a symplectomorphism (since the equations of motion are Hamiltonian), and then $L = \text{graph}\, U_{[t_0,t_1]}$. Being a graph of a symplectomorphism, $L \subset \bar{\Phi}_{t_0} \times \Phi_{t_1}$ is Lagrangian. One can think of L as a set-theoretic relation between Φ_{t_0} and Φ_{t_1} with additional Lagrangian property (such relations are called "canonical relations"). Since L encodes the time evolution of the system (or "dynamics"), it deserves a name of the "evolution relation" or "dynamic relation".

Now we are back to Chern-Simons.

Lemma 2.8. $L_{M,\Sigma} \subset \Phi_\Sigma$ is isotropic.

Proof. Let $A_\Sigma \in L_{M,\Sigma}$ be the boundary value of a flat connection $\tilde{A}$ on M. The tangent space to $L_{M,\Sigma}$ is

$$T_{A_\Sigma} L_{M,\Sigma} = \{\alpha \in \Omega^1(\Sigma, \mathfrak{g}) \text{ s.t. } \alpha = \tilde{\alpha}|_\Sigma \text{ for some } \tilde{\alpha} \in \Omega^1(\Sigma, \mathfrak{g})_{d_{\tilde{A}}-\text{closed}}\}$$

Thus, for $\alpha, \beta \in T_{A_\Sigma} L_{M,\Sigma}$, we have

$$\omega_\Sigma(\alpha, \beta) = \int_\Sigma \text{tr}\, \alpha \wedge \beta \underset{\text{Stokes}'}{=} \int_M \text{tr}\, (d_{\tilde{A}}\tilde{\alpha} \wedge \tilde{\beta} - \tilde{\alpha} \wedge d_{\tilde{A}}\tilde{\beta}) = 0$$

(Note that replacing $d \to d_{\tilde{A}}$ under trace is an innocent operation, as $\text{tr}\, [\tilde{A}, \bullet] = 0$.) Thus, ω_Σ vanishes on $L_{M,\Sigma}$, which is the isotropic property. □

2.2.6. *Reduction of the boundary structure by gauge transformations.* Let $\underline{C}_\Sigma = C_\Sigma/\text{Gauge}_\Sigma$ be the coisotropic reduction of C_Σ (by definition, this is the space of leaves of characterisitc distribution on C_Σ) – the space of classes of flat connections on Σ module gauge transformations. Thus,

$$\underline{C}_\Sigma = \mathcal{M}_\Sigma \simeq \text{Hom}(\pi_1(\Sigma), G)/G$$

is the moduli space of flat connections on Σ.

Note that the tangent space to the moduli space is

$$T_{[A_\Sigma]}\mathcal{M}_\Sigma = \frac{T_{A_\Sigma} C_\Sigma}{(T_{A_\Sigma} C_\Sigma)^\perp} = \frac{\Omega^1(\Sigma, \mathfrak{g})_{d_{A_\Sigma}-\text{closed}}}{\Omega^1(\Sigma, \mathfrak{g})_{d_{A_\Sigma}-\text{exact}}} = H^1_{d_{A_\Sigma}}(\Sigma, \mathfrak{g})$$

– the twisted (by a flat connection A_Σ) first de Rham cohomology.

Symplectic structure $\underline{\omega}_\Sigma$ on $\mathcal{M}_\Sigma$ (the Atiyah-Bott symplectic structure) is:

$$\underline{\omega}_\Sigma([\alpha], [\beta]) = \int_\Sigma \text{tr}\, \alpha \wedge \beta$$

– the standard Poincaré duality pairing (with coefficients in a local system determinaed by A_Σ), $\langle , \rangle_\Sigma : H^1_{d_{A_\Sigma}} \otimes H^1_{d_{A_\Sigma}} \to \mathbb{R}$.

Let $\underline{L}_{M,\Sigma} = L_{M,\Sigma}/\text{Gauge}_\Sigma \subset \underline{C}_\Sigma$ be the reduction of the evolution relation by gauge symmetry, i.e. $\underline{L}_{M,\Sigma}$ is the space of gauge classes of connections on Σ which can be extended as flat connection over all M.

2.2.7. *Lagrangian property of $L_{M,\Sigma}$.*

Lemma 2.9. Submanifold $\underline{L}_{M,\Sigma} \subset \mathcal{M}_\Sigma$ is Lagrangian.

Proof. Fix some $A_\Sigma \in L_{M,\Sigma}$ with $\tilde{A}$ a flat extension into M. Then the tangent space $T_{[A_\Sigma]}\underline{L}_{M,\Sigma} = \frac{\{\alpha\in\Omega^1(\Sigma,\mathfrak{g}) \text{ s.t. } \exists\, \tilde{\alpha}\in\Omega^1(M,\mathfrak{g})_{d_{\tilde{A}}-\text{closed}} \text{ with } \alpha=\tilde{\alpha}|_\Sigma\}}{\Omega^1(\Sigma,\mathfrak{g})_{d_{A_\Sigma}-\text{exact}}} = \mathrm{im}(\Pi)$ the image of the map Π in the long exact sequence of cohomology of the pair (M,Σ):

$$\cdots \to H^1_{d_{\tilde{A}}}(M;\mathfrak{g}) \xrightarrow{\Pi} H^1_{d_{A_\Sigma}}(\Sigma;\mathfrak{g}) \xrightarrow{\varkappa} H^2_{d_{\tilde{A}}}(M,\Sigma;\mathfrak{g}) \to \cdots \tag{24}$$

Let us calculate the symplectic complement $\mathrm{im}(\Pi)^\perp$ in $H^1(\Sigma)$:

$$\mathrm{im}(\Pi)^\perp = \{[\alpha]\in H^1(\Sigma) \text{ s.t. } \langle[\alpha],\Pi[\tilde{\beta}]\rangle_\Sigma = 0\ \forall[\tilde{\beta}]\in H^1(M)\} \tag{25}$$

Note that

$$\langle[\alpha],\Pi[\tilde{\beta}]\rangle_\Sigma = \int_\Sigma \mathrm{tr}\,\alpha\wedge\tilde{\beta}|_\Sigma \underset{\text{Stokes}'}{=} \int_M d\,\mathrm{tr}\,\tilde{\alpha}\wedge\tilde{\beta} = \int_M \mathrm{tr}\, d_{\tilde{A}}\tilde{\alpha}\wedge\tilde{\beta} - \tilde{\alpha}\wedge\underbrace{d_{\tilde{A}}\tilde{\beta}}_{0}$$

Here $\tilde{\alpha}$ is an arbitrary (not necessarily closed) extension of the closed 1-from α into the bulk of M. Note that $d_{\tilde{A}}\tilde{\alpha}$ is a closed 2-form on M vanishing on Σ. The class $[d_{\tilde{A}}\tilde{\alpha}]$ in relative cohomology $H^2(M,\Sigma)$ is $\varkappa[\alpha]$, by construction of the connecting homomorphism $\varkappa$. Thus, we have $\langle[\alpha],\Pi[\tilde{\beta}]\rangle_\Sigma = \langle\varkappa[\alpha],[\tilde{\beta}]\rangle_M$ where $\langle,\rangle_M : H^1(M)\otimes H^2(M,\Sigma)\to\mathbb{R}$ is the Lefschetz pairing between relative and absolute cohomology. We then continue the calculation (25):

$$\begin{aligned}\mathrm{im}(\Pi)^\perp = \{[\alpha]\in H^1(\Sigma) \text{ s.t. } \langle\varkappa[\alpha],[\tilde{\beta}]\rangle_\Sigma = 0\ \forall[\tilde{\beta}]\in H^1(M)\}\\ = \{[\alpha]\in H^1(\Sigma) \text{ s.t. } \varkappa[\alpha]=0\} = \ker\varkappa = \mathrm{im}(\Pi)\end{aligned}$$

Here we used non-degeneracy of the Lefschetz pairing and, in the last step, used exactness of the sequence (24). This finishes the proof that $\underline{L}_{M,\Sigma}\subset\mathcal{M}_\Sigma$ is Lagrangian. □

A corollary of this is the following.

Theorem 2.10. Submanifold $L_{M,\Sigma}\subset\Phi_\Sigma$ is Lagrangian.

Proof. Fix $A_\Sigma\in L_{M,\Sigma}$. Denote $\Theta := T_{A_\Sigma}L_{M,\Sigma}$ and $V := T_{A_\Sigma}\Phi_\Sigma$. We know by Lemma 2.8 that Θ is isotropic in V, i.e. $\Theta\subset\Theta^\perp$. Let also $U := T_{A_\Sigma}C_\Sigma$ and $H := U^\perp \subset U$. We have then a sequence of subspaces

$$H\subset\Theta\subset\Theta^\perp\subset U\subset V$$

Note that $\Lambda := \Theta/H = T_{[A_\Sigma]}\underline{L}_{M,\Sigma}$Note that

$$\begin{aligned}\Lambda^\perp = \{[v]\in U/H \text{ s.t. } \underline{\omega}([v],[\theta]) = 0\ \forall[\theta]\in\Theta/H\}\\ = \{v\in U \text{ s.t. } \omega(v,\theta)=0\ \forall\theta\in\Theta\}/H = \Theta^\perp/H\end{aligned}$$

On the other hand, Λ is the space we have proven to be a Lagrangian subspace $U/H = T_{[A_\Sigma]}\underline{C}_\Sigma$ in Lemma 2.9. Thus

$$\Theta/H = \Lambda = \Lambda^\perp = \Theta^\perp/H$$

which, in combination with $\Theta\subset\Theta^\perp$, proves $\Theta=\Theta^\perp$. □

Lecture 5, 09/07/2016.

2.2.8. *Behavior of S_{CS} under gauge transformations, Wess-Zumino cocycle.* For a manifold M with boundary Σ, Chern-Simons action changes w.r.t. gauge transformation of a connection in following way (result of a straightforward calculation):

$$S_{CS}(A^g) - S_{CS}(A) = \int_\Sigma \mathrm{tr}\, \frac{1}{2} g^{-1} A g \wedge g^{-1} dg \underbrace{- \int_M \mathrm{tr}\, \frac{1}{6} (g^{-1} dg)^{\wedge 3}}_{=: W_\Sigma(g)} \tag{26}$$

The last term here is called the *Wess-Zumino term.*

Lemma 2.11. $W_\Sigma(g) \mod 4\pi^2\mathbb{Z}$ depends only on the restriction of g to the boundary, $g|_\Sigma \in \mathrm{Gauge}_\Sigma$.

Proof. Let M' be a second copy of M and let $\widetilde{M} = M \cup_\Sigma \overline{M'}$ be the closed 3-manifold obtained by gluing M and M' along Σ. Let $g : M \to G$ and $g' : M' \to G$ be two maps to the group which agree on Σ, $g|_\Sigma = g'|_\Sigma$. The pair (g, g') determines a map $\widetilde{g} : \widetilde{M} \to G$. We have

$$W_\Sigma(g) - W_\Sigma(g') = - \int_{\widetilde{M}} \mathrm{tr}\, \frac{1}{6} (\widetilde{g}^{-1} d\widetilde{g})^{\wedge 3} = 4\pi^2 \langle [M] \widetilde{g}^* [\theta] \rangle \quad \in 4\pi^2 \cdot \mathbb{Z}$$

where $[\theta]$ is the class of Cartan's 3-form in $H^3(G)$. □

Denote

$$c_\Sigma^k(A, g) := e^{\frac{ik}{2\pi} (\int_\Sigma \mathrm{tr}\, \frac{1}{2} g^{-1} A g \wedge g^{-1} dg + W_\Sigma(g))}$$

By the Lemma above, for $k \in \mathbb{Z}$, it this is a well-defined function of a pair $(A, g) \in \mathrm{Conn}_\Sigma \times \mathrm{Gauge}_\Sigma$.

In particular, (26) can be rewritten as the gauge transformation rule for the (normalized) exponential of Chern-Simons action $\psi_k(A) = e^{\frac{ik}{2\pi} S_{CS}(A)}$ (which we introduced earlier in the closed case):

$$\psi_k(A^g) = \psi_k(A) \cdot c_\Sigma^k(A|_\Sigma, g|_\Sigma) \tag{27}$$

Remark 2.12. c_Σ^k can be viewed as a 1-cocycle in the cochain complex of the group Gauge_Σ acting on $\mathrm{Map}(\mathrm{Conn}_\Sigma, S^1)$. Group cocycle property amounts to

$$\left(g \circ c_\Sigma^k(A, h)\right) \cdot \left(c_\Sigma^k(A, gh)\right)^{-1} \cdot \left(c_\Sigma^k(A, g)\right) = 1$$

(here $\cdot$ refers to the product in abelian group S^1 and $g \circ \phi(A) = \phi(A^g)$ is the Gauge_Σ action on the module $\{\phi(A)\} = \mathrm{Map}(\mathrm{Conn}_\Sigma, S^1)$). This property in turn follows from (26) by exponentiating the obvious relation

$$0 = \left(S_{CS}(A^{gh}) - S_{CS}(A^g)\right) - \left(S_{CS}(A^{gh}) - S_{CS}(A)\right) + \left(S_{CS}(A^g) - S_{CS}(A)\right)$$

Remark 2.13. The construction of c_Σ^k from ψ_k is similar to the *transgression* in the *inflation-restriction* exact sequence in group cohomology:

$$\cdots \to H^j(\mathcal{G}/\mathcal{N}, \mathcal{A}^\mathcal{N}) \to H^j(\mathcal{G}, \mathcal{A}) \to H^j(\mathcal{N}, \mathcal{A})^{\mathcal{G}/\mathcal{N}} \xrightarrow{\mathbb{T}} H^{j+1}(\mathcal{G}/\mathcal{N}, \mathcal{A}^\mathcal{N}) \to \cdots$$

which holds for $\mathcal{G}$ a group, $\mathcal{N} \subset \mathcal{G}$ a normal subgroup and $\mathcal{A}$ a $\mathcal{G}$-module (this exact sequence is related to the *Lyndon-Hochschild-Serre spectral sequence*). In our case, $\mathcal{G} = \mathrm{Gauge}_M$, $\mathcal{N} = \{g : M \to G \text{ s.t. } g|_\Sigma = 1\}$, with the quotient $\mathcal{G}/\mathcal{N} \cong \mathrm{Gauge}_\Sigma$; the module is $\mathcal{A} = \mathrm{Map}(\mathrm{Conn}_M, S^1)$. In particular, invariants $\mathcal{A}^\mathcal{N}$ are the functionals of connections on M which are gauge-invariant w.r.t. gauge transformations *relative to the boundary* (i.e. fixed to 1 at the boundary). We can view ψ_k as a class in $H^0(\mathcal{N}, \mathcal{A})$ and $c_\Sigma^k =$ "$\mathbb{T}(\psi_k)$" as a class in $H^1(\mathcal{G}/\mathcal{N}, \mathcal{A}^\mathcal{N})$.

Let $\mathcal{L}_M = S^1 \times \mathrm{Conn}_M$ be the trivial circle bundle over Conn_M. We define the action of Gauge_M on $\mathcal{L}_M$ by

$$g : (\lambda, A) \mapsto (\lambda \cdot c^k_\Sigma(A|_\Sigma, g|_\Sigma), A^g)$$

with $\lambda \in S^1$. By (27), ψ_k is a Gauge_M-invariant section of $\mathcal{L}_M$.

Similarly, on the boundary, we have a trivial bundle $\mathcal{L}_\Sigma = S^1 \times \mathrm{Conn}_\Sigma$ with action of Gauge_Σ defined as

$$g_\Sigma : (\lambda, A_\Sigma) \mapsto (\lambda \cdot c^k_\Sigma(A_\Sigma, g_\Sigma), A_\Sigma^{g_\Sigma})$$

The 1-form

$$\alpha^k_\Sigma = \frac{ik}{2\pi} \underbrace{\int_\Sigma \mathrm{tr}\, \frac{1}{2} A_\Sigma \wedge \delta A_\Sigma}_{\alpha_\Sigma} \quad \in \quad \Omega^1(\mathrm{Conn}_\Sigma, \mathfrak{u}(1))$$

defines a Gauge_Σ-invariant connection in $\mathcal{L}_\Sigma$ (here $\mathfrak{u}(1) = i\mathbb{R}$ is the Lie algebra of $S^1 = U(1)$). Its curvature is

$$\omega^k_\Sigma = \frac{ik}{2\pi} \underbrace{\int_\Sigma \mathrm{tr}\, \frac{1}{2} \delta A_\Sigma \wedge \delta A_\Sigma}_{\omega_\Sigma} \quad \in \quad \Omega^2(\mathrm{Conn}_\Sigma, \mathfrak{u}(1))$$

Exponential of the action ψ_k restricted to flact connections satisfies the following property (instead of beilng locally constant as a function on $\mathrm{FlatConn}_M$ as in the case of M closed):

$$(\delta - \pi^* \alpha^k_\Sigma)\psi_k = 0$$

with $\pi : \mathrm{Conn}_M \to \mathrm{Conn}_\Sigma$ the pullback of connections to the boundary; $\pi^*\alpha^k_\Sigma$ is the pullback of an S^1-connection α^k_Σ on Conn_Σ to an S^1-connection on Conn_M.

2.2.9. *Prequantum line bundle on the moduli space of flat connections on the surface.* Restricting the circle bundle $\mathcal{L}_\Sigma$ to flat connections and taking the quotient over gauge transformations, we obtain a non-trivial circle bundle $\underline{\mathcal{L}}^k_\Sigma$ over the moduli space $\mathcal{M}_\Sigma$ with connection $\underline{\alpha}^k_\Sigma$ with curvature $\underline{\omega}^k_\Sigma = \frac{ik}{2\pi}\underline{\omega}_\Sigma$ – a multiple of the standard Atiyah-Bott symplectic structure on the moduli space $\mathcal{M}_\Sigma$. In fact, $\underline{\mathcal{L}}^k_\Sigma = (\underline{\mathcal{L}}^1_\Sigma)^{\otimes k}$ (here we implicitly identify a circle bundle and the associated complex line bundle $\underline{\mathcal{L}} \times_{S^1} \mathbb{C}$). $\underline{\mathcal{L}}^1_\Sigma$ is known as the *prequantum line bundle* on the moduli space of flat connections on the surface.

Another point of view on the line bundle $\underline{\mathcal{L}}^k_\Sigma$ is as follows. Consider C_Σ as a space with Gauge_Σ-action with quotient $\mathcal{M}_\Sigma$. Restriction of the symplectic form $\omega_\Sigma|_{C_\Sigma}$ is a basic form (horizontal and invariant) w.r.t. Gauge_Σ and thus is a pullback of a form $\underline{\omega}_\Sigma$ on the quotient. But ω_Σ before reduction is exact, with primitive 1-form α_Σ. The first question is: can we reduce α_Σ to a primitive 1-form for the reduced symplectic structure? The answer is: NO, because $\alpha_\Sigma|_{C_\Sigma}$ is not basic (in particular, not horizontal).[13]

The solution is to promote α_Σ to a connection ∇ in the trivial circle bundle over C_Σ, then one can identify the circle fibers along Gauge_Σ-orbits on C_Σ. Locally this identification is consistent because ∇ is flat when restricted to the orbit (since $F_\nabla = \omega_\Sigma$ and orbits are isotropic submanifolds). For the identification to be globally consistent, the holonomy of ∇ on the orbit has to be trivial. This turns out to

[13] Also, we could not have succeded in constructing a primitive 1-form for $\underline{\omega}_\Sigma$ because, being a symplectic structure on a compact manifold (for G compact, $\mathcal{M}_\Sigma$ is also compact), it has to define a nontrivial class in $H^2(\mathcal{M}_\Sigma)$.

be true precisely if we normalize the connection 1-form as $\frac{ik}{2\pi}\alpha_\Sigma = \alpha_\Sigma^k$ with k an integer! The resulting consistent identification of circle fibers along gauge orbits on C_Σ yields the circle bundle $\mathcal{L}_\Sigma^k$ over the moduli space $C_\Sigma/\mathrm{Gauge}_\Sigma = \mathcal{M}_\Sigma$.

Remark 2.14. The Chern-Weil representative of the first Chern class of $\underline{\mathcal{L}}_\Sigma^k$ is $\frac{1}{2\pi i}\omega_\Sigma^k = \frac{k}{4\pi^2}\omega_\Sigma$ - the (normalized) curvature of the connection in $\underline{\mathcal{L}}_\Sigma^k$. In particular, this implies that the 2-form $\frac{1}{4\pi^2}\omega_\Sigma$ on the moduli space $\mathcal{M}_\Sigma$ has integral periods.

Exponential of the action ψ_k restricted to flat connections, after reduction modulo gauge symmetry, yields a section $\underline{\psi}_k \in \Gamma(\mathcal{M}_M, (\pi_*)^*\underline{\mathcal{L}}_\Sigma^k)$ which satisfies

$$(\delta - (\pi_*)^*\underline{\alpha}_\Sigma^k)\underline{\psi}_k = 0$$

i.e. is horizontal w.r.t. the connection $\underline{\mathcal{L}}_\Sigma^k$ pulled back to $\mathcal{M}_M$ by the map $\pi_* : \mathcal{M}_M \to \mathcal{M}_\Sigma$ sending the gauge class of a connection on M to the gauge class of its restriction to the boundary.

Remark 2.15. Existence of a global section $\underline{\psi}_k$ of the line bundle $(\pi_*)^*\underline{\mathcal{L}}_\Sigma^k$ over $\mathcal{M}_M$ implies, in particular, that the latter is trivial. Put another way, the pullback of the (nontrivial) first Chern class $c_1(\underline{\mathcal{L}}_\Sigma^k) \in H^2(\mathcal{M}_\Sigma)$ by $\pi_* : \mathcal{M}_M \to \mathcal{M}_\Sigma$ is zero.

2.2.10. *Two exciting formulae.* Symplectic volume of the moduli space of flat connections on a surface of genus $h \geq 2$ is given by

(28)
$$\mathrm{Vol}(\mathcal{M}_\Sigma) := \int_{\mathcal{M}_\Sigma} \frac{(\omega_\Sigma)^{\wedge m}}{m!} = \#Z(G)\cdot(\mathrm{Vol}(G))^{2h-2} \sum_{R\in\{\text{irrep of } G\}} \frac{1}{(\dim R)^{2h-2}}$$

where $m := \frac{1}{2}\dim \mathcal{M}_\Sigma = \dim G \cdot (h-1)$ and $\#Z(G)$ is the number of elements in the center of G; R runs over irreducible representations of G (see [37]).

A related result is the celebrated Verlinde formula for the dimension of the space of holomorphic sections of the line bundle $\underline{\mathcal{L}}_\Sigma^k$ over $\mathcal{M}_\Sigma$ (with respect to some a priori chosen complex structure on the surface Σ which in turn endows $\mathcal{M}_\Sigma$ with a complex structure – and, moreover, makes $\mathcal{M}_\Sigma$ a Kähler manifold). For simplicity, we give the formula for $G = SU(2)$:

$$\dim H^0_{\bar\partial}(\mathcal{M}_\Sigma, \underline{\mathcal{L}}_\Sigma^k) = \left(\frac{k+2}{2}\right)^{h-1} \sum_{j=0}^{k-1} \frac{1}{\left(\sin\frac{\pi(j+1)}{k+2}\right)^{2h-2}} \tag{29}$$

This formula gives the dimension of the space of states which *quantum* Chern-Simons theory assigns to the surface Σ (see [36]). The r.h.s. here is, in fact, a polynomial in k of degree $m = 3h-3$ (in case $h \geq 2$), with the coefficient of the leading term given by (28). This follows from Riemann-Roch-Hirzebruch formula which gives the following for the dimension of the space of holomorphic sections:

$$\begin{aligned}\dim H^0_{\bar\partial}(\mathcal{M}_\Sigma, \underline{\mathcal{L}}_\Sigma^k) &= \int_{\mathcal{M}_\Sigma} \mathrm{Td}(\mathcal{M}_\Sigma)\cdot e^{\frac{k}{2\pi}\omega_\Sigma} \\ &= \left(\frac{k}{2\pi}\right)^m \mathrm{Vol}(\mathcal{M}_\Sigma) + \text{polynomial of degree} < m \text{ in } k\end{aligned}$$

The sum in (29) runs, secretely, over "integrable" irreducible representations of the affine Lie algebra $\hat{\mathfrak{g}}$ (with $\mathfrak{g} = \mathfrak{su}(2)$ in the case at hand) at level k (resp. irreducible representations of the quantum group $SL_q(2)$ with $q = e^{\frac{\pi i}{k+2}}$ a root of unity).

Lecture 6, 09/12/2016.

2.2.11. *Classical field theory as a functor to the symplectic category.*

Definition 2.16. [14] Let (Φ_1,ω_1) and (Φ_2,ω_2) be two symplectic manifolds. A *canonical relation* L between Φ_1 and Φ_2 is a Lagrangian submanifold $L\subset\overline{\Phi}_1\times\Phi_2$ where $\overline{\Phi}_1=(\Phi_1,-\omega_1)$ is the *symplectic dual* of Φ_1, i.e. Φ_1 endowed with symplectic structure of opposite sign. The notation is: $L:\Phi_1\nrightarrow\Phi_2$. Composition of canonical relations $L:\Phi_1\nrightarrow\Phi_2$ and $L':\Phi_2\nrightarrow\Phi_3$ is defined as the set-theoretic composition of relations:

$$
\begin{aligned}
(30)\quad L'\circ L &:= \{(x,z)\in\Phi_1\times\Phi_3 \text{ s.t. } \exists y\in\Phi_2 \text{ s.t. } (x,y)\in L \text{ and } (y,z)\in L'\}\\
&= P\left((L\times L')\cap(\Phi_1\times \mathrm{Diag}_{\Phi_2}\times\Phi_3)\right)
\end{aligned}
$$

where $\mathrm{Diag}_{\Phi_2}=\{(y,y)\in\Phi_2\times\Phi_2\}$ – the diagonal Lagrangian in $\Phi_2\times\overline{\Phi}_2$ and $P:\overline{\Phi}_1\times\Phi_2\times\overline{\Phi}_2\times\Phi_3\to\overline{\Phi}_1\times\Phi_3$ is a projection to the outmost factors.

Composition of canonical relations is guaranteed to be a canonical relation in the context of finite-dimensional symplectic vector spaces. More generally (for symplectic manifolds, possibly infinite-dimensional), the composition is always isotropic but may fail to be Lagrangian, if the intersection in (30) fails to be transversal. Also, the composition may fail to be smooth.

Thus, we have a symplectic category of symplectic manifolds and canonical relations between them with partially-defined composition. Unit morphisms are the diagonal Lagrangians $\mathrm{id}_\Phi=\mathrm{Diag}_\Phi:\Phi\to\Phi$. The monoidal structure is given by direct products and the monoidal unit is the point (regarded as a symplectic manifold).

For $C\subset(\Phi,\omega)$ a coisotropic submanifold, introduce a special canonical relation $r_C:\Phi\nrightarrow\Phi$ defined as the set of pairs $(x,y)\in C\times C$ such that x and y are on the same leaf of the characterictic distribution on C. Note that this relation is an idempotent: $r_C\circ r_C=r_C$. Also note that for $C=\Phi$, r_C is the identity (diagonal) relation on Φ.

One can formulate an n-dimensional classical field theory in the spirit of Atiyah-Segal axiomatics of QFT, as the following association.[15]

- To an $(n-1)$-manifold Σ (possibly with geometric structure), the classical field theory assigns a symplectic manifold $(\Phi_\Sigma,\omega_\Sigma)$ – the *phase space*.[16]
- To an n-cobordism $\Sigma_{\mathrm{in}}\xrightarrow{M}\Sigma_{\mathrm{out}}$, the classical field theory assigns a canonical relation $L_M:\Phi_{\Sigma_{\mathrm{in}}}\nrightarrow\Phi_{\Sigma_{\mathrm{out}}}$.[17]

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[14] See [34].

[15] See [8] for an overview of this approach and examples.

[16] The idea of construction of the phase space from variational calculus data (fields and action functional) of the field theory is to first construct $\Phi^{\mathrm{pre}}_\Sigma$ as normal ∞-jets of fields at Σ on some manifold M containing Σ as a boundary component. Thus, tautologically, one has a projection $\pi^{\mathrm{pre}}:F_M\to\Phi^{\mathrm{pre}}_\Sigma$ – evaluation of the normal jet of a field at Σ. By integrating by parts in the variation of action δS_M, one gets the pre-Nöther 1-form $\alpha^{\mathrm{pre}}_\Sigma\in\Omega^1(\Phi^{\mathrm{pre}}_\Sigma)$. Setting $\omega^{\mathrm{pre}}_\Sigma$, one performs the symplectic reduction by the kernel of $\omega^{\mathrm{pre}}_\Sigma$. Phase space Φ_Σ is the result of this reduction. By construction, it comes with a symplectic structure and a projection $\pi:F_M\to\Phi_\Sigma$.

[17] The idea is to consider the space EL_M of solutions of Euler-Lagrange equations on M (as defined by the bulk term of the variation of action δS_M), and to construct $L_M:=(\pi_{\mathrm{in}}\times\pi_{\mathrm{out}})(EL_M)\subset\overline{\Phi}_{\Sigma_{\mathrm{in}}}\times\Phi_{\Sigma_{\mathrm{out}}}$ – the set of boundary values of solutions of Euler-Lagrange equations. **Warning:** though it is automatic that L_M is isotropic, fact that is Lagrangian has to be proven for individual field theories and there exist (pathological) examples where Lagrangianity fails, e.g. 2-dimensional scalar field on Misner's cylinder [9].

- Composition (gluing) of cobordisms $\Sigma_1 \xrightarrow{M} \Sigma_2 \xrightarrow{M'} \Sigma_3$ is mapped to the set-theoretic composition of relations $\Phi_{\Sigma_1} \xrightarrow{L_M} \Phi_{\Sigma_2} \xrightarrow{L_{M'}} \Phi_{\Sigma_3}$.
- Disjoint unions are mapped to direct products.
- Null $(n-1)$-manifold is mapped to the point as its phase space.
- A short[18] cylinder $\Sigma \times [0,\epsilon]$ is mapped to the relation $r_{C_\Sigma} : \Phi_\Sigma \nrightarrow \Phi_\Sigma$ for some distinguished coisotropic $C_\Sigma \subset \Phi_\Sigma$ – the *Cauchy subspace*.

Thus, from this point of view, a classical field theory, similarly to quantum field theory, is a functor of monoidal categories from the category of n-cobordisms (possibly with geometric structure) to the symplectic category. With two corrections:

- The target category has only partially defined composition. On the other hand, if we know that the space of solutions of Euler-Lagrange equations induces a Lagrangian submanifold in the phase space on the boundary for any spacetime manifold M (which is the case in all but pathological examples), then we know that there is no problem with composition of relations in the image of cobordisms under the given field theory.
- Units do not go to units (if we deal with a gauge theory; for a non-degenerate/unconstrained theory, we have $C_\Sigma = \Phi_\Sigma$ and then units do go to units). One can then pass to a *reduced* field theory, by replacing phase spaces Φ_Σ with coisotropic reductions $\underline{C}_\Sigma =: \Phi_\Sigma^{\mathrm{reduced}}$ and replacing relations L_M with respective reduced relations $L_M^{\mathrm{reduced}} := \underline{L}_M : \underline{C}_{\Sigma_{\mathrm{in}}} \nrightarrow \underline{C}_{\Sigma_{\mathrm{out}}}$ (pushforwards of L_M along the coisotropic reduction). The reduced theory is a functor to the symplectic category and takes units to units, but there may be a problem with reductions not being smooth manifolds.

Example 2.17 (Non-degenerate classical mechanics)**.** This is a 1-dimensional classical field theory. A point with positive orientation pt^+ is mapped to some symplectic manifold Φ and pt^- is mapped to the symplectic dual $\overline{\Phi}$. An interval $[t_0,t_1]$ (our cobordisms are equipped with Riemannian metric and so have length) is mapped to a relation $L_{[t_0,t_1]} : \Phi \nrightarrow \Phi$. Using the gluing axiom, by the argument similar to Example 1.4 (where we considered quantum mechanics as an example of Segal's axioms), we have that

$$L_{[t_0,t_1]} = \{(x,y) \subset \Phi \times \Phi \ \text{ s.t. } y = \mathrm{Flow}_{t_1-t_0}(X) \circ x\}$$

– the graph of the flow, in time $t_1 - t_0$, of a vector field X on Φ preserving symplectic structure. If Φ is simply connected, X has to be a Hamiltonian vector field, $X = \{H, \bullet\}_\omega$ for some Hamiltonian $H \in C^\infty(\mathbb{R})$. On the other hand $L_{[t_0,t_1]}$ is constructed out of the action of the classical field theory (e.g. in the case of second-order Lagrangian, $S[x(\tau)] = \int_{t_0}^{t_1} d\tau \left(\frac{\dot{x}^2}{2m} - U(x(\tau))\right)$) as

$$L_{[t_0,t_1]} = \{(x,y) \in \Phi \times \Phi \ \mid \ \exists \text{ sol. of EL eq. } x(\tau) \text{ s.t. } x(t_0) = x,\ x(t_1) = y\}$$

In particular, evolution in infinitesimal time relates the Lagrangian density and the Hamiltonian (the relation being the Legendre transform).

Example 2.18 (Classical Chern-Simons theory)**.** Classical Chern-Simons theory as we discussed it here is the prototypical example of a functorial classical field theory, with $n = 3$, $\Phi_\Sigma = \mathrm{Conn}_\Sigma$, $C_\Sigma = \mathrm{FlatConn}_\Sigma$ and $L_M = \mathrm{im}(\mathrm{FlatConn}_M \to \overline{\mathrm{Conn}_{\Sigma_{\mathrm{in}}}} \times \mathrm{Conn}_{\Sigma_{\mathrm{out}}})$.

[18]In a topological theory, we can think of a unit cylinder $\Sigma \times [0,1]$ and in a theory e.g. with cobordisms endowed with metric, we should think of taking a limit $\epsilon \to 0$.

3. Feynman diagrams

Here we will discuss how Feynman diagrams arise in the context of finite-dimensional integrals. References: [14, 27].

3.1. Gauss and Fresnel integrals.

Gauss integral:[19]

$$\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi} \quad \text{or more generally} \quad \int_{-\infty}^{\infty} dx\, e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}} \tag{31}$$

with $\mathrm{Re}\,\alpha > 0$ needed for absolute convergence. Multi-dimensional version:

$$\int_{\mathbb{R}^n} d^n x\, e^{-Q(x,x)} = \pi^{\frac{n}{2}} (\det Q)^{-\frac{1}{2}} \tag{32}$$

Here $Q(x,x) = \sum_{i,j=1}^n Q_{ij} x_i x_j$ is a *positive-definite* (as necessary for convergence) quadratic form and $\det Q$ stands for the determinant of the matrix (Q_{ij}).[20]

Fresnel integral is the oscillating version of Gauss integral:

$$\int_{-\infty}^{\infty} dx\, e^{ix^2} = \sqrt{\pi} \cdot e^{\frac{i\pi}{4}}, \qquad \int_{-\infty}^{\infty} dx\, e^{-ix^2} = \sqrt{\pi} \cdot e^{-\frac{i\pi}{4}} \tag{33}$$

To calculate e.g. the first one, one way is to take the limit $\alpha \to -i$ in (31). Equivalently, one views it as an integral over the real line in the complex plane $\mathbb{R} \subset \mathbb{C}$ and rotates the integration contour counterclockwise $\mathbb{R} \to e^{\frac{\pi i}{4}} \cdot \mathbb{R} \subset \mathbb{C}$. On the new contour, the integrand becomes the standard Gaussian integrand (not oscillating but decaying). Note that we could not have rotated the contour clockwise because then the integral would have diverged.

Fresnel integrals are only conditionally convergent, as opposed to Gaussian integrals which are absolutely convergent.

Multi-dimensional Fresnel integral:

$$\int_{\mathbb{R}^n} d^n x\, e^{i\,Q(x,x)} = \pi^{\frac{n}{2}} \cdot e^{\frac{\pi i}{4} \mathrm{sign}\, Q} \cdot |\det Q|^{-\frac{1}{2}} \tag{34}$$

with $Q(x,x) = \sum_{i,j=1}^n Q_{ij} x_i x_j$ a *non-degenerate* quadratic form (not required to be positive-definite); $\mathrm{sign}\, Q$ is the *signature* of Q – the number of positive eigenvalues minus the number of negative eigenvalues. (Proven as in footnote 20, by diagonalization of Q).

Remark 3.1 (On convergence of Fresnel integrals)**.** Although one-dimensional integral can be made sense of as a limit of integrals with cut-off integration domain, $\lim_{\Lambda\to\infty} \int_{-\Lambda}^{\Lambda} dx\, e^{ix^2}$ (the cut-off integral oscillates as a function of Λ but the amplitude of oscillation goes to zero as $\Lambda \to \infty$), in the higher-dimensional case there are problems. E.g. if $Q = x_1^2 + \cdots + x_n^2$, then cutting-off the integration domain to a ball of radius Λ, we obtain $\int_{||x||^2<\Lambda} d^n x\, e^{i||x||^2} \propto \int_0^\infty d\Lambda\, \Lambda^{n-1} e^{i\Lambda^2}$ – here the amplitude of oscillations in Λ does not decrease for $n = 2$ and actually increases for $n \geq 3$. The solution is to say that the limit $\Lambda \to \infty$ exists not pointwise, but in the distributional sense, i.e. convolving with a smoothing function ρ (that is, replacing $\lim_{\Lambda\to\infty} \int^{\Lambda} \cdots$ with $\lim_{\Lambda_0\to\infty} \int d\Lambda\ \rho(\frac{\Lambda}{\Lambda_0}) \int^{\Lambda} \cdots$). This is equivalent to

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[19] Sometimes also called Poisson integral.

[20] (32) is proven e.g. by making an orthogonal change of coordinates on $\mathbb{R}^n$ which diagonalizes Q; then the integration variables split and the problem is reduced to a product of 1-dimensional Gaussian integrals.

replacing an abrupt cut-off of the integration domain by "smeared cut-off" (e.g. multiplying the integrand by a bump function which realizes the smeared cut-off). A technically convenient way of arranging a smeared cut-off is simply to multiply the integrand by $e^{-\epsilon Q_0(x,x)}$ for some fixed positive-definite Q_0, and then take the limit $\epsilon \to 0$. I.e. the meaning of the integral (34) is:

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^n} d^n x\; e^{iQ(x,x)-\epsilon\, Q_0(x,x)}$$

The integral now is absolutely convergent $\forall \epsilon > 0$; the result is independent of Q_0 and is equal to the r.h.s. of (34). Note that, for Q of diagonal (Morse) form $Q = \sum_{i=1}^p x_i^2 - \sum_{i=p+1}^n x_i^2$, our regularization is equivalent to infinitesimally rotating the integration contour for x_i counterclockwise for $i = 1, \ldots, p$ and clockwise for $i = p+1, \ldots, n$.

3.2. **Stationary phase formula.**

Theorem 3.2. Let X be an oriented n-manifold, $\mu \in \Omega^n_c(X)$ a top-degree form with compact support, $f \in C^\infty(X)$ a smooth function which has only non-degenerate critical points $x_0^{(1)}, \ldots, x_0^{(m)}$ on $\mathrm{Supp}\,\mu \subset X$. Then the integral $I(k) := \int_X \mu\, e^{ikf(x)}$ has the following asymptotics at $k \to \infty$:

(35)

$$I(k) \sim \sum_{x_0\in\{\text{crit. points of } f\}} e^{ikf(x_0)} \left(\frac{2\pi}{k}\right)^{\frac{n}{2}} |\det f''(x_0)|^{-\frac12} \cdot e^{\frac{\pi i}{4}\operatorname{sign} f''(x_0)} \cdot \mu_{x_0} + O(k^{-\frac{n}{2}-1})$$

Here:

- We assume that we have chosen, arbitrarily, a coordinate chart $(y_1, \ldots, y_n)$ near each critical point x_0.
- Critical point x_0 of f is said to be non-degenerate if the Hessian matrix $f''(x_0) = \left.\frac{\partial^2}{\partial y_i \partial y_j}\right|_{y=0} f$ is non-degenerate. (In particular, a non-degenerate critical point has to be isolated and therefore there can be only finitely many of them on the compact $\mathrm{Supp}\,\mu$.)
- μ_{x_0} is the density of μ at x_0 in local coordinates $y_1, \ldots, y_n$. I.e., if μ is written in local coordinates as $\mu = \rho(y) dy_1 \cdots dy_n$ for some local density $\rho(y)$, then $\mu_{x_0} := \rho(y = 0)$.

Remark 3.3. Note that, although the Hessian $f''(x_0)$ and the density of μ at a critical point depend on the choice of local coordinates near x_0, this dependence cancels out in the r.h.s. of (35): if we change the coordinate chart $(y_1, \ldots, y_n) \mapsto (y_1', \ldots, y_n')$, then $\det f''(x_0)$ changes by the square of the Jacobian of the transformation at $y = 0$ (we assume that charts are centered at x_0), and μ_{x_0} changes by the Jacobian. Thus the product $|\det f''(x_0)|^{-\frac12} \cdot \mu_{x_0}$ is, in fact, invariant.

Lemma 3.4. Let $g \in C^\infty_c(\mathbb{R})$ a compactly-supported function on $\mathbb{R}$ and let

$$I(k) := \int_{-\infty}^{\infty} dx\; g(x) e^{ikx}$$

Then $I(k)$ decays faster than any power of k as $k \to \infty$,

$$I(k) \underset{k\to\infty}{\sim} O(k^{-\infty})$$

In other words, for any N there exists some $C_N \in \mathbb{R}$ such that $|k^N I(k)| \leq C_N$.

Proof. We have

$$k^N \cdot I(k) = \int_{-\infty}^{\infty} dx\, g(x) \left(-i\frac{\partial}{\partial x}\right)^N e^{ikx} \underset{\mathrm{Stokes}'}{=} \int_{-\infty}^{\infty} dx\, e^{ikx} \left(i\frac{\partial}{\partial x}\right)^N g(x)$$

In the second step we have integrated by parts N times, removing derivatives from the exponential and putting them on g. The integral on the r.h.s. is certainly bounded by $\int_{-\infty}^{\infty} dx\, |\partial^N g(x)| =: C_N$. This proves the Lemma. □

Lemma 3.5. Let $g \in C_c^\infty(\mathbb{R}^n)$ and let $f \in C^\infty(\mathbb{R}^n)$ with no critical points on $\mathrm{Supp}\, g \subset \mathbb{R}^n$. Then

$$I(k) := \int_{\mathbb{R}^n} d^n x\; g(x) e^{ikf(x)} \underset{k\to\infty}{\sim} O(k^{-\infty})$$

Proof. Since f has no critical points on $\mathrm{Supp}\, g$, it defines a submersion $f : \mathrm{Supp}\, g \to \mathbb{R}$. Thus, the pushforward (fiber integral) $f_*(d^n x\, g(x)) \in \Omega_c^1(\mathbb{R})$ is a smooth 1-form on $\mathbb{R}$. Thus, we can calculate $I(k)$ by first integrating over the level sets of f, $f(x) = y$ (the same as computing the pushforward f_*) and then integrating over the values y of f:

$$I(k) = \int_{\mathbb{R}} e^{iky} f_*(g\, d^n x)$$

This integral behaves as $O(k^{-\infty})$ by Lemma 3.4. □

Lemma 3.6. Let $g \in C_c^\infty(\mathbb{R}^n)$ such that g and its derivatives of all orders vanish at $x = 0$. Let $Q(x, x)$ be a non-degenerate quadratic form on $\mathbb{R}^n$. Then:

$$I(k) := \int_{\mathbb{R}^n} d^n x\; g(x)\, e^{ikQ(x,x)} \underset{k\to\infty}{\sim} O(k^{-\infty})$$

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Proof. First consider the case when Q is *positive-definite*. Then $Q : \mathbb{R}^n - \{0\} \to (0, \infty)$ is a submersion; we can calculate $I(k)$, similarly to the proof of Lemma 3.5, by integrating first over the level sets of Q and then over values y of Q:

$$I(k) = \int_0^\infty e^{iky} Q_*(d^n x\; g(x))$$

The pushforward $Q_*(d^n x\; g(x)) \in \Omega_c^1[0, \infty)$ has vanishing ∞-jet at $y = 0$ (because of the assumption on ∞-jet of g at the origin $x = 0$). Thus one can repeat the proof of Lemma 3.4 and no boundary terms at $y = 0$ will appear when performing integration by parts multiple times. Thus we obtain $I(k) \underset{k\to\infty}{\sim} O(k^{-\infty})$.

For Q not positive-definite, we can assume without loss of generality (by making a linear change of coordinates) that Q has Morse form $Q = \sum_{i=1}^p x_i^2 - \sum_{i=p+1}^n x_i^2$. We can present $g(x)$ as a limit of finite sums of functions of form $g'(x_1, \ldots, x_p) \cdot g''(x_{p+1}, \ldots, x_n)$ (since $C_c^\infty(\mathbb{R}^p) \otimes C_c^\infty(\mathbb{R}^{n-p})$ is dense in $C_c^\infty(\mathbb{R}^n)$). For such products we have $\int_{\mathbb{R}^n} d^n x\; g' \cdot g''\, e^{ikQ(x,x)} = \left(\int_{\mathbb{R}^p} dx_1 \cdots dx_p g'(x_1, \ldots, x_p) e^{ik(x_1^2 + \cdots + x_p^2)}\right) \cdot \left(\int_{\mathbb{R}^{n-p}} dx_{p+1} \cdots dx_n g''(x_{p+1}, \ldots, x_n) e^{-ik(x_{p+1}^2 + \cdots + x_n^2)}\right) \sim O(k^{-\infty})$ by the result in the positive-definite case. One can check that the bound we get is uniform and one can pass to the limit.

□

Corollary 3.7. Let $g \in C_c^\infty(\mathbb{R}^n)$ and let $Q(x,x)$ be a non-degenerate quadratic form on $\mathbb{R}^n$. Let

$$I(k) := \int_{\mathbb{R}^n} d^n x\; g(x)\; e^{ikQ(x,x)} \tag{36}$$

Then:

(i) $I(k)$ modulo $O(k^{-\infty})$-terms depends only on the ∞-jet of g at $x=0$.
(ii) In particular $I(k) = g(0)\cdot \left(\frac{\pi}{k}\right)^{\frac{n}{2}} |\det Q|^{-\frac{1}{2}} e^{\frac{\pi i}{4}\operatorname{sign} Q} + O(k^{-\frac{n}{2}-1})$

Proof. (i) is an immediate consequence of Lemma 3.6.

For (ii), write $g(x) = g(0) + (x, dg(0)) + R(x)$ – a constant term, a linear term (which, being an odd function of x, vanishes when integrated with $e^{iQ(x,x)}$), and the "error term" which has zero of order two at $x=0$. Thus, we have

$$I(k) = g(0)\cdot \int_{\mathbb{R}^n} d^n x\; e^{ikQ(x,x)} + \underbrace{\int_{\mathbb{R}^n} d^n x\; R(x) e^{ikQ(x,x)}}_{r(k)}$$

The first term on the r.h.s. is the standard Fresnel integral and we need to show that the error $r(k)$ behaves as $O(k^{-\frac{n}{2}-1})$. Write

$$r(k) = \int_{\mathbb{R}^n} d^n x\; R(x) e^{ikQ(x,x)} = k^{-\frac{n}{2}-1}\int_{\mathbb{R}^n} d^n y\; kR\left(\frac{y}{\sqrt{k}}\right) e^{iQ(y,y)}$$

Here we made a change $x = \frac{y}{\sqrt{k}}$. Integrand on the r.h.s. has a well-defined limit as $k\to\infty$ (since R has a zero of order 2 at the origin) and converges to $e^{iQ(y,y)}$ times some quadratic form in y.[21] Thus $r(k)$ behaves as $k^{-\frac{n}{2}-1}$ times an integral which converges in the sense of Remark 3.1. □

The general idea is that in the integral (36) one can replace g with a piece of its Taylor series at the origin and the error will be estimated by the contribution of the first discarded term of the Taylor series (or the next one if the discarded term was of odd degree).

An afterthought: better/cleaner way (instead of Lemma 3.6 and Corollary 3.7).

Lemma 3.8. Let g be a Schwartz class function on $\mathbb{R}^n$, let g_N be the Taylor series for g truncated at N-th order for arbitrary N, so that $h(x) := g(x) - g_N(x) \underset{x\to 0}{\sim} O(x^{N+1})$, and let $Q(x,x)$ be a non-degenerate quadratic form on $\mathbb{R}^n$. Then

$$I(k) := \int_{\mathbb{R}^n} d^n x\; h(x) e^{ikQ(x,x)} \underset{k\to\infty}{\sim} O(k^{-\frac{n}{2}-\left[\frac{N+2}{2}\right]}) \tag{37}$$

Proof. Consider the differential operator $\mathcal{D} = -\frac{i}{2}\sum_{j,k=1}^n (Q^{-1})_{jk}\frac{1}{x_j}\frac{\partial}{\partial x_k}$ and its transpose $\mathcal{D}^T = \frac{i}{2}\sum_{j,k=1}^n (Q^{-1})_{jk}\frac{\partial}{\partial x_j}\frac{1}{x_k}$, acting on functions on $\mathbb{R}^n$. Operator $\mathcal{D}$ is constructed so that we have the following property: $\mathcal{D}\, e^{ikQ(x,x)} = k\cdot e^{ikQ(x,x)}$. Thus, multiplying $I(k)$ by a power of k, we have

$$k^m I(k) = \int_{\mathbb{R}^n} d^n x\; h(x)\,\mathcal{D}^m e^{ikQ(x,x)} = \int_{\mathbb{R}^n} d^n x\; e^{ikQ(x,x)} (\mathcal{D}^T)^m h(x) \tag{38}$$

Where we have integrated by parts m times (we think of point $x=0$ as being punctured out of the integration domain). Note that $(\mathcal{D}^T)^m h(x) \underset{x\to 0}{\sim} O(x^{N+1-2m})$

[21] This is a bit sketchy: one has to explain why integration and limit can be interchanged; see a better argument below - Lemma 3.8.

and thus on the r.h.s. of (38) we get an integrable singularity at the origin iff $N+1-2m > -n$ (e.g. $m = \left[\frac{N+n}{2}\right]$ satisfies this inequality); convergence at infinity holds in the sense of Remark 3.1. Thus we have proven that $I(k) \sim O(k^{-\left[\frac{N+n}{2}\right]})$.

This is a slightly weaker estimate than claimed in (37); one can get the improved estimate considering a truncation of the Taylor series for g three steps further, g_{N+3}. Then, by the result that we have proven,

$$\int_{\mathbb{R}^n} d^n x\ (g - g_{N+2}) e^{ikQ(x,x)} \sim O(k^{-\left[\frac{N+n+3}{2}\right]}) \tag{39}$$

(which is a better or equivalent estimate to the r.h.s. of (37)). On the other hand $g_N - g_{N+3}$ is a polynomial in x containing monomials of degrees $N+1$, $N+2$ and $N+3$ only. Thus,

$$\int_{\mathbb{R}^n} d^n x\ (g_N - g_{N+2}) e^{ikQ(x,x)} = C_{N+1} k^{-\frac{n+N+1}{2}} + C_{N+2} k^{-\frac{n+N+2}{2}} + C_{N+3} k^{-\frac{n+N+3}{2}} \tag{40}$$

where the constant C_{N+j} vanishes if $N+j$ is odd for $j = 1, 2, 3$. Thus, (39) and (40) together imply (37). □

In particular: (ii) of Corollary 3.7 is the $N = 0$ case of (37). Also note that Lemma 3.6 is a special case of the new Lemma (for g with vanishing jet at the origin and N arbitrarily large) - here we avoid splitting coordinates into positive and negative eigenspaces of Q (and the painful discussion of approximating g by products) by the trick with the differential operator $\mathcal{D}$. **/End afterthought.**

Proof of Theorem 3.2. We can assume without loss of generality that X is compact (since we only care about Supp g anyway which is compact by assumption). Choose a covering $\{U_\alpha\}$ of X by open subsets such that

- each U_α contains at most one critical point of f,
- each critical point of f is contained in exactly one U_α.

Choose a partition of unity $\{\psi_\alpha \in C^\infty(X)\}$ subordinate to the covering $\{U_\alpha\}$, i.e.

- Supp $\psi_\alpha \subset U_\alpha$,
- $\psi_\alpha \geq 0$,
- $\sum_\alpha \psi_\alpha = 1$.

Then $I(k) = \sum_\alpha I_\alpha(k)$ with $I_\alpha(k) = \int_{U_\alpha} \mu \psi_\alpha(x)\, e^{ikf(x)}$. We should consider two case:

(i) U_α does not contain critical points of f. Then $I_\alpha(k) \sim O(k^{-\infty})$ by Lemma 3.5.

(ii) U_α contains a critical point x_0 of f. By Morse Lemma, we can introduce local coordinates $(y_1, \ldots, y_n)$ on U_α such that $f = f(x_0) + \underbrace{y_1^2 + \cdots + y_p^2 - y_{p+1}^2 - \cdots - y_n^2}_{Q(y,y)}$.

Then, by (ii) of Corollary 3.7 (or by Lemma 3.8 for $N = 0$), we have

$$I_\alpha(k) = \int_{\mathbb{R}^n} d^n y\ \rho(y)\ \psi_\alpha\ e^{ikf(x_0) + ikQ(y,y)} \sim$$
$$\sim \rho(0) e^{ikf(x_0)} \left(\frac{\pi}{k}\right)^{\frac{n}{2}} |\det Q|^{-\frac{1}{2}} e^{\frac{\pi i}{4} \operatorname{sign} Q} + O(k^{-\frac{n}{2}-1})$$

where $d^n y\,\rho(y)$ is μ expressed in coordinates y. Note that $Q_{ij} = \frac{1}{2}\frac{\partial^2}{\partial y_i \partial y_j} f$, thus

$$I_\alpha(k) \sim \mu_{x_0} e^{ikf(x_0)} \left(\frac{2\pi}{k}\right)^{\frac{n}{2}} |\det f''(x_0)|^{-\frac{1}{2}} e^{\frac{\pi i}{4}\operatorname{sign} f''(x_0)} + O(k^{-\frac{n}{2}-1})$$

Summing over α, we obtain the stationary phase formula for $I(k)$. Note that, by Remark 3.3, it does not matter that we have chosen the Morse chart around every critical point: the result is independent of this choice.

□

3.3. **Gaussian expectation values. Wick's lemma.** Consider normalized expectation values with respect to Gaussian measure

$$\ll p \gg := \frac{\int_{\mathbb{R}^n} d^n x\, e^{-\frac{1}{2}Q(x,x)} \cdot p(x)}{\int_{\mathbb{R}^n} d^n x\, e^{-\frac{1}{2}Q(x,x)}} \tag{41}$$

with $Q(x,x) = \sum_{i,j} Q_{ij} x_i x_j$ a positive-definite quadratic form on $\mathbb{R}^n$, for $p(x)$ a polynomial on $\mathbb{R}^n$.

Definition 3.9. For H a finite set with even number of elements we call partitions of H into two-element subsets *perfect matchings* on H.

Note that a perfect matching is the same as an involution γ on H with no fixed points. Then the two-element subsets are the orbits of γ.

Example 3.10. On the set $\{1,2,3,4\}$ there exist three different perfect matchings:

$$\{1,2\}\cup\{3,4\}, \qquad \{1,3\}\cup\{2,4\}, \qquad \{1,4\}\cup\{2,3\}$$

More generally, on the set of order $2m$ there are $(2m-1)!! = 1\cdot 3\cdot 5\cdots(2m-1)$ perfect matchings.[22]

The following lemma allows one to calculate the expectation $\ll p \gg$ for any monomial (and hence every polynomial) p.

Lemma 3.11 ("Wick's lemma"). [23]

(i) $\ll 1 \gg = 1$.
(ii) $\ll x_{i_1}\cdots x_{i_{2m-1}} \gg = 0$.
(iii) $\ll x_i x_j \gg = (Q^{-1})_{ij}$ – the (i,j)-th matrix element of the inverse matrix to the matrix of the quadratic form $Q(x,x)$.
(iv)

$$\ll x_{i_1}\cdots x_{i_{2m}} \gg = \tag{42}$$
$$= \sum_{\text{perfect matchings } \{1,\ldots,2m\}=\{a_1,b_1\}\cup\cdots\cup\{a_m,b_m\}} \underbrace{\ll x_{i_{a_1}} x_{i_{b_1}} \gg}_{(Q^{-1})_{i_{a_1} i_{b_1}}} \cdot \cdots \cdot \underbrace{\ll x_{i_{a_m}} x_{i_{b_m}} \gg}_{(Q^{-1})_{i_{a_m} i_{b_m}}}$$

[22] Indeed, the first element of the set has to be matched with one of $2m-1$ other elements, first element among those left has to be matched with one of $(2m-3)$ remaining elements etc.

[23] The original Wick's lemma, though a similar statement, was formulated in the context of expressing words constructed out of creation and annihilation operators in terms of normal ordering.

Remark 3.12. We can identify perfect matchings on the set $H = \{1, \ldots, 2m\}$ with elements of the quotient of the symmetric group S_{2m} of permutations of H by the group of permutations of two-element subsets constituting the partition and transpositions of the elements inside the two-element subsets. In other words, the set of perfect matchings can be presented as $S_{2m}/(S_m \ltimes \mathbb{Z}_2^m)$. Thus, in particular, expectation value (42) can be written as

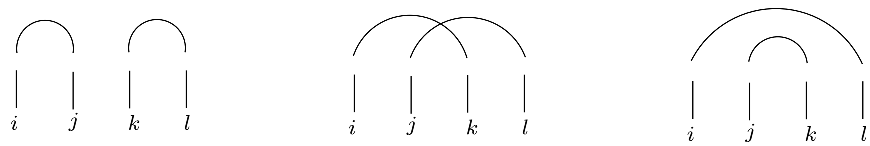
$$\ll x_{i_1} \cdots x_{i_{2m}} \gg = \sum_{\sigma \in S_{2m}/(S_m \ltimes \mathbb{Z}_2^m)} \ll x_{i_{\sigma_1} i_{\sigma_2}} \gg \cdot \cdots \cdot \ll x_{i_{\sigma_{2m-1}} i_{\sigma_{2m}}} \gg \tag{43}$$

Example 3.13.

$$\ll x_i x_j x_k x_l \gg = \ll x_i x_j \gg \cdot \ll x_k x_l \gg + \ll x_i x_k \gg \cdot \ll x_j x_l \gg + \ll x_i x_l \gg \cdot \ll x_j x_k \gg$$

Pictorially, the three terms on the r.h.s. can be drawn as follows:

(44)



Example 3.14. From the count of perfect matchings and Wick's formula, we deduce, for 1-dimensional moment of Gaussian measure $dx\, e^{-\frac{x^2}{2}}$, that

$$\ll x^{2m} \gg = (2m-1)!!$$

or equivalently

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2}} x^{2m} = \sqrt{2\pi} \cdot (2m-1)!!$$

Proof of Lemma 3.11. Item (i) is obvious, and (ii) also (integrand in the numerator of (41) is odd with respect to $x \to -x$, hence the integral is zero). For (iii) and (iv), consider an auxiliary integral

$$W(J) := \int_{\mathbb{R}^n} d^n x\, e^{-\frac{1}{2}Q(x,x) + \langle J, x\rangle} \tag{45}$$

with $J \in \mathbb{R}^n$ the *source*. The integral is easily calculated by completing the expression in the exponential to the full square:

$$W(J) = \int_{\mathbb{R}^n} d^n x\, \underbrace{e^{-\frac{1}{2}Q(x,x) + \langle J,x\rangle - \frac{1}{2}\langle J, Q^{-1}J\rangle}}_{e^{-\frac{1}{2}Q(x-Q^{-1}J,\, x-Q^{-1}J)}} \cdot e^{\frac{1}{2}\langle J, Q^{-1}J\rangle} = \tag{46}$$
$$= e^{\frac{1}{2}\langle J, Q^{-1}J\rangle} \cdot \int_{\mathbb{R}^n} d^n y\, e^{-\frac{1}{2}Q(y,y)} = e^{\frac{1}{2}\langle J, Q^{-1}J\rangle} \cdot (2\pi)^{\frac{n}{2}} (\det Q)^{-\frac{1}{2}}$$

Here in the second step we made a shift $x \mapsto y = x - Q^{-1}J$.

From definition (45), we have

$$\ll x_{i_1} \cdots x_{i_{2m}} \gg = \frac{1}{W(0)} \left| \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_{2m}}} W(J) \right|_{J=0} = \tag{47}$$
$$= \frac{1}{2^m m!} \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_{2m}}} \underbrace{Q^{-1}(J,J) \cdot \cdots \cdot Q^{-1}(J,J)}_{\left(\sum_{j_1,k_1} Q^{-1}_{j_1 k_1} J_{j_1} J_{k_1}\right) \cdots \left(\sum_{j_m,k_m} Q^{-1}_{j_m k_m} J_{j_m} J_{k_m}\right)}$$

Here in the second step we replaced $W(J)$ by m-th term of the Taylor series for the exponential in the explicit formula (46) for $W(J)$ (lower terms do not contribute because they are killed by the $2m$ derivatives in the source J and higher terms do not contribute as tehy are killed by setting $J = 0$ after taking the derivatives). Then (iv) follows by evaluating the multiple derivative in the source in (47) by Leibniz rule. Item (iii) is the trivial $m = 1$ case of this computation.

□

Remark 3.15. In a slightly more invariant language, replace $\mathbb{R}^n$ by an abstract finite-dimensional $\mathbb{R}$-vector space V. Our input is a positive-definite quadratic form $Q \in \mathrm{Sym}^2 V^*$. We are interested in the map $\ll - \gg$: $\mathrm{Sym}V^* \to \mathbb{R}$ sending

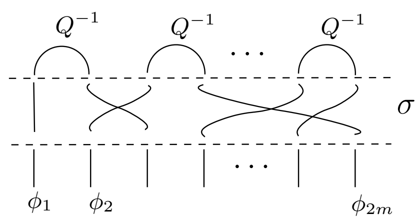
$$p \quad \mapsto \quad \ll p \gg = \frac{\int_V \mu\, e^{-\frac{1}{2}Q}\; p}{\int_V \mu\, e^{-\frac{1}{2}Q}}$$

with $\mu \in \wedge^{\mathrm{top}} V^*$ a fixed constant volume form (irrelevant for the normalized expectation values). Then the Wick's lemma (43) can be formulated as

$$\ll \phi_1 \odot \cdots \odot \phi_{2m} \gg = \sum_{\sigma \in S_{2m}/(S_m \ltimes \mathbb{Z}_2^m)} \langle \sigma \circ (Q^{-1})^{\otimes m}, \phi_1 \otimes \cdots \otimes \phi_{2m} \rangle \tag{48}$$

Here $\phi_1, \ldots, \phi_{2m} \in V^*$ are linear functions on V, $\odot$ is the commutative product in $\mathrm{Sym}V^*$. We understand the inverse to Q as an element in the symmetric square of V, $Q^{-1} \in \mathrm{Sym}^2 V$; σ acts on $V^{\otimes 2m}$ by permuting the copies of V; the pairing in the r.h.s. is the pairing between $V^{\otimes 2m}$ and $(V^*)^{\otimes 2m}$

Remark 3.16. Another visualization (as opposed to (44)) of the terms on the r.h.s. of Wick's lemma, corresponding to the presentation (48) is like as follows:



Here the lower strip presents $\phi_1 \otimes \cdots \otimes \phi_{2m} \in (V^*)^{\otimes 2m}$, the upper strip presents pairing with $(Q^{-1})^{\otimes m} \in V^{\otimes 2m}$ and middle strip presents the action of σ by permuting the V-factors (if we read the diagram from top to bottom), or equivalently the action of σ^{-1} by permuting V^*-factors (if we read te diagram from bottom to top).

Lecture 8, 09/19/2016.

Remark 3.17. In the setup of Remark 3.15, the value of the Gaussian integral itself, $\int_V \mu\, e^{-\frac{1}{2}Q}$, can be understood as follows (without referring explicitly to the matrix of Q or, in other words, without identifying bilinears on V with endomorphisms). To $Q \in \mathrm{Sym}^2 V^*$, there is an associated sharp map $Q^\# : V \to V^*$. Raising it to the maximal exterior power, we obtain a map of determinant lines $\wedge^n Q^\# : \wedge^n V \to \wedge^n V^*$ (with $n = \dim V$) or equivalently, dualizing the domain line and putting it to the right side, $\mathrm{Det}\, Q := \wedge^n Q^\# \in (\wedge^n V^*)^{\otimes 2}$.[24] Thus, $\mathrm{Det}\, Q$ in

[24] Here we implicitly used the identification $(\wedge^n V)^* \cong \wedge^n V^*$. It is induced by the pairing $\wedge^n V \otimes \wedge^n V^* \to \mathbb{R}$ which sends $(v_1 \wedge \cdots \wedge v_n) \otimes (\theta_1 \wedge \cdots \wedge \theta_n) \mapsto \det \left(\langle v_i, \theta_j \rangle \right)_{i,j=1}^n$, where on the r.h.s.$\langle , \rangle$ is the canonical pairing between V and V^*.

this context is not a number, but an element of the line $(\wedge^n V^*)^{\otimes 2}$. (Whenever a basis in V is chosen, we have a trivialization $(\wedge^n V^*)^{\otimes 2} \simeq \mathbb{R}$, and then $\mathrm{Det}\, Q$ gets assigned the number value, which coincides with the determinant of the matrix of the bilinear Q in the chosen basis). Note that $\mu^{\otimes 2}$ is a nonzero element of the same line, thus we can form a quotient $\frac{\mathrm{Det}\, Q}{\mu^{\otimes 2}} \in \mathbb{R}$. Value of the Gaussian integral is then

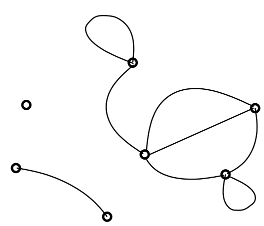
$$\int_V \mu\, e^{-\frac{1}{2}Q} = (2\pi)^{\frac{n}{2}} \left(\frac{\mathrm{Det}\, Q}{\mu^{\otimes 2}}\right)^{-\frac{1}{2}}$$

3.4. **A reminder on graphs and graph automorphisms.**

Definition 3.18. A graph is the following set of data:

- A set V of *vertices*.
- A set HE of *half-edges*.
- A map $i : HE \to V$ – *incidence*.
- A perfect matching E on HE, i.e. a partition of E into two-element subsets – *edges*. Put differently, we have a fixed-point-free involution γ on HE and its orbits are the edges.

We will only consider *finite* graphs, i.e. with V and HE finite. Here is a picture of a generic graph.



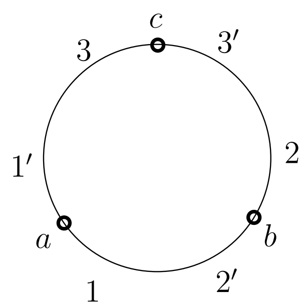
Definition 3.19. For $v \in V$ a vertex, one calls $i^{-1}(v) \subset HE$ the *star* (or *corolla*) of v and the number of incident halh-edges to the vertex, $\# i^{-1}(v)$, is called the *valency* of v.

Definition 3.20. For two graphs $\Gamma = (V, HE, i, E)$, $\Gamma' = (V', HE', i', E')$, a *graph isomorphism* $\Gamma \xrightarrow{\sim} \Gamma'$ is a pair of bijections $\sigma_V : V \xrightarrow{\sim} V'$, $\sigma_{HE} : HE \xrightarrow{\sim} HE'$ commuting with the incidence maps (satisfying $i' \circ \sigma_{HE} = \sigma_V \circ i$) and preserving the partition into edges (i.e. $\gamma' \circ \sigma_{HE} = \sigma_{HE} \circ \gamma$ with γ, γ' the respective involutions on half-edges).

Example 3.21. Vertices: $V = \{a, b, c\}$, half-edges: $HE = \{1, 1', 2, 2', 3, 3'\}$, incidence:

$$i : \begin{array}{ccc} 1 & \mapsto & a \\ 1' & \mapsto & a \\ 2 & \mapsto & b \\ 2' & \mapsto & b \\ 3 & \mapsto & c \\ 3' & \mapsto & c \end{array}$$

Edges: $E = \{1, 2'\} \cup \{2, 3'\} \cup \{3, 1'\}$. Equivalently, the invloution is $\gamma : 1 \leftrightarrow 2', 2 \leftrightarrow 3', 3 \leftrightarrow 1'$. Here is the picture:



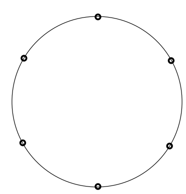
Example of an automorphism of this graph:

$$\sigma_V : (a, b, c) \mapsto (b, a, c), \quad \sigma_{HE} : (1, 1', 2, 2', 3, 3') \mapsto (2', 2, 1', 1, 3', 3)$$

(Check explicitly that this pair of permutations commutes with incidence maps and with involutions!) On the picture of the graph above, this automorphism corresponds to reflection w.r.t. the vertical axis.

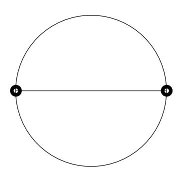
We will be interested in the group of automorphisms $\mathrm{Aut}(\Gamma)$ of a graph Γ.

Example 3.22 (Automorphism groups)**.** (i) A "polygon graph" with $n \geq 3$ vertices and n edges:



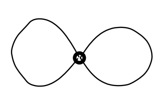
Automorphism group: $\mathrm{Aut}(\Gamma) = \mathbb{Z}_2 \ltimes \mathbb{Z}_n$.

(ii) "Theta graph":



Automorphism group: $\mathrm{Aut}(\Gamma) = \mathbb{Z}_2 \times S_3$.

(iii) "Figure-eight graph":



Automorphism group: $\mathrm{Aut}(\Gamma) = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Remark 3.23. A graph automorphism has to preserve valencies of vertices, in particular it permutes vertices of any given valency and maps the star of a source vertex to the star of a target vertex (via some permutation). Therefore, for a graph Γ which has V_d vertices of valency d for $d = 0, \ldots, D$, the automorphism group can be seen as a subgroup of permutations of vertices for each valency d and permutations of incident half-edges for each vertex:

$$\mathrm{Aut}(\Gamma) \subset \prod_{d=0}^{D} S_{V_d} \ltimes S_d^{\times V_d}$$

Remark 3.24. Graphs naturally form a groupoid, with morphisms given by graph isomorphisms. Consider the restriction $\mathrm{Graph}_{V_0,\ldots,V_D}$ of this groupoid to graphs with number of vertices of valency d fixed to V_d for $d=0,\ldots,D$ (and no vertices of higher valency than D). One can realize objects of $\mathrm{Graph}_{V_0,\ldots,V_D}$ as all $(2m-1)!!$ (for $2m=\sum_{d=1}^D d\cdot V_d$) perfect matchings on the set of half-edges constituting the given vertex stars. The total group of isomorphisms is then $\prod_{d=0}^D S_{V_d}\ltimes S_d^{\times V_d}$. Thus the groupoid volume of $\mathrm{Graph}_{V_0,\ldots,V_D}$ is:

$$\mathrm{Vol}\left(\mathrm{Graph}_{V_0,\ldots,V_D}\right) = \underbrace{\sum_\Gamma \frac{1}{|\mathrm{Aut}(\Gamma)|}}_{\mathrm{Vol}\ \pi_0\left(\mathrm{Graph}_{V_0,\ldots,V_D}\right)} = \frac{(2m-1)!!}{\prod_{d=0}^D V_d!\cdot d!^{V_d}} \tag{49}$$

where Γ runs over *isomorphism classes* of graphs; on the r.h.s. the numerator and denominator are the numbers of objects and morphisms of $\mathrm{Graph}_{V_0,\ldots,V_D}$, respectively.

Remark 3.25. One can also define graphs as 1-dimensional CW complexes. From this point of view, the automorphism group of Γ is π_0 of the group of cellular homeomorphisms of Γ viewed as a CW complex.

3.5. **Back to integrals: Gaussian expectation value of a product of homogeneous polynomials.** Fix $Q\in\mathrm{Sym}^2V^*$ a positive-definite quadratic form on $V=\mathbb{R}^n$. Let $\Psi_a\in\mathrm{Sym}^{d_a}V^*$ for $a=1,\ldots,r$ be a collection of homogeneous polynomials of degrees $d_1,\ldots,d_r$ on V. In coordinates, $\Psi_a=\sum_{i_1,\ldots,i_{d_a}=1}^n(\Psi_a)_{i_1\cdots i_{d_a}}x_{i_1}\cdots x_{i_{d_a}}$. Consider the Gaussian expectation value $\ll \frac{1}{d_1!}\Psi_1\cdots\frac{1}{d_r!}\Psi_r \gg$. Denote $2m=\sum_{a=1}^r d_a$. Also denote

$$\mathrm{Matchings}_{2m}:=S_{2m}/(S_m\ltimes\mathbb{Z}_2^m)$$

the set of perfect mathcings on $2m$ elements. We have the following:

$$\begin{aligned}\ll \frac{1}{d_1!}\Psi_1\cdots\frac{1}{d_r!}\Psi_r \gg &= \frac{1}{d_1!\cdots d_r!}\sum_{\sigma\in\mathrm{Matchings}_{2m}}\langle\sigma\circ(Q^{-1})^{\otimes m},\Psi_1\otimes\cdots\otimes\Psi_r\rangle\\ &=\sum_{[\sigma]\in(\prod_{a=1}^r S_{d_a})\backslash\,\mathrm{Matchings}_{2m}}\frac{1}{|\mathrm{Stab}_{[\sigma]}|}\langle\sigma\circ(Q^{-1})^{\otimes m},\Psi_1\otimes\cdots\otimes\Psi_r\rangle\end{aligned}$$

Here in the first step we have applied the Wick's lemma to calculate the Gaussian expectation value and in the second step we collected similar terms in the sum. In the second sum $[\sigma]$ runs over classes of perfect matchings under the action of $\prod_{a=1}^r S_{d_a}\subset S_{2m}$ (in other words, $[\sigma]$ is a class in the two-sided quotient of the symmetric group, $[\sigma]\in(\prod_{a=1}^r S_{d_a})\backslash S_{2m}/(S_m\ltimes\mathbb{Z}_2^m)$). This action is not free and has stabilizer subgroups $\mathrm{Stab}_{[\sigma]}\subset\prod_{a=1}^r S_{d_a}$. Note that the coefficient $\frac{1}{|\mathrm{Stab}_{[\sigma]}|}$ arises as

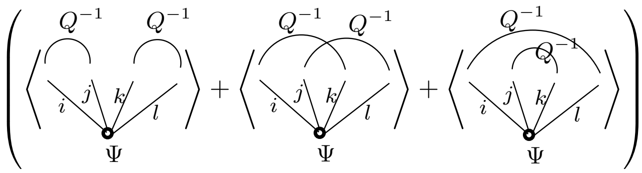
$$\frac{1}{|\mathrm{Stab}_{[\sigma]}|}=\frac{\#\{\text{orbit of }\sigma\text{ under }S_{d_1}\times\cdots\times S_{d_r}\text{−action}\}}{|S_{d_1}\times\cdots\times S_{d_r}|}$$

where the denominator is $d_1!\cdots d_r!$.

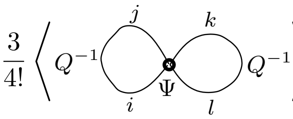
Example 3.26. Let $\Psi=\sum_{i,j,k,l=1}^n\Psi_{ijkl}\,x_ix_jx_kx_l\in\mathrm{Sym}^4V^*$ be a quartic polynomial. Then we have

$$\ll \frac{1}{4!}\Psi \gg = \frac{1}{4!}\sum_{\sigma\in\mathrm{Matchings}_4} \ll \sigma\circ (Q^{-1})^{\otimes 2},\Psi \gg =$$

$$= \frac{1}{4!}$$



$$= \frac{3}{4!}$$



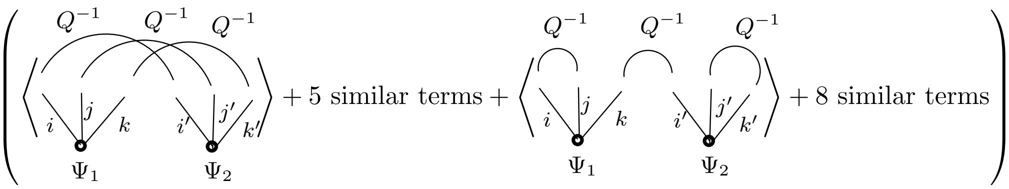
$$= \frac{1}{8}\sum_{i,j,k,l=1}^{n} \Psi_{ijkl}\,(Q^{-1})_{ij}(Q^{-1})_{kl}$$

Here all three matchings give the same contribution to the expectation value (correspondingly, $S_4\backslash \mathrm{Matchings}_4 \ni [\sigma]$ consists of a single class).

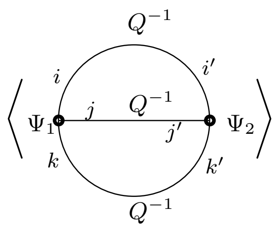
Example 3.27. Let $\Psi_1 = \sum_{i,j,k=1}^{n} (\Psi_1)_{ijk}\, x_i x_j x_k,\ \Psi_2 = \sum_{i',j',k'=1}^{n} (\Psi_2)_{i'j'k'}\, x_{i'} x_{j'} x_{k'} \in \mathrm{Sym}^3 V^*$ be two cubic polynomials. Then we have

$$\ll \frac{1}{3!}\Psi_1 \cdot \frac{1}{3!}\Psi_2 \gg = \frac{1}{3!\,3!}\sum_{\sigma\in\mathrm{Matchings}_6} \ll \sigma\circ (Q^{-1})^{\otimes 3}, \Psi_1\otimes\Psi_2 \gg =$$

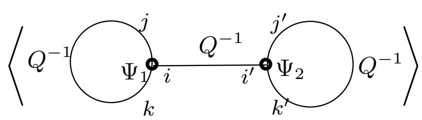
$$= \frac{1}{3!\,3!}$$



$$= \frac{6}{3!3!}$$



$$+ \frac{9}{3!3!}$$



$$= \frac{1}{6}\sum_{i,j,k,i',j',k'=1}^{n} (\Psi_1)_{ijk}(\Psi_2)_{i'j'k'}(Q^{-1})_{ii'}(Q^{-1})_{jj'}(Q^{-1})_{kk'} +$$

$$+ \frac{1}{4}\sum_{i,j,k,i',j',k'=1}^{n} (\Psi_1)_{ijk}(\Psi_2)_{i'j'k'}(Q^{-1})_{ii'}(Q^{-1})_{jk}(Q^{-1})_{j'k'}$$

Here $(S_3\times S_3)\backslash \mathrm{Matchings}_6 \ni [\sigma]$ consists of two different classes:

- one with 6 representatives in $\mathrm{Matchings}_6$ (i.e. with stabilizer subgroup of order $\frac{3!\,3!}{6} = 6$), corresponding to the "theta graph";
- the second with 9 representatives in $\mathrm{Matchings}_6$ (i.e. with stabilizer subgroup of order $\frac{3!\,3!}{9} = 4$), corresponding to the "dumbbell graph".

3.6. **Perturbed Gaussian integral.** Fix again $Q(x,x)$ a positive-definite quadratic form on $V = \mathbb{R}^n$. We are interested in the integrals of form

$$\int_V d^n x\; e^{-\frac{1}{2}Q(x,x) + p(x)} \tag{50}$$

with p a small polynomial perturbation of the quadratic form in the exponential. More precisely, consider the perturbation $p(x)$ of the form

$$p(x) = \sum_{d=0}^{D} \frac{g_d}{d!} P_d(x) \tag{51}$$

with D some fixed degree, $P_d = \sum_{i_1,\ldots,i_d=1}^n (P_d)_{i_1\cdots i_d} x_{i_1}\cdots x_{i_d} \in \mathrm{Sym}^d V^*$ a homogeneous polynomial of degree d, and $g_0,\ldots,g_D$ – infinitesmial formal parameters ("coupling constants"). Note that then the exponential of the perturbation $e^{p(x)}$ is a formal power series in the couplings $g_0,\ldots,g_D$ where the coefficient of each monomial $g_0^{v_0}\cdots g_D^{v_D}$ is a finite-degree polynomial in x, i.e.

$$e^{p(x)} \quad \in \mathrm{Sym}V^*\otimes \mathbb{R}[[g_0,\ldots,g_D]] = \mathrm{Sym}V^*[[g_0,\ldots,g_D]]$$

Definition 3.28. We define the *perturbative evaluation* of the integral (50) as follows:

$$\int_V^{\mathrm{pert}} d^n x\, e^{-\frac12 Q(x,x)+p(x)} := \underbrace{\left(\int_V d^n x\, e^{-\frac12 Q(x,x)}\right)}_{(2\pi)^{\frac{n}{2}}(\det Q)^{-\frac12}} \cdot \ll e^{p(x)} \gg \tag{52}$$

where the symbol $\ll e^{p(x)} \gg$ is to be understood as the evaluation on $e^{p(x)} \in \mathrm{Sym}V^*[[g_0,\ldots,g_D]]$ of the Gaussian expectation value $\ll \cdots \gg\colon \mathrm{Sym}V^* \to \mathbb{R}$, extended by linearity to a map $\ll\cdots\gg\colon \mathrm{Sym}V^*[[g_0,\ldots,g_D]] \to \mathbb{R}$.

Remark 3.29. Perturbative integral (52) is well-defined for any perturbation $p(x)$ of form (51), while (50) as a measure-theoretic integral may fail to exist for non-zero coupling constants. E.g. the integral

$$\int_{\mathbb{R}} dx\, e^{-\frac{x^2}{2}+\frac{\alpha}{3!}x^3}$$

diverges for any non-zero coefficient $\alpha = g_3$ (except for the case of $\alpha\in i\cdot\mathbb{R}$ purely imaginary), while

$$\int_{\mathbb{R}} dx\, e^{-\frac{x^2}{2}+\frac{\lambda}{4!}x^4}$$

converges for $\lambda = g_4$ negative (or, more generally, for $\mathrm{Re}\,\lambda\le 0$) and diverges for λ positive (resp. $\mathrm{Re}\,\lambda > 0$).

Lecture 9, 09/26/2016.

Definition 3.30. Let Γ be a graph ("Feynman diagram"). Fix a collection of symmetric tensors (the "Feynman rules"):

- The "propagator"

$$\eta = \sum_{i,j=1}^n \eta_{ij}\, e_i\odot e_j \quad \in \mathrm{Sym}^2 V$$

  with $\{e_i\}$ the standard basis in $\mathbb{R}^n$ (or, more abstractly, *a* basis in V).
- "Vertex functions[25] for vertices of valency d",

$$p_d = \sum_{i_1,\ldots,i_d=1}^n (p_d)_{i_1\cdots i_d}\, x_{i_1}\cdots x_{i_d} \quad \in \mathrm{Sym}^d V^*$$

  for $d=0,\ldots,D$; $\{x_i\}$ is the basis in V^* dual to $\{e_i\}$.

[25] Or, more appropriately, "vertex tensors".

We define the *Feynman weight* (or the "value of the Feynman diagram") of Γ as

$$\frac{1}{|\mathrm{Aut}(\Gamma)|}\Phi_{\eta;p_0,\ldots,p_D}(\Gamma)$$

where $\Phi_{\eta;p_0,\ldots,p_D}(\Gamma)$ is defined as the following *state sum*.

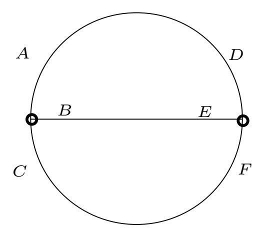
- We define a *state* s on Γ as a decoration of all half-edges of Γ by numbers in $\{1,\ldots,n\}$.
- To a state $s: HE\to\{1,\ldots,n\}$ we assign a *weight*

  $$w_s := \prod_{\text{edges } e=(h,h')} \eta_{s(h)s(h')} \times \prod_{\text{vertices } v} (p_d)_{s(h_1)\cdots s(h_d)}$$

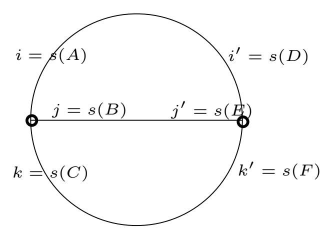
  In the first product, h,h' are the two constituent half-edges of the edge e. In the second product, d is the valency of the vertex v and $h_1,\ldots,h_d$ are the half-edges adjacent to v.
- We define Φ as the sum over states on Γ:

  $$\Phi_{\eta;p_0,\ldots,p_D}(\Gamma) := \sum_{\text{states } s\colon HE\to\{1,\ldots,n\}} w_s$$

Example 3.31. Consider Γ the theta-graph; we label the half-edges by $\{A,B,C,D,E,F\}$:



A state s on Γ maps half-edges to numbers $s:(A,B,C,D,E,F)\mapsto(i,j,k,i',j',k')$ each of which can take values from 1 to n:



The weight of the state is:

$$w_s = \eta_{ii'}\eta_{jj'}\eta_{kk'} \times (p_3)_{ijk}(p_3)_{i'j'k'}$$

And thus the Feynman value of the theta graph is

$$\frac{1}{12}\sum_{i,j,k,i',j',k'=1}^{n} \eta_{ii'}\eta_{jj'}\eta_{kk'} \times (p_3)_{ijk}(p_3)_{i'j'k'}$$

Theorem 3.32 (Feynman)**.** For Q a positive-definite quadratic form on $V=\mathbb{R}^n$ and $p(x)=\sum_{d=0}^{D}\frac{g_d}{d!}P_d(x)$ a polynomial perturbation with homogeneous terms $P_d\in\mathrm{Sym}^d V^*$, the perturbative evaluation of the integral (50) is given by the sum

over all finite graphs (up to graph isomorphism) of their Feynman weights:
(53)
$$\int_V^{\mathrm{pert}} d^n x\; e^{-\frac12 Q(x,x)+p(x)} = (2\pi)^{\frac{n}{2}} (\det Q)^{-\frac12} \sum_{\text{graphs }\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} \Phi_{Q^{-1};g_0P_0,\ldots,g_DP_D}(\Gamma)$$

Proof. By definition (52), we need to compute the Gaussian expectation value $\ll e^{p(x)} \gg$. Writing $e^p = \prod_{d=1}^D e^{\frac{g_d}{d!}P_d(x)}$ and expanding each exponential in Taylor series, we obtain

(54)
$$\ll e^p \gg = \ll \prod_{d=1}^D e^{\frac{g_d}{d!}P_d(x)} \gg = \sum_{v_0,\ldots,v_D\geq 0} \prod_{d=1}^D \frac{g_d^{v_d}}{v_d!\, d!^{v_d}} \ll P_0(x)^{v_0}\cdots P_D(x)^{v_D} \gg$$
$$\underset{\text{Wick's lemma}}{=} \sum_{v_0,\ldots,v_D\geq 0} \frac{g_0^{v_0}\cdots g_D^{v_D}}{|\mathcal{V}_{v_0\ldots v_D}|} \sum_{\sigma\in\mathrm{Matchings}_{2m}} \left\langle \sigma\circ (Q^{-1})^{\otimes m}, \otimes_{d=0}^D P_d^{\otimes v_d}\right\rangle \cdot \left\langle \sigma\circ (Q^{-1})^{\otimes m}, \otimes_{d=0}^D P_d^{\otimes v_d}\right\rangle$$

Here we denoted $\mathcal{V}_{v_0\cdots v_D} = \prod_{d=0}^D S_{v_d} \ltimes (S_d)^{\times v_d}$ – group of "vertex symmetries" which we understand as a subgroup of S_{2m} with $2m = \sum_{d=0}^D d\, v_d$. The picture is that for each $d = 0,1,\ldots,D$, we have v_d of d-valent stars decorated with P_d (the vertex tensors); thus, in total, we have $2m = \sum_{d=0}^D d\, v_d$ half-edges. Then we attach m edges decorated by Q^{-1} according to all possible perfect matchings σ of half-edges. The sum over matchings contains many similar terms, collecting which we get:

$$\ll e^p \gg =$$
$$= \sum_{v_0,\ldots,v_D\geq 0} g_0^{v_0}\cdots g_D^{v_D} \sum_{[\sigma]\in \mathcal{V}_{v_0\cdots v_D}\backslash \mathrm{Matchings}_{2m}} \frac{|\text{orbit of }\sigma\text{ in Matchings}_{2m}\text{ under }\mathcal{V}_{v_0\cdots v_D}|}{|\mathcal{V}_{v_0\cdots v_D}|}\cdot$$
$$\cdot \left\langle \sigma\circ (Q^{-1})^{\otimes m}, \otimes_{d=0}^D P_d^{\otimes v_d}\right\rangle$$

Equivalence classes of matchings

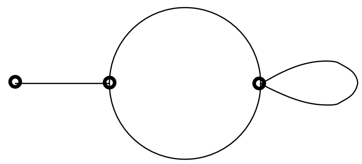
$$[\sigma]\in \mathcal{V}_{v_0\cdots v_D}\backslash \mathrm{Matchings}_{2m} = \left(\prod_{d=0}^D S_{v_d}\ltimes (S_d)^{\times v_d}\right) \backslash\, S_{2m}/(S_m \ltimes \mathbb{Z}_2^{\times m})$$

are in bijection with isomorphism classes of graphs with v_0 of 0-valent vertices, ..., v_D of D-valent vertices; the weight of the class $[\sigma]$ is easily seen to be the Feynman weight of the corresponding graph:

$$\ll e^p \gg = \sum_{v_0,\ldots,v_D\geq 0} g_0^{v_0}\cdots g_D^{v_D} \sum_{\text{graphs }\Gamma\text{ with }v_d\ d-\text{valent vertices},\, d=0,\ldots,D} \frac{1}{|\mathrm{Aut}(\Gamma)|}\Phi_{Q^{-1},\{P_d\}_{d=0}^D}(\Gamma)$$

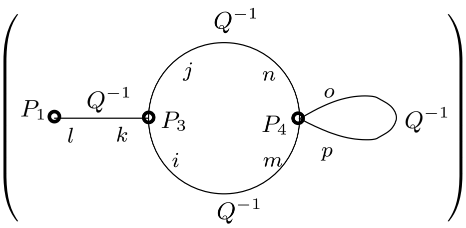
We can absorb g_d-factors into the normalization of vertex tensors, getting the r.h.s. of (53). □

Example 3.33. The contribution of the following graph



to the r.h.s. of (53)is:

$$\frac{g_1\, g_3\, g_4}{|\mathrm{Aut}|}\ \Phi$$



$$= \frac{g_1\, g_3\, g_4}{4} \sum_{i,j,k,l,m,n,o,p=1}^{n} (Q^{-1})_{kl} (Q^{-1})_{im} (Q^{-1})_{jn} (Q^{-1})_{op} \times (P_3)_{ijk} (P_1)_l (P_4)_{mnop}$$

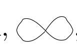
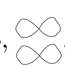
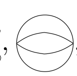
Remark 3.34. We can see the sum over graphs in the r.h.s. of (53) as the volume of the groupoid of graphs with standard groupoid measure $\frac{1}{|\mathrm{Aut}(\Gamma)|}$ on objects (graphs) deformed by Feynman rules to $\frac{1}{|\mathrm{Aut}(\Gamma)|}\Phi_{Q^{-1},\{g_d\, P_d\}}(\Gamma)$.

Example 3.35. Consider

$$I(\lambda) = \int_{\mathbb{R}} dx\ e^{\frac{x^2}{2} + \frac{\lambda}{4!}x^4} \tag{55}$$

By (53), the perturbative evaluation yields the sum over 4-valent graphs:

$$\begin{aligned} \int_{\mathbb{R}}^{\mathrm{pert}} dx\ e^{\frac{x^2}{2} + \frac{\lambda}{4!}x^4} &= \sqrt{2\pi} \sum_{4-\text{valent graphs } \Gamma} \frac{\lambda^{\#\text{vertices}}}{|\mathrm{Aut}(\Gamma)|} \\ &= \sqrt{2\pi}\left(1 + \frac{1}{8}\lambda + \left(\frac{1}{2\cdot 8^2}\lambda^2 + \frac{1}{2\cdot 4!}\lambda^2 + \frac{1}{16}\lambda^2\right) + \cdots\right) \end{aligned} \tag{56}$$

The first contributing graphs here are: the empty graph, , , , .
Note that, using (49), we can evaluate the total coefficient of λ^n:

$$\int_{\mathbb{R}}^{\mathrm{pert}} dx\ e^{\frac{x^2}{2} + \frac{\lambda}{4!}x^4} = \sqrt{2\pi} \sum_{n=0}^{\infty} \lambda^n \frac{(4n-1)!!}{n!\, 4!^n} \tag{57}$$

Coefficients of this power series in λ grow super-exponentially (roughly, as $n!$), therefore the convergence radius in λ is zero! On the other hand, for $\lambda = -\nu < 0$ the integral (55) converges, as a usual measure-theoretic integral, to the function

$$\sqrt{\frac{3}{\nu}} \cdot e^{\frac{3}{4\nu}} K_{\frac{1}{4}}\left(\frac{3}{4\nu}\right) \tag{58}$$

where $K_\alpha(x) = \int_0^\infty dt\ e^{-x\cosh t}\cosh(\alpha t)$ is the modified Bessel's function. The relation between formal power series (57) and the measure-theoretic evaluation(58) is that the former is the *asymptotic series* for the latter at $\lambda = -\nu \to -0$ (i.e. λ approaching zero along the negative half-axis).

Definition 3.36. Let $\phi(z) \in C^\infty(0,\infty)$ a function on the open positive half-line and let $f_n(z) \in C^\infty(0,\infty)$ be a collection of functions for $n = 0, 1, \ldots$. One says that $\sum_n f_n(z)$ is a *Poincaré asymptotic series* for the function $\phi(z)$ at $z = 0$ (notation: $\phi(z) \underset{z\to 0}{\sim} \sum_n f_n(z)$) if:

(i) $\phi(z) - \sum_{n=0}^N f_n(z) \underset{z\to 0}{\sim} O(f_{N+1}(z))$ for any $N \geq 0$ and

(ii) $f_{n+1}(z) \underset{z\to 0}{\sim} o(f_n(z))$ for any $n \geq 0$, i.e. $\lim_{z\to +0} \frac{f_{n+1}(z)}{f_n(z)} = 0$.

Lecture 10, 09/28/2016.

3.6.1. *Aside: Borel summation.* Introduce an operation which assigns to a power series $f(z) = \sum_{n\geq 0} a_n z^n$ a new power series $\mathcal{B}f(t) := \sum_{n\geq 0} \frac{a_n}{n!} t^n$.

We can recover $f(z)$ from $\mathcal{B}f(t)$ by certain integral transform $\mathbb{T}$ (the Laplace transform, up to a change of variable):

$$\mathbb{T}(\mathcal{B}f)(z) := \int_0^\infty dt\; e^{-t}\mathcal{B}f(tz) \qquad = \sum_{n\geq 0} \frac{a_n}{n!} \underbrace{\int_0^\infty dt\; e^{-t}(tz)^n}_{n!z^n} = f(z)$$

Note that the map $f(z) \mapsto \mathcal{B}f(t)$ improves convergence properties: if $f(z)$ has finite convergence radius in z, then $\mathcal{B}f(t)$ is an entire function in t.

Borel's summation method amounts to taking a possibly divergent series as $f(z)$ (e.g. with zero convergence radius); then $\mathcal{B}f(t)$ can still be convergent (possibly, with a finite convergence radius but possessing an analytic continuation). Then one can define $f_{\mathrm{Borel}}(z)$ – the *Borel summation* of $f(z)$, as a function which can be evaluated for nonzero z, rather than just a formal power series, as $\mathbb{T}(\mathcal{B}f)$.

Example 3.37. Consider the power series $f(z) = \sum_{n\geq 0}(-1)^n n! z^n$ – it clearly has zero convergence radius in z. We have $\mathcal{B}f(t) = \sum_{n\geq 0}(-1)^n t^n$ – this power series converges to $\frac{1}{1+t}$ with convergence radius 1 and extends to an analytic function in $t \in \mathbb{C}\backslash\{-1\}$. Thus, the Borel summation of $f(z)$ is:

$$f_{\mathrm{Borel}}(z) := \mathbb{T}\left(\frac{1}{1+t}\right) = \int_0^\infty dt\; e^{-t}\frac{1}{1+tz} = z^{-1}\, e^{z^{-1}} E_1(z^{-1})$$

where $E_1(x) = \int_x^\infty ds\; \frac{e^{-s}}{s}$ is the *exponential integral.*

General fact: Original power series $f(z)$ is the asymptotic series for the Borel summation $f_{\mathrm{Borel}}(z)$ at $z \to 0$.

In application to perturbative integral, the idea is that one may be able to recover the value of the integral at finite value of coupling constants from the perturbation series by means of Borel summation (which is particularly interesting for path integrals where a direct measure theoretic definition at finite coupling constants/Planck constant is not accessible and one only has the perturbative expansion).

If $F(z)$ is a function and $f(z) = \sum_{n\geq 0} a_n z^n$ is the asymptotic series for F at $z \to 0$ then under some assumptions it is guaranteed that the Borel summation of $f(z)$ gives back $F(z)$ (i.e. the question is when is the function uniquely determined by its asymptotic expansion).

Theorem 3.38 (Watson)**.** Assume that, for some positive constants $R, \varkappa, \epsilon, c$, we have the following:

- $F(z)$ is holomorphic in the region

$$\mathcal{D} := \{z \in \mathbb{C} \mid \; |z| < R, |\arg(z)| < \varkappa\frac{\pi}{2} + \epsilon\}$$

- In this region $F(z)$ is "well approximated" by its asymptotic series $f(z)$:

$$\left|F(z)-\sum_{n=0}^{N-1} a_n z^n\right| < c^N (\varkappa n)!\, z^N$$

Then, in the region $\mathcal{D}$, we $F(z)$ coincides with Borel summation of its asymptotic series $f(z)=\sum_{n\geq 0} a_n z^n$.

Example 3.39. Function $F(z)=e^{-\frac{1}{z}}$ has zero asymptotic series $f(z)=0$ and thus cannot be recovered by Borel summation of $f(z)$. On the other hand, $F(z)$ fails the assumptions of Watson's theorem for any value of $\varkappa$. (Check this!)

3.6.2. *Connected graphs.* It turns out, one can reformulate the r.h.s. of Feynman's formula (53) in terms of summation over *connected* graphs only.

Theorem 3.40. For a positive-definite quadratic form Q and a polynomial perturbation $p(x)=\sum_{d=0}^{D}\frac{g_d}{d!}P_d(x)$ as in Theorem 3.32, we have
(59)

$$\int_V^{\mathrm{pert}} d^n x\; e^{-\frac12 Q(x,x)+p(x)} = (2\pi)^{\frac{n}{2}}(\det Q)^{-\frac12}\cdot\exp\left(\sum_{\text{connected graphs }\gamma}\frac{1}{|\mathrm{Aut}(\gamma)|}\Phi_{Q^{-1},\{g_d P_d\}}(\gamma)\right)$$

Proof. Note that any graph Γ can be uniquely split into connected components:

$$\Gamma=\gamma_1^{\sqcup r_1}\sqcup\cdots\sqcup\gamma_k^{\sqcup r_k} \tag{60}$$

where $\gamma_1,\ldots,\gamma_k$ are pairwise non-isomorphic connected graphs and $r_1,\ldots,r_k$ are multiplicities with which they appear in the graph Γ. Automorphisms of Γ are generated by automorphisms of individual connected components and permutations of connected components of same isomorphism type:

$$\mathrm{Aut}(\Gamma)=\prod_{i=1}^{k} S_{r_i}\ltimes \mathrm{Aut}(\gamma_i)^{\times r_i} \tag{61}$$

Choose some total ordering on the set of isomorphism classes of connected graphs. Let us calculate $\exp\sum_{\gamma\text{ connected}}\frac{1}{|\mathrm{Aut}(\gamma)|}\Phi(\gamma)$ by expanding the exponential in the Taylor series:

$$\begin{aligned}(62)\quad \exp\sum_{\gamma\text{ connected}}\frac{1}{|\mathrm{Aut}(\gamma)|}\Phi(\gamma) &= \prod_{\gamma\text{ connected}}\sum_{r=0}^{\infty}\frac{1}{|\mathrm{Aut}(\gamma)|^r\, r!}\Phi(\gamma)^r = \\ &=\sum_{k=0}^{\infty}\sum_{\gamma_1<\ldots<\gamma_k}\sum_{r_1,\ldots,r_k=1}^{\infty}\frac{1}{\prod_{i=1}^k r_i!|\mathrm{Aut}(\gamma_i)|^{r_i}}\Phi(\gamma_1)^{r_1}\cdots\Phi(\gamma_k)^{r_k} \\ &=\sum_{k=0}^{\infty}\sum_{\gamma_1<\ldots<\gamma_k}\sum_{r_1,\ldots,r_k=1}^{\infty}\frac{1}{|\mathrm{Aut}(\Gamma)|}\Phi(\Gamma)\end{aligned}$$

where in the last step we set $\Gamma:=\gamma_1^{\sqcup r_1}\sqcup\cdots\sqcup\gamma_k^{\sqcup r_k}$ and we used (61) and multiplicativity of Feynman state sum on graphs: $\Phi(\Gamma_1\sqcup\Gamma_2)=\Phi(\Gamma_1)\cdot\Phi(\Gamma_2)$. The sum in the final expression in (62) corresponds simply to summing over all Γ (by uniqueness of decomposition (60)). Thus, we have proven that

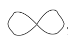
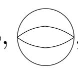
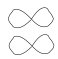
$$\exp\sum_{\gamma\text{ connected}}\frac{1}{|\mathrm{Aut}(\gamma)|}\Phi(\gamma)=\sum_{\Gamma}\frac{1}{|\mathrm{Aut}(\Gamma)|}\Phi(\Gamma) \tag{63}$$

which, together with Feynman's formula (53) implies (59).

□

Example 3.41. Returning to the Example 3.35, we can now rewrite (56) as a sum over *connected* graphs with 4-valent vertices:

$$\int_{\mathbb{R}}^{\text{pert}} dx\, e^{-\frac{1}{2}x^2+\frac{\lambda}{4!}x^4} = \sqrt{2\pi}\cdot\exp\left(\sum_{\gamma\ \text{connected, }4-\text{valent}} \frac{\lambda^{\#\text{vertices}}}{|\mathrm{Aut}(\gamma)|}\right) = \sqrt{2\pi}\cdot\exp\left(\frac{\lambda}{8}+\frac{\lambda^2}{2\cdot 4!}+\frac{\lambda^2}{16}+\cdots\right)$$

where the first contributing graphs are , , . Note that the empty graph and  are disconnected and do not contribute here.[26]

3.6.3. *Introducing the "Planck constant" and bookkeeping by Euler characteristic of Feynman graphs.* Consider the integral

$$\int_V d^n x\, e^{\frac{1}{\hbar}(-\frac{1}{2}Q(x,x)+p(x))} \tag{64}$$

with $\hbar$ an infinitesimal parameter, Q a positive-definite quadratic form and $p(x) = \sum_{d=3}^{D}\frac{1}{d!}P_d(x)$ with $P_d\in \mathrm{Sym}^d V^*$. Note that here, unlike in (51), we did not scale terms of the perturbation $p(x)$ with coupling constants, however here we only allow at least cubic terms in $p(x)$. We define the perturbative evaluation of (64) by rescaling the integration variable $x=\sqrt{\hbar}\,y$ which converts it to the perturbative integral of the type defined in (52):

$$\int_V^{\text{pert}} d^n x\, e^{\frac{1}{\hbar}(-\frac{1}{2}Q(x,x)+p(x))} := \hbar^{\frac{n}{2}}\int_V^{\text{pert}} d^n y\, e^{-\frac{1}{2}Q(y,y)+\sum_{d=3}^{D}\frac{\hbar^{\frac{d}{2}-1}}{d!}P_d(y)} \quad \in \hbar^{\frac{n}{2}}\mathbb{R}[[\hbar^{\frac{1}{2}}]] \tag{65}$$

Note that, in the integral on the r.h.s., the terms of the perturbation got scaled with "coupling constants" $\hbar^{\frac{d}{2}-1}$ – positive powers of $\hbar$ (as we only allowed terms with $d\geq 3$ in $p(x)$). Moreover, there are finitely many Feynman graphs contributing to each order in $\hbar$.

Lemma 3.42 ("Loop expansion")**.** We have

$$\int_V^{\text{pert}} d^n x\, e^{\frac{1}{\hbar}(-\frac{1}{2}Q(x,x)+p(x))} = (2\pi\hbar)^{\frac{n}{2}}(\det Q)^{-\frac{1}{2}}\sum_{\text{graphs }\Gamma}\frac{\hbar^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\Phi_{Q^{-1},\{P_d\}_{d=3}^D}(\Gamma) = (2\pi\hbar)^{\frac{n}{2}}(\det Q)^{-\frac{1}{2}}\exp\left(\frac{1}{\hbar}\sum_{\gamma\text{ connected}}\frac{\hbar^{l(\gamma)}}{|\mathrm{Aut}(\gamma)|}\Phi_{Q^{-1},\{P_d\}_{d=3}^D}(\gamma)\right) \tag{66}$$

where $\chi(\Gamma)$ is the Euler characteristic of the graph and $l(\gamma)=B_1(\gamma)$ is the "number of loops" (the first Betti number of a connected graph). Feynman graphs in these expansions are assumed to have valency ≥ 3 for all vertices (in particular, this implies $l(\gamma)\geq 2$).

[26]Empty graph is regarded as disconnected: it has zero connected components whereas a connected graph should have one connected component.

Proof. Applying Feynman's formula (3.32) to the r.h.s. of the definition (65), we get the following Feynman wights of graphs:

$$\frac{1}{|\mathrm{Aut}(\Gamma)|}\Phi_{Q^{-1},\{\hbar^{\frac{d}{2}-1}P_d\}_{d=3}^D}(\Gamma)=\hbar^{\sum_{\text{vertices } v}(\frac{\mathrm{val(v)}}{2}-1)}\frac{1}{|\mathrm{Aut}(\Gamma)|}\Phi_{Q^{-1},\{P_d\}_{d=3}^D}(\Gamma)$$

with $\mathrm{val}(v)$ the valency of a vertex v of Γ. Note that $\sum_{\text{vertices } v}\mathrm{val}(v)=\#HE$ – the number of half-edges, therefore

$$\sum_{\text{vertices } v}(\frac{\mathrm{val(v)}}{2}-1)=\#E-\#V=-\chi(\Gamma)$$

Thus the Feynman weight of a graph is $\hbar^{-\chi(\Gamma)}\frac{1}{|\mathrm{Aut}(\Gamma)|}\Phi(\Gamma)$ which proves the first equality in (66). For the second equality, we simply notice that, for γ connected, $\hbar^{-\chi(\gamma)}=\frac{1}{\hbar}\cdot\hbar^{l(\gamma)}$.

□

Remark 3.43. An intuitive way to recover the result (66) is to interpret the normalization of the integrand of l.h.s. of (66) by $\hbar$ as a change of normalization of the quadratic form $Q\mapsto\hbar^{-1}Q$ (and thus $Q^{-1}\mapsto\hbar Q^{-1}$), $p(x)\mapsto\hbar^{-1}p(x)$ with respect to (53). Thus, each edge of a graph picks a factor $\hbar$ and each edge picks a factor $\hbar^{-1}$ which results in the value of the entire graph being scaled with the factor $\hbar^{-\chi(\Gamma)}$.

Remark 3.44. If we allow terms of degree <3 in $p(x)$, in the integral (65) (denote it by $I(\hbar)$) there will be infinitely many terms contributing in each order in $\hbar$, also, $I(\hbar)\in\hbar^{\frac{n}{2}}\mathbb{R}[[\hbar^{-1},\hbar]]$ – a two-sided formal Laurent series; more precisely, $I(\hbar)\in\hbar^{\frac{n}{2}}\exp\left(\hbar^{-1}\mathbb{R}[[\hbar]]\right)$.[27]

3.6.4. *Expectation values with respect to perturbed Gaussian measure.* We can consider graphs with vertices marked by elements of a set of colors $\mathcal{C}$. Then we only allow those graph automorphisms which preserve the vertex colors.

Here is the modification of Feynman's Theorem 3.32 for expectation values w.r.t. perturbed Gaussian measure:

Theorem 3.45. Let Q be a positive-definite quadratic form, let $p(x)=\sum_{d=0}^D\frac{g_d}{d!}P_d(x)$ be a polynomial perturbation and let $\Psi_j=\sum_{d\geq 0}\frac{1}{d!}\Psi_{j,d}\in\mathrm{Sym}V^*$ for $j=1,\ldots,r$ be a collection of r polynomials ("observables") with $\Psi_{j,d}$ their respective homogeneous pieces of degree d. Then we have:

(i)

$$\int_V^{\text{pert}}d^nx\,e^{-\frac{1}{2}Q(x,x)+p(x)}\Psi_1(x)\cdots\Psi_r(x)=\\=(2\pi)^{\frac{n}{2}}(\det Q)^{-\frac{1}{2}}\sum_\Gamma\frac{1}{|\mathrm{Aut}(\Gamma)|}\Phi_{Q^{-1};\{g_dP_d\},\{\Psi_{j,d}\}_{j=1}^r}(\Gamma)\tag{67}$$

where in the r.h.s. we sum over graphs with vertices colored with elements of $\mathcal{C}=\{0;1,2,\ldots,r\}$ with the condition that vertices of each color $\neq 0$ occur in the graph exactly once (and there are arbitrarily many vertices of color 0 – the "neutral color"). Vertices of color 0 and valency d are assigned the vertex

[27] Note however that not every power series of form $\sum_{n\geq -1}a_n\hbar^n$ can be exponentiated to a formal Laurent series – certain convergence condition needs to hold for a_n for the coefficients of $\exp(\sum_{n\geq -1}a_n\hbar^n)$ to be finite.

tensor $g_d P_d$, while a vertex of color $j \in \{1,\ldots,r\}$ and valency d is assigned the vertex tensor $\Psi_{j,d}$.

The normalized expectation value of the product of observables w.r.t. the perturbed the Gaussian measure is:

$$(68)\qquad \ll \Psi_1\cdots\Psi_r \gg_\mathrm{pert} := \frac{\int_V^\mathrm{pert} d^n x\; e^{-\frac12 Q(x,x)+p(x)}\Psi_1(x)\cdots\Psi_r(x)}{\int_V^\mathrm{pert} d^n x\; e^{-\frac12 Q(x,x)+p(x)}} = \\ = \sum_\Gamma \frac{1}{|\mathrm{Aut}(\Gamma)|}\Phi_{Q^{-1};\{g_d P_d\},\{\Psi_{j,d}\}_{j=1}^r}(\Gamma)$$

where the sum over graphs is as in (67) with additional requirement that each connected component Γ should contain at least one vertex of nonzero color. (Thus, Γ can have at most r connected components.)

The proof is a straightforward modification of the proof of Theorem 3.32.

Lecture 11, 09/30/2016.

Remark 3.46. If we normalize the perturbed Gaussian measure in Theorem 3.45 by a Planck constant, as $e^{\frac{1}{\hbar}(-\frac12 Q(x,x)+p(x))}$, then Feynman graphs will get weighed with $\hbar^{r-\chi(\Gamma)}$. We can interpret the power of $\hbar$ here as minus the Euler characteristic of the graph with vertices marked by nonzero colors removed (but the adjacent edges retained as half-open intervals).

3.6.5. *Fresnel (oscillatory) version of perturbative integral.* Instead of considering perturbed Gaussian integrals, one can consider perturbed Fresnel integrals in the exact same manner. E.g. Fresnel version of (67), with normalization by Planck constant, is as follows:

$$(69)\qquad \int_V^\mathrm{pert} d^n x\; e^{\frac{i}{\hbar}(\frac12 Q(x,x)+p(x))}\Psi_1(x)\cdots\Psi_r(x) = \\ = (2\pi\hbar)^{\frac n2}|\det Q|^{-\frac12} e^{\frac{\pi i}{4}\mathrm{sign}Q}\sum_{\text{graphs }\Gamma}\frac{\hbar^{r-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\Phi_{iQ^{-1};\{iP_d\};\{\Psi_{j,d}\}}$$

Here Q is a non-degenerate (not necessarily positive-definite) quadratic form and $p(x)=\sum_{d=3}^D \frac{1}{d!}P_d(x)$ a polynomial perturbation. Note that the effect of passing to Fresnel version (i.e. introducing the factor i in the exponential in the integrand) amounts to introducing a factor i in the Feynman rules for edges and vertices of neutral color (and the appearance of phase $e^{\frac{\pi i}{4}\mathrm{sign}Q}$ which comes from bare Fresnel integral and has nothing to do with perturbation).

3.6.6. *Perturbation expansion via exponential of a second order differential operator.* For a non-degenerate quadratic form $Q(x,x)$, introduce a second order differential operator $Q^{-1}(\frac{\partial}{\partial x},\frac{\partial}{\partial x}) := \sum_{i,j=1}^n (Q^{-1})_{ij}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}$.

One can rewrite perturbation expansion (53) as follows:

$$(70)\qquad \frac{1}{(2\pi)^{\frac n2}(\det Q)^{-\frac12}}\int_V^\mathrm{pert} d^n x\; e^{-\frac12 Q(x,x)+p(x)} = e^{\frac12 Q^{-1}(\frac{\partial}{\partial x},\frac{\partial}{\partial x})}\circ e^{p(x)}\Big|_{x=0}$$

Here on the l.h.s. both exponentials are to be understood via expanding them in the Taylor series.

This follows from the fact that Wick's lemma can be rewritten as

$$\ll x_{i_1}\cdots x_{i_{2m}} \gg = \frac{1}{2^m m!}\left(Q^{-1}(\frac{\partial}{\partial x},\frac{\partial}{\partial x})\right)^m \circ (x_{i_1}\cdots x_{i_{2m}}) = e^{\frac12 Q^{-1}(\frac{\partial}{\partial x},\frac{\partial}{\partial x})}\circ(x_{i_1}\cdots x_{i_{2m}})\Big|_{x=0}$$

And, consequently, for any $f \in \mathrm{Sym}V^*$, the Gaussian expectation value can be written as

$$\ll f(x) \gg = e^{\frac12 Q^{-1}(\frac{\partial}{\partial x},\frac{\partial}{\partial x})}\circ f(x)\Big|_{x=0}$$

Setting $f(x) = e^{p(x)}$, we get (70).

Remark 3.47. Pictorially, the mechanism of producing Feynman graphs from the r.h.s. of (70) is as follows: e^p produces, upon Taylor expansion, collections of stars of vertices (decorated with $g_d P_d$ for a d-valent vertex). Applying the operator $e^{\frac12 Q^{-1}(\frac{\partial}{\partial x},\frac{\partial}{\partial x})}$ connects some of the half-edges of those stars by arcs, into edges marked by Q^{-1}. Then, setting $x = 0$, we kill all pictures where some half-edges were left unpaired, thus retaining only the perfect matchings on all available half-edges.

3.7. **Stationary phase formula with corrections.** The following version of the stationary phase formula (Theorem 3.2) explains that formal perturbative integrals we studied in Section 3.6 do indeed provide asymptotic expansions for measure-theoretic oscillating integrals in the limit of fast oscillation.

Theorem 3.48. Let X be an n-manifold, let $\mu \in \Omega^n_c(X)$ be a compactly supported top-degree form, and let $f \in C^\infty(X)$ be a function with only non-degenerate critical points on $\mathrm{Supp}\,\mu$. Let $I(\hbar) := \int_X \mu\, e^{\frac{i}{\hbar}f}$ – a smooth complex-valued function on $\hbar \in (0,\infty)$. Then the behavior of $I(\hbar)$ at $\hbar \to 0$ is given by the following asymptotic series:

(71)

$$I(\hbar) \underset{\hbar\to 0}{\sim} \sum_{\text{crit. points } x_0 \text{ of } f \text{ on } \mathrm{Supp}\,\mu} e^{\frac{i}{\hbar}f(x_0)}(2\pi\hbar)^{\frac{n}{2}}|\det f''(x_0)|^{-\frac12}\cdot e^{\frac{\pi i}{4}\mathrm{sign}\, f''(x_0)}\mu_{x_0}\cdot$$
$$\cdot \exp \hbar^{-1}\left(\sum_{\gamma \text{ conn. graphs with vertices of val}\geq 3} \frac{\hbar^{l(\gamma)}}{|\mathrm{Aut}(\gamma)|}\Phi_{if''(x_0)^{-1};\{i\partial^d f|_{x_0}\}_{d\geq 3}}(\gamma)\right)$$

Here we assumed that around every critical point x_0 of f on $\mathrm{Supp}\,\mu$ we have chosen some coordinate chart $(y_1,\ldots,y_n)$ with the property that locally near x_0 we have $\mu = d^n y\, \mu_{x_0}$ with μ_{x_0} a constant. Total d-th partial derivative appearing in the Feynman rules on the r.h.s. is understood as a symmetric tensor $\partial^d f|_{x_0} \in \mathrm{Sym}^d V^*$ with components $\frac{\partial}{\partial y_{i_1}}\cdots\frac{\partial}{\partial y_d} f\Big|_{y=0}$.

For the proof, see e.g. [18, 17, 38].

Remark 3.49. One can drop the assumption that the density of μ in the local coordinates $(y_1,\ldots,y_n)$ around a critical point x_0 is constant. Let $\mu = \rho(y)\cdot d^n y$ with possibly non-constant $\rho(y)$. Then (71) becomes

(72)

$$I(\hbar) \underset{\hbar\to 0}{\sim} \sum_{\text{crit. points } x_0 \text{ of } f \text{ on } \mathrm{Supp}\,\mu} e^{\frac{i}{\hbar}f(x_0)}(2\pi\hbar)^{\frac{n}{2}}|\det f''(x_0)|^{-\frac12}\cdot e^{\frac{\pi i}{4}\mathrm{sign}\, f''(x_0)}\cdot$$
$$\cdot\sum_\Gamma \frac{\hbar^{1-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\Phi_{if''(x_0)^{-1};\underbrace{\{i\partial^d f|_{x_0}\}_{d\geq 3}}_{\text{color 0}};\underbrace{\{\partial^d\rho|_{y=0}\}_{d\geq 0}}_{\text{color 1}}}(\Gamma)$$

where the sum on the r.h.s. is over (possibly disconnected) graphs Γ with vertices of valency ≥ 3 colored by neutral color 0 and a single marked vertex, of arbitrary valency, colored by 1.

3.7.1. *Laplace method.* Laplace method applies to integrals of form $I(\hbar) = \int dx\, e^{-\frac{1}{\hbar}f(x)}$. The idea is that the integrand is concentrated around the minimum x_0 of f, in the neighborhood of x_0 of size $\sim \sqrt{\hbar}$; in this neighborhood the integrand is well approximated by a Gaussian (given by expanding f at x_0 in Taylor series and retaining only the constant and quadratic terms; higher Taylor terms may be accounted for as a perturbation, to obtain higher corrections in powers of $\hbar$).

Simplest version of this asymptotic result is as follows.

Theorem 3.50 (Laplace)**.** Let $f \in C^\infty[a,b]$ be a function on an interval attaining a unique absolute minimum on $[a,b]$ at an interior point $x_0 \in (a,b)$, with $f''(x_0) > 0$. Let $g \in C^\infty[a,b]$ be another function on the interval with $g(x_0) \neq 0$. Then the integral

$$I(\hbar) := \int_a^b dx\; g(x) e^{-\frac{1}{\hbar}f(x)}$$

as a smooth function of $\hbar > 0$ has the following asymptotics as $\hbar \to 0$:

$$I(\hbar) \underset{\hbar\to 0}{\sim} e^{-\frac{1}{\hbar}f(x_0)} \sqrt{\frac{2\pi\hbar}{f''(x_0)}} \cdot g(x_0) \tag{73}$$

A more general multi-dimensional version, with $\hbar$-corrections is as follows.

Theorem 3.51 (Feynman-Laplace)**.** Let X be a compact n-manifold, possibly with boundary, and let $f \in C^\infty(X)$ be a function attaining a unique minimum on X at an interior point $x_0 \in \mathrm{int}(X)$ and assume that the Hessian $f''(x_0)$ is non-degenerate (thus, automatically, positive-definite); also, let $\mu \in \Omega^n(X)$ be a top-degree form. Assume that we have chosen some local coordinates $(y_1, \ldots, y_n)$ near x_0 and in these coordinates $\mu = \rho(y)\, d^n y$. Then the integral

$$I(\hbar) := \int_X \mu\, e^{-\frac{1}{\hbar}f(x)} \qquad \in C^\infty(0,\infty)$$

has the following asymptotic expansion at $\hbar \to 0$:

$$I(\hbar) \underset{\hbar\to 0}{\sim} e^{-\frac{1}{\hbar}f(x_0)} (2\pi\hbar)^{\frac{n}{2}} (\det f''(x_0))^{-\frac{1}{2}} \cdot \\ \cdot \sum_\Gamma \frac{\hbar^{1-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \Phi_{f''(x_0)^{-1};\{-\partial^d f|_{x_0}\}_{d\geq 3};\{\partial^d \rho|_{y=0}\}_{d\geq 0}}(\Gamma) \tag{74}$$

where, as in (72), the sum is over graphs with arbitrarily many vertices of color 0 and valency ≥ 3 and a single vertex of color 1 and arbitrary valency.

Lecture 12, 10/03/2016.

Example 3.52 (Stirling's formula with corrections)**.** Consider $z \to \infty$ asymptotics of the Euler's Gamma function

$$\Gamma(z) = \int_0^\infty dt\; t^{z-1} e^{-t}$$

It is convenient to make a change of the integration variable $t = z\, e^x$, yielding

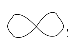
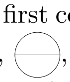
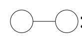
$$\Gamma(z) = z^z \int_{-\infty}^\infty dx\, e^{-z f(x)}$$

with $f(x) = e^x - x$; f has unique absolute minimum at $x = 0$ with Taylor expansion $f(x) = 1 + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$. The asymptotics of this integral at $z \to \infty$ can evaluated using Laplace's theorem (73), with $\hbar := \frac{1}{z}$:

$$\Gamma(z) \underset{z\to\infty}{\sim} z^z e^{-z} \sqrt{\frac{2\pi}{z}}$$

Using (74), we can find corrections to this asymptotics in powers of $\frac{1}{z}$:

$$\Gamma(z) \underset{z\to\infty}{\sim} z^z e^{-z} \sqrt{\frac{2\pi}{z}} \exp \sum_{n=1}^{\infty} \frac{c_n}{z^n}$$

with $c_n = \sum_\Gamma \frac{(-1)^{\# \text{vertices}}}{|\mathrm{Aut}(\Gamma)|}$ where the sum goes over connected graphs with $n-1$ loops (all valencies ≥ 3 allowed). E.g. the first coefficient c_1 gets contributions from the three connected 2-loop graphs: , , : $c_1 = -\frac{1}{8} + \frac{1}{12} + \frac{1}{8} = \frac{1}{12}$.[28] In particular, this implies that the factorial of a large number $n! = n\Gamma(n)$ behaves as

$$n! \underset{n\to\infty}{\sim} \sqrt{2\pi n}\, n^n e^{-n} \left(1 + \frac{1}{12n} + O(\frac{1}{n^2})\right)$$

3.8. **Berezin integral.**

3.8.1. *Odd vector spaces.* Fix $n \geq 1$. Consider the "odd $\mathbb{R}^n$", denoted as $\Pi\mathbb{R}^n$ or $\mathbb{R}^{0|n}$, – space with anti-commuting[29] coordinates $\theta_1, \ldots, \theta_n$. I.e. $\Pi\mathbb{R}^n$ is defined by its algebra of functions

$$\mathrm{Fun}(\Pi\mathbb{R}^n) := \mathbb{R}\langle \theta_1, \ldots, \theta_n \rangle \,/\, \theta_i\theta_j = -\theta_j\theta_i$$

More abstractly, for V a vector space over $\mathbb{R}$, its odd version ΠV has the algebra of functions

$$\mathrm{Fun}(\Pi V) = \wedge^\bullet V^*$$

– the exterior algebra of the dual (viewed as a super-commutative associative algebra), whereas for an even vector space $\mathrm{Fun}(V) = \widehat{\mathrm{Sym}} V^*$ – the (completed) symmetric algebra of the dual.

3.8.2. *Integration on the odd line.* Consider the case $n = 1$ – the odd line $\Pi\mathbb{R}$ with coordinate θ subject to relation $\theta^2 = 0$. Functions on $\Pi\mathbb{R}$ have form $a + b\theta$ with $a, b \in \mathbb{R}$ arbitrary coefficients. We define the integration map $\int_{\Pi\mathbb{R}} D\theta\, (\cdots) : \mathrm{Fun}(\Pi\mathbb{R}) \to \mathbb{R}$ by

$$\int_{\Pi\mathbb{R}} D\theta\, (a + b\theta) := b \tag{75}$$

I.e. the integration simply picks the coefficient of θ in the function being integrated. Integration as defined above is uniquely characterized by the following properties:

- integration maps is $\mathbb{R}$-linear,

[28] In fact, as can be obtained independently, e.g., from Euler-Maclaurin formula, $c_n = \frac{B_{n+1}}{n(n+1)}$, with B_{n+1} the $(n+1)$-st Bernoulli number.

[29] "Odd" or "Grassman" or "fermionic" variables.

- "Stokes' theorem": $\int_{\Pi\mathbb{R}} D\theta\, \frac{\partial}{\partial\theta} g(\theta) = 0$ for $g(\theta)$ an arbitrary function on $\Pi\mathbb{R}$.[30] This implies that the integral of a constant function has to vanish.
- Normalization convention: $\int_{\Pi\mathbb{R}} D\theta\, \theta = 1$.

3.8.3. *Integration on the odd vector space.* A function on $\Pi\mathbb{R}^n$ can be written as

$$f(\theta_1, \ldots, \theta_n) = \sum_{k=0}^{n} \sum_{1\leq i_1 < \cdots < i_k \leq n} f_{i_1\cdots i_k} \theta_{i_1} \cdots \theta_{i_k}$$

with $f_{i_1\cdots i_k} \in \mathbb{R}$ the coefficients. Berezin integral on $\Pi\mathbb{R}^n$ is defined as follows:

$$\int_{\Pi\mathbb{R}^n} D\theta_n \cdots D\theta_1\, f := f_{1\cdots n} \qquad = \text{coefficient of } \theta_1 \cdots \theta_n \text{ in } f \tag{76}$$

This definition can be obtained from the definition (75) for the 1-dimensional case by formally imposing the Fubini theorem, e.g. for $n = 2$ and $f(\theta_1, \theta_2) = f_\varnothing + f_1\theta_1 + f_2\theta_2 + f_{12}\theta_1\theta_2$ we have

$$\int D\theta_2\, D\theta_1\, f = \int D\theta_2 \underbrace{\left(\int D\theta_1\, f\right)}_{f_1 + f_{12}\theta_2} = f_{12}$$

Case of general n is treated similarly, by inductively integrating over odd variables θ_i, in the order of increasing i.

Remark 3.53. Berezin integral can also be seen as an iterated derivative:

$$\int_{\Pi\mathbb{R}^n} D\theta_n \cdots D\theta_1\, f = \frac{\partial}{\partial\theta_n} \cdots \frac{\partial}{\partial\theta_1} f \Big|_{\theta=0}$$

More abstractly, $f \in \mathrm{Fun}(\Pi V)$ a function on an odd vector space ΠV (for V of dimension n) and $\mu \in \wedge^n V$ a "Berezinian" (a replacement of the notion of integration measure or volume form in the context of integration over odd vector spaces), Berezin integral is defined as

$$\int_{\Pi V} \mu \cdot f := \langle \mu, f \rangle$$

– the pairing between the top component of f in $\wedge^n V^*$ and $\mu \in \wedge^n V$. The pairing between $\wedge^n V$ and $\wedge^n V^*$ is defined by

$$\langle \psi_n \wedge \cdots \wedge \psi_1, \theta_1 \wedge \cdots \wedge \theta_n \rangle := \det \langle \psi_i, \theta_j \rangle$$

for $\psi_i \in V$ vectors, $\theta_j \in V^*$ covectors and $\langle \psi_i, \theta_j \rangle$ the canonical pairing between V and V^*.

Note that constant volume forms on an even space V are (nonzero) elements of $\wedge^n V^*$ whereas Berezinians are elements of $\wedge^n V$. Note that there is no dual in the second case!

Given a basis $e_1, \ldots, e_n$ in V and the associated dual basis regarded as coordinate functions on the odd space $\theta_1, \ldots, \theta_n \in V^* \subset \mathrm{Fun}(\Pi V)$, we have a "coordinate Berezinian"

$$\mu = D\theta_n \cdots D\theta_1 := e_n \wedge \cdots \wedge e_1 \quad \in \wedge^n V$$

[30] Derivatives are defined on $\Pi\mathbb{R}^n$ in the following way: $\frac{\partial}{\partial\theta_i}$ is an odd derivation of $\mathrm{Fun}(\Pi\mathbb{R}^n)$ (i.e. a linear map $\mathrm{Fun}(\Pi\mathbb{R}^n) \to \mathrm{Fun}(\Pi\mathbb{R}^n)$ satisfying the Leibniz rule with appropriate sign $\frac{\partial}{\partial\theta_i}(f \cdot g) = (\frac{\partial}{\partial\theta_i} f) \cdot g + (-1)^{|f|} f \cdot (\frac{\partial}{\partial\theta_i} g)$) and defined on generators by $\frac{\partial}{\partial\theta_i}\theta_j = \delta_{ij}$.

Note that, if we have a change of coordinates on ΠV, $\theta_i = \sum_j A_{ij}\theta'_j$, the respective coordinate Berezinians are related by

$$D^n\theta = (\det A)^{-1} D^n\theta' \tag{77}$$

where $D^n\theta$ is a shorthand for $D\theta_n \cdots D\theta_1$ and similarly for $D^n\theta'$. Then we have a change of coordinates formula for the Berezin integral:

$$\int_{\Pi V} D^n\theta f(\theta) = \int_{\Pi V} (\det A)^{-1} D^n\theta' \; f(\theta_i = \sum_j A_{ij}\theta'_j)$$

Observe the difference from the case of a change of variables $x_i = \sum_j A_{ij}x'_j$ in an integral over an even space:

$$\int_V d^n x \; f(x) = \int_V |\det A| \; d^n x' \; f(x_i = \sum_j A_{ij}x'_j)$$

In even case we have the absolute value of the Jacobian of the transformation,[31] whereas in the odd case we have the inverse of the Jacobian, without taking the absolute value.

3.9. **Gaussian integral over an odd vector space.** Let $Q(\theta,\theta) = \sum_{i,j=1^n} Q_{ij}\theta_i\theta_j$ be a quadratic form on $\Pi\mathbb{R}^n$ with Q_{ij} an anti-symmetric matrix, so that $\frac{1}{2}Q(\theta,\theta) = \sum_{i<j} Q_{ij}\theta_i\theta_j$. We assume that $n = 2s$ is even. Then we have the following version of Gaussian integral over $\Pi\mathbb{R}^n$:

$$\int_{\Pi\mathbb{R}^n} D^n\theta \; e^{\frac{1}{2}Q(\theta,\theta)} = \frac{1}{2^s s!} \sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^{s} Q_{\sigma_{2i-1}\sigma_{2i}} = \mathrm{pf}(Q) \tag{78}$$

– the Pfaffian of the anti-symmetric matrix Q_{ij}; here $(-1)^\sigma$ is the sign of permutation σ. We obtain the Pfaffian simply by expanding $e^{\frac{1}{2}Q}$ in Taylor series, picking the top monomial in θ-s and evaluating its coefficients (as per definition of Berezin integral (76)).[32] Note that, for n odd, the integral on the l.h.s. of (78) vanishes identically (the exponential contains only monomials of even degree in θ, hence there is no monomial of top degree).

Recall the basic properties of Pfaffians:

- $\mathrm{pf}(Q)^2 = \det Q$,
- for A any $n \times n$ matrix, $\mathrm{pf}(A^T Q A) = \det A \cdot \mathrm{pf}(Q)$,
- $\mathrm{pf}(Q_1 \oplus Q_2) = \mathrm{pf}(Q_1) \cdot \mathrm{pf}(Q_2)$,
- $\mathrm{pf}(\lambda Q) = \lambda^s \mathrm{pf}(Q)$.

[31] In the even case we either think of an integral over an oriented space against a top form, or of an integral over a non-oriented space against a measure (density). A measure transforms with the absolute value of the Jacobian, while a top form transform just with the Jacobian itself – but then one has to take the change of orientation into account separately.

[32] Recall that an alternative definition of Pfaffian is as the coefficient on the r.h.s. of $\frac{1}{s!}(\sum_{i<j} Q_{ij}\theta_i\theta_j)^s = \mathrm{pf}(Q) \cdot \theta_1 \cdots \theta_n$, which is precisely what we need to evaluate the Berezin integral (78).

Example 3.54.

$$\mathrm{pf}\begin{pmatrix} 0 & a_1 & & & & & \\ -a_1 & 0 & & & & & \\ & & 0 & a_2 & & & \\ & & -a_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & a_s \\ & & & & & -a_s & 0 \end{pmatrix} = a_1\cdot\ldots\cdot a_s$$

Example 3.55.

$$\mathrm{pf}\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = a_{12}a_{34}-a_{13}a_{24}+a_{14}a_{23}$$

Remark 3.56. Consider a special instance of Berezin Gaussian integral where odd variables come in pairs $\theta_i,\bar\theta_i$ (the bar does not stand for complex conjugation: $\bar\theta_i$ is an independent variable from θ_i):

$$\int_{\Pi\mathbb{R}^n\oplus\Pi\mathbb{R}^n}(D\theta_n D\bar\theta_n)\cdots(D\theta_1 D\bar\theta_1)\; e^{B(\bar\theta,\theta)}=\det B \tag{79}$$

here $B(\bar\theta,\theta)=\sum_{i,j=1}^n B_{ij}\bar\theta_i\theta_j$ where B_{ij} is a matrix which does not have to be symmetric or anti-symmetric. The fact that the integral above is equal to $\det B$ is a simple calculation of the Berezin integral:

$$\text{l.h.s.}=\frac{1}{n!}\sum_{\sigma,\sigma'\in S_n}(-1)^\sigma(-1)^{\sigma'}B_{\sigma_1\sigma'_1}\cdots B_{\sigma_n\sigma'_n}=\sum_{\sigma''\in S_n}(-1)^{\sigma''}B_{1\sigma''_1}\cdots B_{n\sigma''_n}=\det B$$

where $\sigma''=\sigma'\cdot\sigma^{-1}$. More invariantly, for endomorphism $B\in\mathrm{End}(V)\simeq V^*\otimes V\subset\mathrm{Fun}(\Pi V\oplus\Pi V^*)$, we have

$$\int_{\Pi V\oplus\Pi V^*}\mu^{\mathrm{can}}_{\Pi V\oplus\Pi V^*}\; e^B=\det B$$

where $\mu^{\mathrm{can}}_{\Pi V\oplus\Pi V^*}$ is the canonical Berezinian on $\Pi V\oplus\Pi V^*$, which, for any choice of coordinates $\theta_1,\ldots,\theta_n$ on V and *dual* coordinates $\bar\theta_1,\ldots,\bar\theta_n$ on V^*, takes the form $(D\theta_n D\bar\theta_n)\cdots(D\theta_1 D\bar\theta_1)$.

3.10. **Perturbative integral over a vector superspace.**

3.10.1. *"Odd Wick's lemma".* We have the following version of Wick's lemma for integration over an odd vector space.

Lemma 3.57. Let V be a vector space over $\mathbb{R}$ of even dimension $n=2s$. Let $Q\in\wedge^2V^*$ be a non-degenerate anti-symmetic bilinear, viewed as a quadratic form on ΠV, and let $\xi_1,\ldots,\xi_{2m}$ be a collection of elements of V^* (viewed as linear functions on ΠV). Then the expectation value

$$\ll\xi_1\cdots\xi_{2m}\gg:=\frac{\int_{\Pi V}\mu\, e^{-\frac12 Q}\xi_1\cdots\xi_{2m}}{\int_{\Pi V}\mu\, e^{-\frac12 Q}}$$

(here μ is an arbitrary non-zero Berezinian on ΠV; the expectation value is clearly independent of μ) is equal to the sum over perfect matchings with signs:

$$\ll \xi_1 \cdots \xi_{2m} \gg = \sum_{\sigma \in S_{2m}/S_m \ltimes \mathbb{Z}_2^m} (-1)^\sigma \left\langle \sigma \circ (Q^{-1})^{\otimes m}, \xi_1 \otimes \cdots \otimes \xi_{2m} \right\rangle \tag{80}$$

It is proven by the same technique as the usual Wick's lemma for an even Gaussian integral: one introduces a source J (which is now odd) and obtains the expectation values as derivatives in J of the Gaussian integral modified by the source term.

Lecture 13, 10/05/2016.

Example 3.58. Gaussian expectation value of a quartic monomial on $\Pi\mathbb{R}^n$ is:

$$\ll \theta_i\theta_j\theta_k\theta_l \gg = \ll \theta_i\theta_j \gg \cdot \ll \theta_k\theta_l \gg - \ll \theta_i\theta_k \gg \cdot \ll \theta_j\theta_l \gg + \ll \theta_i\theta_l \gg \cdot \ll \theta_j\theta_k \gg$$

where e.g. the sign of the second term in the r.h.s. is $(-1)^{\left(\begin{array}{cccc} i & j & k & l \\ i & k & j & l \end{array}\right)} = -1$. Quadratic expectation values in turn are the matrix element of the inverse of Q:

$$\ll \theta_i\theta_j \gg = (Q^{-1})_{ij}$$

3.10.2. *Perturbative integral over an odd vector space.* Perturbed Gaussian integral over an odd space can be treated similarly to the even case. Let Q be a non-degenerate quadratic form on $\Pi V = \Pi\mathbb{R}^n$ and let $p(\theta) = \sum_{d=0}^{D} \frac{g_d}{d!} P_d(\theta)$ be a polynomial perturbation where we allow only even degrees d for the homogeneous components $P_d \in \wedge^d V^*$. Consider the integral

$$I := \int_{\Pi V} D^n\theta \; e^{-\frac{1}{2}Q(\theta,\theta)+p(\theta)}$$

Evaluating it be expanding $e^{p(\theta)}$ in Taylor series and applying Wick's lemma termwise, we obtain:

$$\int_{\Pi V} D^n\theta \; e^{-\frac{1}{2}Q(\theta,\theta)+p(\theta)} = \tag{81}$$

$$= \mathrm{pf}(-Q) \cdot \sum_{v_0,\ldots,v_D=0}^{\infty} \sum_{[\sigma] \in (\prod_d S_{v_d} \ltimes S_d^{\times v_d}) \backslash \, S_{2m} \, / S_m \ltimes \mathbb{Z}_2^m} \frac{1}{\mathrm{Stab}_{[\sigma]}} (-1)^\sigma \left\langle \sigma \circ (Q^{-1})^{\otimes m}, \prod_{d=0}^{D} (g_d P_d)^{\otimes v_d} \right\rangle$$

$$= \mathrm{pf}(-Q) \cdot \sum_\Gamma \frac{1}{|\mathrm{Aut}(\Gamma)|} \Phi_{Q^{-1};\{g_d P_d\}}(\Gamma)$$

Here $2m = \sum_d d \cdot v_d$; the sum runs over all graphs Γ with vertices of even valency ranging between 0 and D. Feynman state sum $\Phi(\Gamma)$ for a graph now contains the sign of a permutation $\sigma \in S_{2m}$ representing Γ.

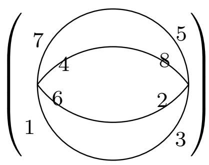
Remark 3.59.
- Note that (81) is an exact evaluation of a Berezin integral (i.e. the perturbative evaluation and exact evaluation automatically coincide for integrals over finite-dimensional odd vector spaces).
- Since sufficiently high powers of $p(\theta)$ vanish identically, the r.h.s. of (81) is a finite-degree polynomial in $g_0, \ldots, g_D$.
- Graphs with $> n$ half-edges are guaranteed to cancel out on the r.h.s. (note that individual graphs with $\#HE > n$ can be still nonzero, but cancel out once all graphs are summed over).

- R.h.s. of (81) can be rewritten as

$$\mathrm{pf}(-Q)\cdot\exp\sum_{\gamma}\frac{1}{|\mathrm{Aut}(\gamma)|}\Phi_{Q^{-1};\{g_d P_d\}}(\gamma) \tag{82}$$

  where the sum is over connected graphs γ. Here the sum in the exponential is, generally, not a polynomial in g_d and contributions of connected graphs do not cancel out for graphs of high complexity.

Example 3.60. Here is an example of a weight of a Feynman graph in the r.h.s. of (81):

$$\frac{1}{48}\Phi\left(\right.$$


$$\left.\right) =$$

$$= \frac{(g_4)^2}{48}\sum_{s_1,\ldots,s_8=1}^{n}(Q^{-1})_{s_1s_3}(Q^{-1})_{s_6s_2}(Q^{-1})_{s_4s_8}(Q^{-1})_{s_7s_5}\cdot(P_4)_{s_1s_6s_4s_7}(P_4)_{s_3s_2s_8s_5}\cdot$$

$$\cdot(-1)^{\left(\begin{smallmatrix}1&3&6&2&4&8&7&5\\1&6&4&7&3&2&8&5\end{smallmatrix}\right)}$$

Here we assigned arbitrary labels (from 1 to 8) to the half-edges; the sign factor is the sign of the permutation taking the order of labels for edges to the order of labels for the vertices.

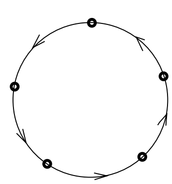
Example 3.61. Let $B\in GL(V)$ and $P\in\mathrm{End}(V)$. Consider the following perturbation of the integral (79):

$$I(\alpha)=\int_{\Pi V\oplus\Pi V^*}\overleftarrow{\prod_{j=1}^{n}}D\theta_j D\bar\theta_j\; e^{-\sum_{i,j}B_{ij}\bar\theta_i\theta_j+\alpha\sum_{i,j}P_{ij}\bar\theta_i\theta_j} \tag{83}$$

with α a coupling constant. Using (81,82) we find that

$$I(\alpha)=\det(-B)\cdot\exp\left(-\sum_{k=1}^{\infty}\frac{\alpha^k}{k}\mathrm{tr}\,(B^{-1}P)^k\right) \tag{84}$$

Terms in the exponential correspond to *oriented* polygon graphs with k vertices and k edges



Oriented graphs appear if we label half-edges corresponding to variables θ_i as *outgoing* and half-edges corresponding to $\bar\theta_i$ as *incoming*. In the sum in the exponential in (84) we can recognize the Taylor expansion of $\log(1-x)$, thus we obtain

$$I(\alpha)=\det(-B)\cdot\exp\mathrm{tr}\,\log(1-\alpha B^{-1}P)=\det(-B)\cdot\det(1-\alpha B^{-1}P)=\det(-B+\alpha P)$$

which is what we would have obtained if we evaluated (83) directly as a Gaussian integral with quadratic form $B-\alpha P$ rather than treating αP as a perturbation.

Note that the series in the exponential in (84) has a finite convergence radius $|\alpha| < \frac{1}{||B^{-1}P||}$ where $||A|| = \max_\lambda |\lambda|$ with λ going over eigenvalues of A.

3.10.3. *Perturbative integral over a superspace.* Consider a vector superspace

$$\mathcal{V} = V^e \oplus \Pi V^o$$

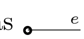
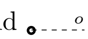
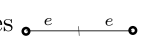
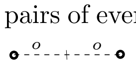
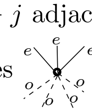
for V^e, V^o two vector spaces of dimensions n, m (superscripts e, o stand for "even", "odd"), with the algebra of functions $\mathrm{Fun}(\mathcal{V}) := C^\infty(V^e) \otimes \wedge^\bullet (V^o)^*$. Let $x_1, \ldots, x_n$ be coordinates on V^e and $\theta_1, \ldots, \theta_m$ be coordinates on ΠV^o. Let Q_e be a quadratic form on V^e and Q_o a quadratic form on ΠV^o, and let $p(x,\theta) = \sum_{j,k} \frac{g_{jk}}{j!k!} P_{jk}(x,\theta)$ be a perturbation, with $P_{jk} \in \mathrm{Sym}^j (V^e)^* \otimes \wedge^k (V^o)^*$ the homogeneous parts; degree k here is only allowed to take even values. Consider the perturbative integral

$$I := \int^{\mathrm{pert}}_{V^e \oplus \Pi V^o} d^n x\, D^m\theta\; e^{-\frac{1}{2} Q_e(x,x) - \frac{1}{2} Q_0(\theta,\theta) + p(x,\theta)} \tag{85}$$

It is understood by formally imposing Fubini theorem: we first integrate over the odd variables and then – perturbatively – over even variables. The result is the following generalization of Feynman's theorem (Theorem 3.32) for integration over a superspace:

$$I = (2\pi)^{\frac{n}{2}} (\det Q_e)^{-\frac{1}{2}} \mathrm{pf}(-Q_o) \sum_\Gamma \frac{1}{|\mathrm{Aut}(\Gamma)|} \Phi(\Gamma)$$

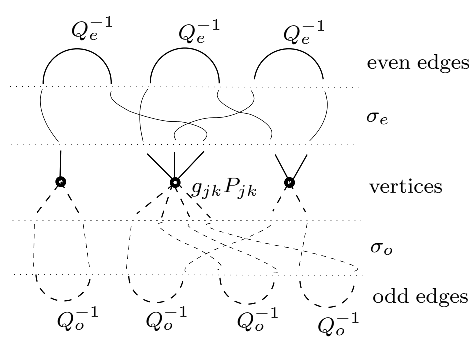
Feynman rules for evaluating $\Phi(\Gamma)$ are as follows:

- Graphs Γ are allowed to have half-edges marked as  (even) and  (odd).
- Edges are pairs of even half-edges  (assigned Q_e^{-1}) or pairs of odd half-edges  (assigned Q_o^{-1}).
- Vertices have bi-valency (j,k) – j adjacent even half-edges and k (an even number) adjacent odd half-edges  (assigned $g_{jk}P_{jk}$).

Put another way, a graph Γ, with E_e, E_o the numbers of even/odd half-edges and with v_{jk} the number of vertices of bi-valency (j,k), is identified with the class of a pair of permutations (σ_e, σ_o) in the double coset

$$\prod_{j,k} S_{v_{jk}} \ltimes (S_j \times S_k)^{v_{jk}} \backslash \quad S_{2E_e} \times S_{2E_o} \quad / (S_{E_e} \ltimes \mathbb{Z}_2^{E_e}) \times (S_{E_o} \ltimes \mathbb{Z}_2^{E_o})$$

Pictorially:



Note that, when defining automorphisms of a graph, we now only allow permutations of half-edges which preserve the parity. The Feynman state sum of a graph is

$$\Phi(\Gamma)=(-1)^{\sigma_o}\left\langle\left(\sigma_e\circ(Q_e^{-1})^{\otimes E_e}\right)\otimes\left(\sigma_o\circ(Q_o^{-1})^{\otimes E_o}\right),\bigotimes_{j,k}(g_{jk}P_{jk})^{\otimes v_{jk}}\right\rangle \tag{86}$$

Example 3.62 ("Faux quantum electrodynamics" integral). Fix $V\simeq\mathbb{R}^n$, $U\simeq\mathbb{R}^m$ two vector spaces. Let $\mathcal{V}=V\oplus\Pi(U\oplus U^*)$ with coordinates $x_i,\theta_a,\bar\theta_a$ – "photon", "electron" and "positron" variables. We also need the following input data:

- quadratic form $Q_e(x,x)\in\mathrm{Sym}^2V^*$,
- quadratic form $Q_o(\bar\theta,\theta)=\langle\bar\theta,\mathfrak{D}\theta\rangle$ with $\mathfrak{D}\in GL(U)$ – "faux Dirac operator",
- a tensor $P(x,\theta,\bar\theta)\in V^*\otimes U^*\otimes U$ – "photon-electron interaction".

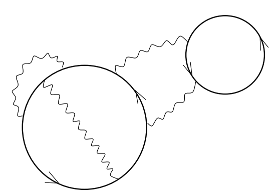
We then consider the following perturbative integral

$$\int_{\mathcal{V}}^{\mathrm{pert}} d^nx\; D^m\theta\; D^m\bar\theta\; e^{-\frac12 Q_e(x,x)-\langle\bar\theta,\mathfrak{D}\theta\rangle+gP(x,\theta,\bar\theta)}=\left(\det\frac{Q_e}{2\pi}\right)^{-\frac12}\cdot\det(-\mathfrak{D})\cdot\sum_\Gamma\frac{1}{|\mathrm{Aut}(\Gamma)|}\Phi(\Gamma) \tag{87}$$

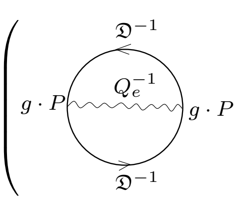
Here g is a coupling constant ("charge of the electron"). Graphs Γ in the r.h.s. of (87) have three types of half-edges:

(i) for "photon" variables x_i,
(ii) for "electron" variables θ_a,
(iii) for "positron" variables $\bar\theta_a$

Admissible edges are: (non-oriented, assigned the propagator Q_e^{-1}) and (oriented, assigned the propagator $\mathfrak{D}^{-1}$). The only admissible vertex is (assigned $g\cdot P$). Typical graph contributing to the r.h.s. of (87) looks like this:



An admissible Γ is always a collection of oriented solid (elctron/positron) cycles arbitrarily interconnected by photon edges. Here is an example of evaluation of a simple admissible graph:

$\frac12\,\Phi\Big($  $\Big)$

$$=-\frac{g^2}{2}\langle Q_e^{-1},\underbrace{\mathrm{tr}_U(\mathfrak{D}^{-1}P\mathfrak{D}^{-1}P)}_{\in(V^*)^{\otimes 2}}\rangle$$

Here we understand P as an element of $V^*\otimes\mathrm{End}(U)$ and take compositions of endomorphisms of U. The minus sign here is $(-1)^{\sigma_o}$, cf. (86).

3.11. **Digression: the logic of perturbative path integral.** In the case of finite-dimensional integrals of oscillatory type $I(\hbar) = \int_X \mu\; e^{\frac{i}{\hbar}f}$, asymptotics of the measure-theoretic integral (which exists for finite $\hbar$) at $\hbar \to 0$ is given by the expansion in Feynman diagrams (Theorem 3.48).

On the other hand a path (functional) integral

$$I(\hbar) = \text{“}\int_{\Gamma(M,\mathsf{Fields})\ni\phi} \mathcal{D}\phi\; e^{\frac{i}{\hbar}S(\phi)}\;\text{”} \tag{88}$$

with M the spacetime manifold and Fields the sheaf of fields on M, and with action $S = \frac{1}{2}\int_M \langle\phi, \mathfrak{D}\phi\rangle + \int_M \mathcal{L}_{\mathrm{int}}(\phi)$ (here $\mathfrak{D}$ is some differential operator), is a heuristic expression which is *defined* as an asymptotic series in $\hbar$ by its expansion in Feynman diagrams,

$$I(\hbar) := (\det \mathfrak{D})^{-\frac{1}{2}} \cdot \sum_\Gamma \frac{\hbar^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\Phi(\Gamma) \tag{89}$$

Here $\Phi(\Gamma)$ is given as an integral over $M^{\times V}$ (V is the number of vertices in Γ) of certain differential form on $M^{\times V}$ (which we view as the space of configurations of V points on M) which depends on Γ and is constructed in terms of the *propagator* – the integral kernel of the inverse operator $\mathfrak{D}^{-1}$ assigned to edges and *vertex functions*, read off from $\mathcal{L}_{\mathrm{int}}$, assigned to vertices. Expansion (89) is obtained by treating (88) following the logic of finite-dimensional perturbed Gaussian integral: one expands $e^{\frac{i}{\hbar}\int_M \mathcal{L}_{\mathrm{int}}(\phi)}$ in Taylor series, thereby producing integrals over configuration spaces of V points on M (with V the term in the Taylor series for the exponential); then one averages individual terms with (Fresnel version of) Gaussian measure $\mathcal{D}\phi\; e^{\frac{i}{2\hbar}\int_M \langle\phi,\mathfrak{D}\phi\rangle}$ using (formally) Wick's lemma.

3.11.1. *Example: scalar theory with ϕ^3 interaction.* Let (M, g) be a compact Riemannian manifold. Consider the path integral

$$I(\hbar) = \int_{C^\infty(M)} \mathcal{D}\phi\; e^{\frac{i}{\hbar}\int_M \left(\frac{1}{2}\langle d\phi, d\phi\rangle_{g^{-1}} + \frac{m^2}{2}\phi^2 + \frac{g}{3!}\phi^3\right) d\mathrm{vol}} \tag{90}$$

where $m > 0$ is a parameter of the theory – the “mass” (of the field quanta); g is a coupling constant and we treat the ϕ^3 as perturbing the Gaussian integral. Perturbative evaluation of (90) yields

$$I_{\mathrm{pert}}(\hbar) = \det{}^{-\frac{1}{2}}(\Delta + m^2) \cdot \sum_\Gamma \frac{\hbar^{-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|}\Phi(\Gamma) \tag{91}$$

where the sum goes over 3-valent graphs Γ, with

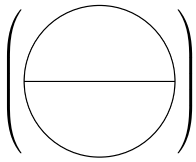
$$\Phi(\Gamma) = g^V \int_{M^{\times V}} d^n x_1 \cdots d^n x_V \prod_{e=(v_1,v_2)} G(x_{v_1}, x_{v_2}) \tag{92}$$

where V is the number of vertices in Γ, dx_i stands for the Riemannian volume element on i-th copy of M, the product goes over edges e of Γ and v_1, v_2 are the vertices adjacent to the edge; $G(x, y)$ is the Green's function for the differential operator $\Delta + m^2$.

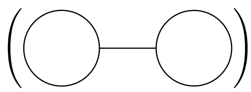
Remark 3.63. One can represent the Green's function $G(x, y)$ by Feynman-Kac formula, as an integral over paths on M going from y to x. Then $\Phi(\Gamma)$ becomes represented as an integral over the mapping space $\mathrm{Map}(\Gamma, M)$. Note that this mapping space is fibered over $M^{\times V}$ (by evaluating the map at the vertices of Γ)

and r.h.s. of (92) can be viewed as the result of the fiber integral over fibers of $\mathrm{Map}(\Gamma, M) \to M^{\times V}$ (i.e. over paths on M representing the edges of Γ, between vertices fixed at points $x_1, \ldots, x_V$ on M).

Example 3.64. The contribution of theta graph to the r.h.s. of (91) is:

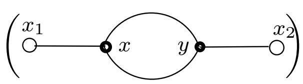
(93) $\frac{\hbar}{12}\,\Phi$  $= \frac{\hbar\, g^2}{12} \int_{M\times M} d^n x\, d^n y\, G(x,y)^3$

And the contribution of the dumbbell graph is:

(94) $\frac{\hbar}{8}\,\Phi$  $= \frac{\hbar\, g^2}{8} \int_{M\times M} d^n x\, d^n y\, G(x,x)\, G(x,y)\, G(y,y)$

Similarly, one can calculate expectation values, e.g. of products $\prod_{i=1}^m \phi(x_i)$ of the values of the field ϕ in several fixed points on M, with respect to the perturbed Gaussian measure (the integrand of (90)). The result is again given as a sum over graphs, with several unique marked vertices.

Example 3.65. The following Feynman graph gives a contribution to the normalized expectation value (w.r.t. to the perturbed measure) $\ll \phi(x_1)\phi(x_2) \gg_{\mathrm{pert}}$:

(95) $\frac{\hbar^2}{2}\,\Phi$  $= \frac{\hbar^2\, g^2}{2} \int_{M\times M} d^n x\, d^n y\, G(x_1,x)\, G(x,y)^2\, G(y,x_2)$

Here the two marked vertices are fixed at points x_1, x_2 whereas the unmarked vertices move around and we integrate over their possible positions on M.

3.11.2. *Divergencies!* **Problem:** Green's function $G(x,y)$ for the operator $\Delta + m^2$ an n-dimensional Riemannian manifold M behaves,as the points x and y approach each other, as

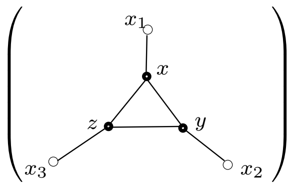
$$G(x,y) \underset{x\to y}{\sim} \frac{\mathrm{const}}{|x-y|^{n-2}}$$

(Case $n = 2$ is special: then $G(x,y) \sim C \cdot \log|x-y|$.) This implies that the integrals over $M^{\times V}$ on the r.h.s. of (92) are, typically, (depending on $n = \dim M$ and on the combinatorics of Γ, see examples below) divergent: the integrand typically has non-integrable singularities near diagonals of $M^{\times V}$. Lecture 14, 10/10/2016.

Examples.

(i) for $n = 2$ and Γ any graph without "short loops" (edges connecting a vertex to itself), there is no divergency.
(ii) The integrand in (93) behaves as $\frac{1}{|x-y|^{3n-6}}$ near the diagonal $x = y$; this singularity is non-integrable iff $3n - 6 \geq n$ or, equivalently, if $n \geq 3$. So, for M of dimension ≥ 3, theta graph for scalar ϕ^3 theory is divergent.
(iii) By a similar argument, graph (95) diverges iff $2 \cdot (n-2) \geq n$ or equivalently $n \geq 4$.
(iv) For the graph (94), singularity of $G(x,y)$ on the diagonal $x = y$ is always integrable but evaluations of the propagator at coinciding points $G(x,x)$ and $G(y,y)$, corresponding to short loops of the graph, are ill-defined for $n \geq 2$.

(v) Consider the graph

Φ  $=$

$$= g^3 \int_{M\times M\times M} d^n x\, d^n y\, d^n z\, \underbrace{G(x_1,x)G(x_2,y)G(x_3,z)G(x,y)G(y,z)G(z,x)}_{\psi}$$

contribution to the 3-point correlation function $\ll \phi(x_1)\phi(x_2)\phi(x_3) \gg_{\mathrm{pert}}$. The integrand ψ has integrable singularities at all diagonals where *pairs* of points collide. However,near the diagonal $x = y = z$, when x, y, z are within distance of order $r \to 0$ of each other, we have $\psi \sim \frac{1}{r^{3(n-2)}}$, and we think of the integral as $\int_M d^n x \int_{M\times M} d^n y\, d^n z$. The internal integral over y, z for fixed x diverges iff $3(n-2) \geq 2n$ or equivalently $n \geq 6$.

Generally, one can say whether the graph diverges or not by analyzing the behavior of the integrand at all diagonals. The answer is as follows. Define the **weight** $w(\Gamma')$ of a graph Γ' with $E_{\Gamma'}$ edges and $V_{\Gamma'}$ vertices as

$$w(\Gamma') := E_{\Gamma'} \cdot (n-2) - (V_{\Gamma'} - 1) \cdot n$$

Lemma 3.66. $\Phi(\Gamma)$ diverges iff the graph Γ contains a subgraph $\Gamma' \subset \Gamma$ with non-negative weight $w(\Gamma') \geq 0$.

This lemma applies to scalar theory with arbitrary polynomial interaction $p(\phi)$, not necessarily ϕ^3 (monomials present in $p(\phi)$ restrict admissible valencies of vertices of contributiong graphs Γ).

Remark 3.67. Consider ϕ^3 theory on a manifold of dimension n.

- For $n = 3$, a graph Γ diverges iff Γ either contains a short loop or contains a theta graph (93) as a subgraph (a corollary of Lemma 3.66).
- More generally, for $n < 6$, there is a finite list of subgraphs with non-negative weight.
- For $\Gamma' \subset \Gamma$, let us call "leaves" of Γ' the edges connecting vertices of Γ' to vertices of Γ not belonging to Γ'. For $n = 6$, the weight of Γ' is non-negative, iff the number of leaves of Γ' is ≤ 3. (There are infinitely many such subgraphs.)
- For $n > 6$, there are infinitely many divergent Γ' and there is nor restriction on the number of leaves for them.

3.11.3. *Regularization and renormalization.* The logic of dealing with divergencies of Feynman graphs for the path integral is to first introduce a

Step I: Regularization. We want to replace the path integral $I(\hbar)$ by a regularized version $I_\epsilon(\hbar)$ with a small parameter ϵ the *regulator*. Here are some of the ideas of regularization.

a) Replace $M \to M_\epsilon$ – a lattice or trianglation or cellular decomposition with spacing/typical cell size ϵ. Space of fields F gets replaces by a finite-dimensional space F_ϵ (modelled on functions on the set vertices or, e.g., cellular cochains of M_ϵ). Action S gets replaced by a finite-difference approximation S_ϵ. Then

$I_\epsilon(\hbar) = \int_{F_\epsilon} e^{\frac{i}{\hbar}S_\epsilon}$ is a well-defined finite-dimensional integral. It can be developed in Feynman graphs, $I_\epsilon(\hbar) \propto \sum_\Gamma \Phi_\epsilon(\Gamma)$ with $\Phi_\epsilon(\Gamma)$ the regularized (finite) weights of Feynman graphs.

b) Regularize the Feynman weights of graphs directly (without deriving this regularization from a regularization of the path integral itself), $\Phi(\Gamma) \to \Phi_\epsilon(\Gamma)$. E.g. regularize the propagator $G(x,y)$ as follows (some of the possible options):
   1. Proper time cut-off: $G_\epsilon(x,y) = \int_\epsilon^\infty dt\, K(x,y|t)$ with $K(x,y|t)$ the *heat kernel* – the integral kernel of the operator $e^{-t(\Delta+m^2)}$.
   2. Spectral cut-off: $G_\Lambda(x,y) = \sum_{\lambda<\Lambda} \frac{1}{\lambda}\Psi_\lambda(x)\overline{\Psi_\lambda(y)}$ where λ runs over eigenvalues of the operator $\Delta + m^2$ (up to Λ) and Ψ_λ are the corresponding eigenfunctions. Here the cut-off $\Lambda = 1/\epsilon$ is large rather than small.
   3. Momentum cut-off (case of $M = \mathbb{R}^n$): $G_\Lambda(x,y) = \int_{|k|<\Lambda} d^n k\, \frac{e^{i(k,x-y)}}{k^2+m^2}$ where the integral is over a ball of large radius $\Lambda = \epsilon^{-1}$ in the momentum space $(\mathbb{R}^n)^* \ni k$.
   4. Regularization $G_\epsilon(x,y) = \int_0^\infty dt\, t^\epsilon\, K(x,y|t)$, with ϵ the regulator. The integral over t is convergent for $\mathrm{Re}(\epsilon) > \frac{n}{2} - 1$, and possesses a meromorphic continuation to the entire $\mathbb{C} \ni \epsilon$; we are interested in the limit $\epsilon \to 0$ of the continuation.

Remark 3.68. The functional determinant in (91) also has to be regularized, e.g. via zeta-regularization, as $\det_{\zeta-\mathrm{reg}}(\Delta+m^2) := e^{-\zeta'(0)}$ with $\zeta(s) = \sum_\lambda \lambda^{-s}$ the zeta function of the operator $\Delta + m^2$ (λ runs over the eigenvalues and it is implied that we take the analytic continuation of the zeta function to $s = 0$).

Whichever way we go about regularization, we get regularized weights of Feynman graphs $\Phi_\epsilon(\Gamma)$. However, the limit of removing the regulator $\lim_{\epsilon\to 0}\Phi_\epsilon(\Gamma)$ typically does not exist. To deal with this, we introduce

Step II: Renormalization.

We replace the action with the renormalized action

$$S(\phi) \to \widetilde{S}_\epsilon(\phi) = S(\phi) + \sum_i c_i(\epsilon) A_i(\phi) \tag{96}$$

where corrections $A_i(\phi) = \int_M d^n x\ \mathcal{A}_i(\phi)$ are local expressions in the field ϕ – *counterterms*, with coefficients $c_i(\epsilon)$ diverging as ϵ^{-k} (for some positive k) or $\log\epsilon$ as $\epsilon \to 0$. Replacement (96) should be such that when we compute Feynman diagrams for the renormalized action $\widetilde{\Phi}_\epsilon(\Gamma)$, the limit $\epsilon \to 0$ exists.[33]

Thus, local action $S(\phi)$ is replaced by $\widetilde{S}_\epsilon(\phi)$ with counterterms divergent as the regulator $\epsilon \to 0$, but the path integral is now perturbatively well-defined:

$$\lim_{\epsilon\to 0} \widetilde{I}_\epsilon(\hbar) =: \widetilde{I}(\hbar)$$

where l.h.s. is defined by regularized Feynman diagrams for the renormalized action.

In practice, counterterms in (96) correspond to the possible divergent subgraphs (cf. Lemma 3.66) and are introduced in order to compensate for these divergencies. E.g. in scalar theory with polynomial perturbation $p(\phi)$, one can assign to a divergent subgraph Γ' of weight $w(\Gamma') \geq 0$ with d leaves the counterterm $\mathcal{A}_{\Gamma'}(\phi) = \phi(x)^d$

[33] To be more precise: counterterms in the renormalized action produce new vertices (with ϵ-dependent coefficients) for the Feynman rules. Contributions of graphs containing these new vertices compensate for the divergence, in the limit $\epsilon \to 0$, of the graphs of original theory.

with coefficient $c_{\Gamma'}(\epsilon) = c_{\Gamma'} \cdot \epsilon^{-w(\Gamma')}$ if the weight $w(\Gamma') > 0$ and $c_{\Gamma'}(\epsilon) = c_{\Gamma'} \cdot \log \epsilon$ if $w(\Gamma') = 0$ with $c_{\Gamma'}$ a constant.

Remark 3.69. In particular, by Remark 3.67, for ϕ^3 scalar theory in dimension < 6, we need finitely many counterterms of form ϕ^d for some values of $d \geq 0$ (number of leaves of Γ') in (96). In dimension 6 there are infinitely many divergent subgraphs, but we only need counterterms ϕ^d with $0 \leq d \leq 3$. In dimension > 6, we need counterterms of form ϕ^d for all d. Thus, one says that in dimensions up to 6, scalar ϕ^3 theory is *renormalizable* (finitely many counterterms) and in dimensions > 6 it is *non-renormalizable*.

3.11.4. *Wilson's picture of renormalization ("Wilson's RG flow").* In Wilson's picture [35], one considers the *tower* of spaces of fields F_Λ with different values of cut-off Λ (originally, the momentum cut-off, though other realizations are possible, see below), equipped with associated actions S_Λ "at cut-off Λ" ("Wilson's effective actions"):

(97)
$$\underbrace{F = F_\infty, S}_{\text{local theory}} \cdots \twoheadrightarrow \underbrace{F_\Lambda, S_\Lambda}_{\text{theory at finite }\Lambda} \twoheadrightarrow F_{\Lambda'}, S_{\Lambda'} \twoheadrightarrow \cdots \twoheadrightarrow \underbrace{F_0, S_0}_{\text{effective theory on zero-modes}}$$

For $\Lambda > \Lambda'$, we have a projection

$$P^{\Lambda \to \Lambda'} : F_\Lambda \twoheadrightarrow F_{\Lambda'} \tag{98}$$

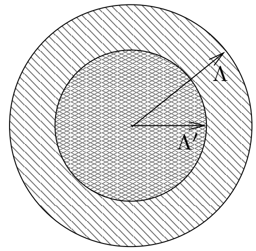
and the actions are related by a pushfroward (fiber integral) $S_{\Lambda'} = P_*^{\Lambda\to\Lambda'} S_\Lambda$ defined by

$$e^{\frac{i}{\hbar} S_{\Lambda'}(\phi')} := \int \mathcal{D}\widetilde{\phi}\; e^{\frac{i}{\hbar} S_\Lambda(\phi' + \widetilde{\phi})} \tag{99}$$

where we are integrating over $\widetilde{\phi}$ in the fiber $\widetilde{F}_{\Lambda,\Lambda'}$ of the projection (98) – "fields between Λ and Λ'".

Examples of realizations:

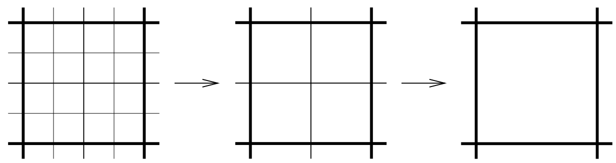
(1) Wilson's original realization. For $M = \mathbb{R}^n$, take F_Λ to be the space of functions of form $\phi(x) = \int_{B_\Lambda \subset (\mathbb{R}^n)^*} d^n k\;\; e^{i(k,x)} \psi(k)$ where $B_\Lambda = \{k \in (\mathbb{R}^n)^* \text{ s.t. } ||k|| \leq \Lambda\}$. I.e. F_Λ consists of functions whose Fourier transform is supported inside the ball of radius Λ in the momentum space $(\mathbb{R}^n)^* \ni k$. Then, for $\Lambda \to \Lambda'$, pushforward $P_*^{\Lambda\to\Lambda'}$ corresponds to integrating out fields in a spherical layer $\Lambda' < ||k|| \leq \Lambda$ in the momentum space.



Picture of Wilson's "renormalization group (RG) flow" amounts to "flowing" from theory at large Λ_{big} (the cut-off) to theory at small Λ by successively integrating out thin spherical layers in the momentum space.

(2) For M compact, we can take $F_\Lambda = \text{Span}_{\lambda \leq \Lambda}\{\Psi_\lambda\}$ – the span of eigenfunctions of the operator $\Delta + m^2$ with eigenvalues $\lambda \leq \Lambda$.

(3) Let $\cdots \succ T_{i+1} \succ T_i \succ \cdots$ be a sequence of CW decompositions of M such that T_{i+1} is a subdivision T_i (the we say that T_i is an *aggregation* of T_{i+1}) and mesh (typical size of cells) of T_i decays fast enough as $i \to \infty$.



We can set $F_i = C^0(T_i)$ – zero-cochains (functions on vertices of T_i), and $S_i \in \mathrm{Fun}(F_i)$ a suitable finite-difference replacement of the action satisfying the compatibility condition w.r.t. aggregations $S_i = P_*(S_{i+1})$.[34]

Remark 3.70. Pushforwards out of the top tier F, S of the tower (97) are ill-defined, and it has to be replaced with the asymptotic "tail" of the tower $F_{\Lambda_{\mathrm{big}}}, S_{\Lambda_{\mathrm{big}}}$ with $S_{\Lambda_{\mathrm{big}}}(\phi) \underset{\Lambda_{\mathrm{big}}\to\infty}{\sim} \widetilde{S}_{\Lambda_{\mathrm{big}}}(\phi) = S(\phi) + \sum_i c_i(\Lambda_{\mathrm{big}}) A_i(\phi)$ the renormalized action (96). Then, if e.g. F_0 is a point, S_0 is given by the sum of connected Feynman diagrams for the renormalized action.

Lecture 15, 10/12/2016.

4. Batalin-Vilkovisky formalism

4.1. **Faddeev-Popov construction.** Faddeev-Popov construction appeared in [?] as a way to resolve the problem of degeneracy of critical points of the Yang-Mills action, in order to construct the perturbative path integral (Feynman diagrams) for the Yang-Mills theory. The construction in fact applies to a large class of gauge theories. Here we study a finite-dimensional model for this situation.

Let G be a compact Lie group of dimension m acting freely on a finite-dimensional n-manifold X with

$$\gamma : G \times X \to X \tag{100}$$

the action map. Let $\mathfrak{g} = \mathrm{Lie}(G)$ be the Lie algebra of G and assume that we have chosen a basis $\{T_a\}$ in $\mathfrak{g}$. Denote by $v_a \in \mathfrak{X}(X)$ the fundamental vector fields on X by which the generators T_a act on X.

Let $S \in C^\infty(X)^G$ be a G-invariant function on X, and let $\mu \in \Omega^n_c(X)^G$ be a G-invariant top form with compact support.

We are interested in the integral

$$I = \int_X \mu\, e^{\frac{i}{\hbar} S} \tag{101}$$

We can rewrite it as the integral over the quotient X/G:

$$I = \mathrm{Vol}(G) \int_{X/G} \widetilde{\mu}\, e^{\frac{i}{\hbar}\widetilde{S}} \tag{102}$$

Where $\widetilde{S} \in C^\infty(X/G)$ is such that

$$S = p^* \widetilde{S} \tag{103}$$

[34] See [24, 25, 12] for an example; there we need cochains of all degrees in F_i.

where $p : X \to X/G$ is the quotient map; $\widetilde{\mu} \in \Omega^{n-m}(X/G)$ is a top form on the quotient constructed in such a way that

$$\iota_{v_m} \cdots \iota_{v_1} \mu = p^* \widetilde{\mu}$$

Note that the $(n-m)$-form on the l.h.s. here is *basic* (invariant and horizontal w.r.t G-action) and hence is a pullback from the quotient. Note that we can write

$$\mu = p^* \widetilde{\mu} \wedge \chi \tag{104}$$

where $\chi \in \Omega^m(X)$ is a (any) form on X with the property that its restrictions to G-orbits in X yield the volume form on the orbits induced from Haar measure on G (via the identification of an orbit with G by picking a base point on the orbit). The normalization of χ should be such that $\iota_{v_m \wedge \cdots \wedge v_1} \chi = 1$. Note that (103,104) together imply (102).

Let $\phi : X \to \mathfrak{g}$ be a $\mathfrak{g}$-valued function on X such that:

- zero is a regular value of ϕ,
- $\sigma = \phi^{-1}(0) \subset X$ intersects every G-orbit transversally, exactly N times, for some fixed $N \geq 1$.[35]

We think of σ as a (local) section of G-orbits. We refer to σ as the **gauge-fixing** (and to ϕ as the *gauge-fixing function*).

Since $\sigma \subset X$ is an N-fold covering of the quotient X/G, (102) implies

$$I = \frac{\mathrm{Vol}(G)}{N} \int_\sigma \iota_{v_m \wedge \cdots \wedge v_1} \mu \;\; e^{\frac{i}{\hbar} S} \Big|_\sigma = \frac{\mathrm{Vol}(G)}{N} \int_X \delta^{(m)}(\phi) \;\; \iota_{v_m \wedge \cdots \wedge v_1} \mu \;\; e^{\frac{i}{\hbar} S} \tag{105}$$

Here $\delta^{(m)}(\phi) = \delta(\phi) \cdot \bigwedge_a d\phi^a$ is the distributional m-form supported on σ; $\delta(\phi) = \prod_a \delta(\phi^a(x))$ is the delta-distribution (not a form) supported on $\sigma \subset X$. We can think of $\delta(\phi)$ and $\delta^{(m)}(\phi)$ as the pullbacks by ϕ of the standard Dirac delta function and delta form, respectively, centered at the origin in $\mathfrak{g}$.

Note that, generally, for $C \subset X$ a k-cycle, we have a distributional form $\delta_C^{(n-k)} : \Omega^k(X) \to \mathbb{R}$ mapping

$$\omega \mapsto \int_C \omega|_C =: \text{“} \int_X \delta_C^{(n-k)} \wedge \omega \text{ ”}$$

Formula (105) is a special case of this, for $C = \sigma$.

Remark 4.1. The delta form $\delta^{(m)}(\phi)$ depends only on zero-locus of ϕ and, in particular, does not change under rescaling $\phi \mapsto \lambda \cdot \phi$ with $\lambda \neq 0$ a constant. On the other hand, the delta function $\delta(\phi)$ changes with rescaling of ϕ, by λ^{-m}.

Let J be a function on X such that

$$\bigwedge_a d\phi^a \wedge \iota_{v_m \wedge \cdots \wedge v_1} \mu = J \cdot \mu \tag{106}$$

Lemma 4.2. The coefficient J in (106) is:

$$J(x) = \det{}_{\mathfrak{g}} FP(x) \tag{107}$$

where

$$FP(x) = d_x \phi \circ d_{1,x} \gamma : \; \mathfrak{g} \to \mathfrak{g} \tag{108}$$

[35]Ideally, we would like to have a single intersection, i.e. $N = 1$, but typically, for G compact, there are topological obstructions for having a global section of $p : X \to X/G$ defined as a zero locus of a globally defined function. E.g. for $G = U(1)$, orbits are circles, thus ϕ has to have some even number of zeroes on an orbit.

is an endomorphism of $\mathfrak{g}$ depending on a point $x \in X$; here $d_{1,x}\gamma : \mathfrak{g} \to T_x X$ is the infinitesimal action of $\mathfrak{g}$ on X viewed as a derivative of the group action (100); $d_x\phi : T_x X \to \mathfrak{g}$ is the derivative of ϕ. In components, we have

$$FP(x)^a_b = \langle d\phi^a(x), v_b(x)\rangle = v^b(\phi_a)|_x \tag{109}$$

One calls $J(x)$ given by (107) the *Faddeev-Popov determinant.*

Proof. First note that nondegeneracy of $FP(x)$ is equivalent to $\phi^{-1}(\phi(x)) \subset X$ intersecting the G-orbit through X transversally. If the intersection is nontransversal, then l.h.s. of (106) is obviously vanishing ant the statement is trivial. So, we assume that the intersection is transversal, i.e. theat $FP(x)$ is non-degenerate.

Let $V = \mathrm{im} d_{1,x}\gamma = \mathrm{Span}(v_a(x)) \subset T_x X$ be the tangent space to G-orbit through x and let $\mathrm{Ann}(V) \subset T^*_x X$ be its annihilator in the cotangent space. Let $\alpha_1, \ldots, \alpha_{n-m}$ be a basis in $\mathrm{Ann}(V)$. We have a basis $(d\phi^1(x), \ldots, d\phi^m(x), \alpha_1, \ldots, \alpha_{n-m})$ in $T^*_x X$ (fact that this is a basis is equivalent to non-degeneracy of $FP(x)$ which we assumed). Without loss of generality (by normalizing αs appropriately), we may assume $\mu = \bigwedge_{a=1}^m d\phi^a(x) \wedge \alpha_1 \wedge \cdots \wedge \alpha_{n-m}$. Contracting with $v_m \wedge \cdots \wedge v_1$ and using orthogonality of vs and αs, we have

$$\iota_{v_m\wedge\cdots\wedge v_1}\mu = \left(\sum_{s\in S_m} (-1)^s \prod_{a=1}^m \left\langle d\phi^a, v_{s(a)}\right\rangle\right) \alpha_1\wedge\cdots\wedge\alpha_{n-m} = \det_{\mathfrak{g}} FP(x)\cdot\alpha_1\wedge\cdots\wedge\alpha_{n-m}$$

Wedging with $\bigwedge_{a=1}^m d\phi^a(x)$, we get the statement of the Lemma. □

Thus, we have the following.

Theorem 4.3 (Faddeev-Popov)**.**

$$\int_X \mu\, e^{\frac{i}{\hbar}S} = \frac{\mathrm{Vol}(G)}{N} \int_X \mu\, \delta(\phi(x)) \cdot \det_{\mathfrak{g}} FP(x) \cdot e^{\frac{i}{\hbar}S} \tag{110}$$

Next, we would like to deal with integrals of stationary phase type, i.e. with integrands of form $e^{\frac{i}{\hbar}(\cdots)}$. We can achieve that, at the cost of introducing auxiliary integration variables, by using integral presentations for the delta function (as a Fourier transform on the unit) and for the determinant (as a Gaussian integral over odd variables):

$$\delta(\phi(x)) = \frac{1}{(2\pi\hbar)^m} \int_{\mathfrak{g}^*} d^m\lambda\; e^{\frac{i}{\hbar}\langle\lambda,\phi(x)\rangle} \tag{111}$$

$$\det_{\mathfrak{g}} FP(x) = \left(\frac{\hbar}{i}\right)^m \int_{\Pi(\mathfrak{g}\oplus\mathfrak{g}^*)} \prod_{a=1}^m (Dc_a D\bar{c}_a)\; e^{\frac{i}{\hbar}\langle\bar{c},FP(x)c\rangle} \tag{112}$$

Here the auxiliary odd variables c_a, $\bar{c}_a$ are called *Faddeev-Popov ghosts*; λ is the even *Lagrange multiplier* variable. For brevity, we will denote the odd Berezin measure in (112) by $D^m c\; D^m\bar{c}$. Plugging integral presentations (111,112) into (110), we obtain the following.

Theorem 4.4 (Faddeev-Popov)**.**

$$\int_X \mu\, e^{\frac{i}{\hbar}S} = \frac{\mathrm{Vol}(G)}{N\cdot(2\pi i)^m} \int_{X\times\mathfrak{g}^*\times\Pi(\mathfrak{g}\oplus\mathfrak{g}^*)} \mu\; d^m\lambda\; D^m c\; D^m\bar{c}\;\; e^{\frac{i}{\hbar}S_{FP}(x,\lambda,c,\bar{c})} \tag{113}$$

where

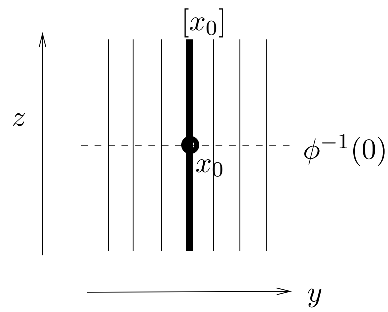
$$S_{FP}(x,\lambda,c,\bar{c}) = S(x) + \langle\lambda,\phi(x)\rangle + \langle\bar{c},FP(x)c\rangle \tag{114}$$

is the *Faddeev-Popov action* associated to the gauge-fixing ϕ.

The point of replacing the integral (101) with the r.h.s. of (113) is that the former cannot be calculated, in the asymptotics $\hbar \to 0$, by stationary phase formula, since the critical points of S are not isolated but rather come in G-orbits (hence the Hessian of S at a critical point is always degenerate and one cannot construct Feynman rules in this case). On the other hand the integral in the r.h.s. of (113) has isolated critical points with non-degenerate Hessians of the extended action S_{FP} and the stationary phase formula is applicable.

4.1.1. *Hessian of S_{FP} in an adapted chart.* Let x_0 be a critical point of S lying on a critical G-orbit $[x_0] \subset X$ and satisfying $\phi(x_0) = 0$. Let $(y_1, \ldots, y_{n-m}; z^1, \ldots, z^m)$ be an adapted local coordinate chart on X near x_0, such that:

(i) x_0 is given by $y = z = 0$.
(ii) $[x_0]$ is given by $y = 0$; moreover, G-orbits are locally given by $y =$ const.
(iii) Locally ϕ is given by $\phi^a = z^a$.



For instance, G-invariance of S implies that $S = S(y)$ and $\frac{\partial}{\partial z^a} S = 0$.

Hessian of S has the form

$$\partial^2 S|_{x_0} = \left(\begin{array}{c|c} \left.\frac{\partial^2 S}{\partial y_i \partial y_j}\right|_{x_0} & 0 \\ \hline 0 & 0 \end{array}\right)$$

where first $(n-m)$ rows/columns correspond to y_i variables and the last m rows/columns correspond to z^a variables. We are assuming that the block $\left.\frac{\partial^2 S}{\partial y_i \partial y_j}\right|_{x_0}$ is non-degenerate, i.e. that all degeneracy of the Hessian of S comes from G-invariance. In other words, we assume that $\mathrm{rank}(\partial^2 S|_{x_0}) = n - m$.

The Hessian $\partial^2 S|_{x_0}$ is, obviously, degenerate. However, let us consider

$$\underbrace{\partial^2 (S + \langle \lambda, \phi(x) \rangle)\big|_{x_0, \lambda=0}}_{\in \mathrm{Sym}^2 (T_{x_0} X \oplus \mathfrak{g}^*)^*} = \left(\begin{array}{c|c|c} \left.\frac{\partial^2 S}{\partial y_i \partial y_j}\right|_{x_0} & 0 & 0 \\ \hline 0 & 0 & \delta^a_b \\ \hline 0 & \delta^b_a & 0 \end{array}\right) \qquad (115)$$

Here rows correspond to y_i, z^a, λ_a and columns correspond to y_j, z^b, λ_b. Note that this Hessian is *non-degenerate*! The $z - \lambda$ blocks that appeared because of the new $\langle \lambda, \phi(x) \rangle$ term make the matrix non-degenerate.

Next, note that assumption (ii) above implies that fundamental vector fields v_a locally have the form $v_a = \sum_b f^b{}_a(y, z) \frac{\partial}{\partial z^b}$ with $(f^b{}_a)(y, z)$ a non-degenerate $m \times m$ matrix. Thus, by (109), we have $FP(x)^a_b = f^a{}_b(y, z)$. Therefore, the part of the

Hessian corresponding to the ghost part of Faddeev-Popov action is:

$$\partial^2 \left\langle \bar{c}, FP(x)c \right\rangle\Big|_{x_0,c=\bar{c}=0} = \left(\begin{array}{c|c} 0 & -f^b{}_a \\ \hline f^a{}_b & 0 \end{array}\right) \tag{116}$$

where rows correspond to $c^a, \bar{c}_a$ and columns correspond to c^b, $\bar{c}_b$.

Assembling (115) and (116), we get the full Hessian of Faddeev-Popov action

$$\partial^2 S_{FP}\Big|_{x_0,\lambda=c=\bar{c}=0} = \left(\begin{array}{c|c|c||c|c} \frac{\partial^2 S}{\partial y_i \partial y_j}\Big|_{x_0} & & & & \\ \hline & & \delta^a_b & & \\ \hline & \delta_a^b & & & \\ \hline\hline & & & & -f^b{}_a \\ \hline & & & f^a{}_b & \end{array}\right) \tag{117}$$

with row variables $y_i, z^a, \lambda_a, c^a, \bar{c}_a$ and column variables $y_j, z^b, \lambda_b, c^b, \bar{c}_b$. All the non-filled blocks are zero. From this explicit form it is obvious that the full Hessian of the Faddeev-Popov action is non-degenerate.

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4.1.2. *Stationary phase evaluation of Faddeev-Popov integral.* Critical point (Euler-Lagrange) equations for Faddeev-Popov action $S_{FP}(x, \lambda, c, \bar{c})$ (114) read:

$$c = \bar{c} = 0 \tag{118}$$

$$\frac{\partial}{\partial x_i} S(x) + \left\langle \lambda, \frac{\partial}{\partial x_i}\phi \right\rangle = 0 \tag{119}$$

$$\phi(x) = 0 \tag{120}$$

Here (118) is equivalent to the Euler-Lagrange equations $\frac{\partial}{\partial c^a} S_{FP} = 0$, $\frac{\partial}{\partial \bar{c}_a} S_{FP} = 0$, whereas (119) corresponds to $\frac{\partial}{\partial x_i} S_{FP} = 0$ (where we dropped the term bilinear in c and $\bar{c}$ which is excluded by (118)); last equation (120) is $\frac{\partial}{\partial \lambda_a} S_{FP} = 0$.

Note that equations (119,120) together correspond to the fact that x is a conditional extremum of S restricted to submanifold $\sigma = \phi^{-1}(0) \subset X$ with λ the Lagrange multiplier. On the other hand, G-invariance of S together with transversality of the local section σ and G-orbits, implies that a conditional extremum of S on σ is in fact a non-conditional extremum (i.e. dS vanishes on the whole tangent space $T_x X$, not just on $T_x \sigma \subset T_x X$). Therefore, (119) implies $\lambda = 0$. Thus, a critical point of S_{FP} has a form $(x_0, \lambda = c = \bar{c} = 0)$ with x_0 an intersection point of the critical G-orbit of $S(x)$ with the gauge-fixing submanifold $\sigma = \phi^{-1}(0)$.

The Hessian of S_{FP} at a critical point (written without using the adapted chart as in (117)), is

$$\partial^2 S_{FP}\Big|_{x_0,\lambda=c=\bar{c}=0} = \tag{121}$$

$$\left(\begin{array}{c|c||c|c} \frac{\partial^2 S}{\partial x^2}\Big|_{x_0} & (d\phi|_{x_0})^T : \mathfrak{g}^* \to T^*_{x_0} X & & \\ \hline d\phi|_{x_0} : T_{x_0} X \to \mathfrak{g} & 0 & & \\ \hline\hline & & & -FP(x_0)^T : \mathfrak{g}^* \to \mathfrak{g}^* \\ \hline & & FP(x_0) : \mathfrak{g} \to \mathfrak{g} & \end{array}\right)$$

with blocks corresponding to variables $x, \lambda, c, \bar{c}$. Its inverse has the following structure:

$$(\partial^2 S_{FP}\big|_{x_0})^{-1} = \left(\begin{array}{c|c||c|c} \mathcal{D} & \beta & & \\ \hline \beta^T & 0 & & \\ \hline\hline & & & FP(x_0)^{-1} \\ \hline & & -FP(x_0)^{-1T} & \end{array}\right) \tag{122}$$

Here $\beta : \mathfrak{g} \to T_{x_0}X$ is the section of the projection $d\phi|_{x_0} : T_{x_0}X \to \mathfrak{g}$ constructed as

$$\beta = d_{1,x_0}\gamma \circ FP(x_0)^{-1} : \mathfrak{g} \to T_{x_0}X$$

where $d_{1,x_0}\gamma : \mathfrak{g} \to T_{x_0}X$ is, as in (108), the infinitesimal action of the Lie algebra $\mathfrak{g}$ on X specialized at the point x_0. Thus, β and $d\phi|_{x_0}$ together give us a splitting

$$T_{x_0}X \simeq T_{x_0}\phi^{-1}(0) \oplus \mathfrak{g} \tag{123}$$

The block $\mathcal{D} \in \mathrm{Sym}^2 T_{x_0}X$ in (122) is the image of $\widetilde{\mathcal{D}} \in \mathrm{Sym}^2 T_{x_0}\phi^{-1}(0)$ under the splitting (123), where $\widetilde{\mathcal{D}}$ is the inverse of $\partial^2_{x_0}(\,S|_{\phi^{-1}(0)})$ – the (invertible) Hessian of S restricted to gauge-fixing submanifold $\phi^{-1}(x_0)$.

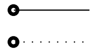
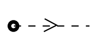
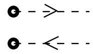
We say that $\mathcal{D}$ is the "propagator" or "Green's function" for $\partial^2 S|_{x_0}$ in the gauge $\phi(x) = 0$.

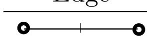
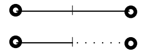
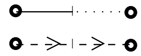
Applying the stationary phase formula to the Faddeev-Popov integral (113), we obtain the following.

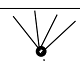
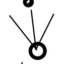
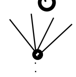
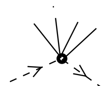
Theorem 4.5 (Stationary phase formula for Faddeev-Popov integral)**.**

$$\int_X e^{\frac{i}{\hbar}S(x)}\mu = \tag{124}$$
$$\frac{\mathrm{Vol}(G)}{(2\pi i)^m} \sum_{\text{crit. } G\text{-orbits } [x_0] \text{ of } S} (2\pi\hbar)^{\frac{n+m}{2}} \left(\frac{i}{\hbar}\right)^m e^{\frac{i}{\hbar}S(x_0)} \left|\det \partial^2_{x_0} S\big|_{\phi^{-1}(0)}\right|^{-\frac{1}{2}} \cdot \det{}_{\mathfrak{g}} FP(x_0) \cdot e^{\frac{\pi i}{4}\,\mathrm{sign}\,\partial^2_{x_0} S\big|_{\phi^{-1}(0)}} \times$$
$$\times \sum_\Gamma \frac{\hbar^{1-\chi(\Gamma)}}{|\mathrm{Aut}(\Gamma)|} \cdot \Phi(\Gamma)$$

Here in the r.h.s. we pick, for every critical G-orbit $[x_0]$ of S, a single representative x_0 – one intersection point of $[x_0]$ with $\phi^{-1}(0)$. The Feynman rules for calculating $\Phi(\Gamma)$ are as follows.

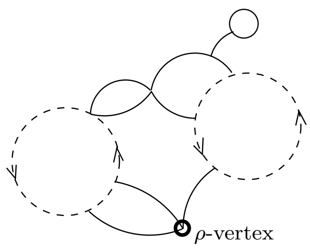
Half-edge	field
	y_i
	λ_a
	c^a
	c_a

Edge	propagator
	$i\mathcal{D} \in \mathrm{Sym}^2 T_{x_0}X$
	$i\beta : \mathfrak{g} \to T_{x_0}X$
	$iFP(x_0)^{-1} : \mathfrak{g} \to \mathfrak{g}$

Vertex	y-valency	vertex tensor
	$k \geq 3$	$i\partial^k S\big|_{x_0} \in \mathrm{Sym}^k T^*_{x_0}X$
	$l \geq 0$	$i\partial^l \rho\big|_{x_0} \in \mathrm{Sym}^l T^*_{x_0}X$
	$j \geq 2$	$i\partial^j \phi\big|_{x_0} \in \mathrm{Sym}^j T^*_{x_0}X \otimes \mathfrak{g}$
	$q \geq 1$	$i\partial^q FP\big|_{x_0} \in \mathrm{Sym}^q T^*_{x_0}X \otimes \mathrm{End}(\mathfrak{g})$

Here we assume that local coordinates y_i on X are introduced near the critical point x_0. "y-valency" refers to the number of adjacent solid (y-)half-edges. The second vertex is the marked vertex that should appear in Γ exactly once; ρ is the density of the volume form μ in the local coordinates y_i, i.e. $\mu = \rho(y) d^n y$.

Remark 4.6. In the special case when the gauge-fixing ϕ is linear in local coordinates y_i,[36] the third vertex above vanishes, and thus λ-half-edges do not appear in admissible graphs in the r.h.s. of (124) at all. Here is a typical graph Γ in such situation:



Remark 4.7. Assume that, in addition to ϕ being linear in y_i, fundamental vector fields have constant coefficients in local coordinates y_i near x_0.[37] Then both third and fourth vertex in the Feynman rules above vanish. In this case one has only solid y-edges in admissible graphs Γ.

Remark 4.8. In order to define invariantly (cf. Remark 3.17) the determinant of the restricted Hessian $\det \partial^2_{x_0} S|_{\phi^{-1}(0)}$ appearing in the r.h.s. of (124), we need a volume element on $T_{x_0}\phi^{-1}(0)$, i.e. an element in $\mathrm{Det}\, T^*_{x_0}\phi^{-1}(0)$.[38] To construct it, we use the short exact sequence $T_{x_0}\phi^{-1}(0) \hookrightarrow T_{x_0} X \xrightarrow{d\phi|_{x_0}} \mathfrak{g}$ which induces a canonical isomorphism of determinant lines

$$\mathrm{Det}\, T^*_{x_0} X \cong \mathrm{Det}\, \mathfrak{g}^* \otimes \mathrm{Det}\, T^*_{x_0}\phi^{-1}(0)$$

Using it, we can take the (canonically defined) "ratio" of $\mu|_{x_0} \in \mathrm{Det}\, T^*_{x_0} X$ (the volume form on X evaluated at x_0) and $\mu_{\mathfrak{g}} \in \mathrm{Det}\, \mathfrak{g}^*$ – the Lebesgue measure on $\mathfrak{g}$, to obtain $\nu = \frac{\mu|_{x_0}}{\mu_{\mathfrak{g}}} \in \mathrm{Det}\, T^*_{x_0}\phi^{-1}(0)$.

Remark 4.9. In Theorem 4.5, instead of choosing the gauge-fixing $\phi : X \to \mathfrak{g}$ *globally* on X, we can choose individual (*local*) gauge-fixing $\phi_j : U_j \to \mathfrak{g}$ in a tubular neighborhood U_j of j-th critical orbit $[x_0^{(j)}]$ of S, with j going over all critical orbits.

4.1.3. *Motivating example: Yang-Mills theory.* For M a Riemannian (or pseudo-Riemannian) manifold, classical Yang-Mills theory on M with structure group G (a compact group with Lie algebra $\mathfrak{g}$) has the space of fields

$$F = \mathrm{Conn}_{M,G} \simeq \Omega^1(M) \otimes \mathfrak{g}$$

[36]This is the finite-dimensional model for, e.g., the Lorentz gauge $d^*A = 0$ in Yang-Mills theory, see Section 4.1.3 below

[37]This is the finite-dimensional model for the Lorentz gauge in QED (abelian Yang-Mills theory) and explains why Faddeev-Popov ghosts do not appear in the Feynman diagrams for QED (but do appear in non-abelian Yang-Mills theory).

[38]Recall that, for V a vector space, the determinant line $\mathrm{Det}\, V$ is the top exterior power of V, $\mathrm{Det}\, V = \wedge^{\dim V} V$.

– the space of connections in a trivial G-bundle on M.[39] The space of fields is acted on by the group of gauge transformations (principal bundle automorphisms), $\mathrm{Gauge}_{M,G} = C^\infty(M,G)$ and the action is given by $A \mapsto A^g = g^{-1}Ag + g^{-1}dg$. Infinitesimally, the Lie algebra of gauge transformations $\mathrm{gauge}_{M,G} \simeq \Omega^0(M,\mathfrak{g})$ acts by

$$A \mapsto d_A\alpha = d\alpha + [A,\alpha] \in T_A F \tag{125}$$

for $\alpha \in \mathrm{gauge}_{M,G}$ the generator of the infinitesimal transformation.

Yang-Mills action is given by

$$S_{YM}(A) = \frac{1}{2}\int_M \mathrm{tr}\, F_A \wedge *F_A \tag{126}$$

with $F_A = dA + \frac{1}{2}[A,A] \in \Omega^2(M,\mathfrak{g})$ the curvature of the connection; $*$ is the Hodge star associated to the metric on M; tr is the trace in the adjoint representation of $\mathfrak{g}$.

Volume form μ on F (thought of the "Lebesgue measure on the space of connections") and the Haar measure on $\mathrm{Gauge}_{M,G}$ are parts of the functional integral measure for Yang-Mills theory and are, certainly, problematic. One works around them by considering *perturbative* Faddeev-Popov integral, as given by the Feynman graph expansion in the r.h.s. of (124).

For the gauge-fixing $\phi : \mathrm{Conn}_{M,G} \to \mathrm{gauge}$, one of the possible choices is the *Lorentz* gauge, corresponding to

$$\phi(A) = d^*A \tag{127}$$

In this case, Faddeev-Popov endomorphism of gauge is:

$$FP(A) = d^*d_A :\ \Omega^0(M,\mathfrak{g}) \to \Omega^0(M,\mathfrak{g}) \tag{128}$$

– as follows from (125) and (127).

We are interested in evaluating the perturbative contribution of the gauge orbit of zero connection. The fact that the intersection of $\phi^{-1}(0)$ and the gauge orbit through $A = 0$ is transversal at $A = 0$ follows from the Hodge decomposition theorem (which implies $\Omega^1(M,\mathfrak{g}) = \Omega^1(M,\mathfrak{g})_{\mathrm{exact}} \oplus \Omega^1(M,\mathfrak{g})_{\mathrm{coclosed}} = \mathrm{im}(d_{1,A=0}\gamma) \oplus T_{A=0}\phi^{-1}(0)$).

The formal Faddeev-Popov integral for Yang-Mills theory in Lorentz gauge is:

$$Z = \int_{\mathrm{Conn}\oplus\mathrm{gauge}^*\oplus\Pi(\mathrm{gauge}\oplus\mathrm{gauge}^*)} \mathcal{D}A\,\mathcal{D}\lambda\,\mathcal{D}c\,\mathcal{D}\bar{c}\ \ e^{\frac{i}{\hbar}S_{FP}(A,\lambda,c,\bar{c})} \tag{129}$$

with

$$S_{FP}(A,\lambda,c,\bar{c}) = S_{YM}(A) + \int_M \langle\lambda, d^*A\rangle + \int_M \langle\bar{c}, d^*d_A c\rangle \tag{130}$$

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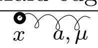
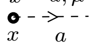
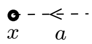
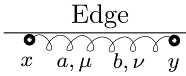
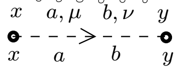
Here $\lambda \in \Omega^{\mathrm{top}}(M,\mathfrak{g}^*)$ where the r.h.s. is our model for the dual of the Lie algebra of gauge transformations. Likewise, $\bar{c} \in \Pi\,\Omega^{\mathrm{top}}(M,\mathfrak{g}^*)$ and $c \in \Pi\,\Omega^0(M,\mathfrak{g})$.

[39] We restrict our discussion to the case of a trivial G-bundle for simplicity. This assumption can be relaxed.

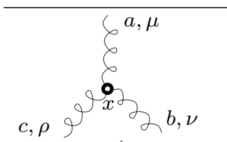
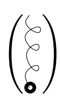
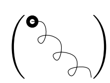
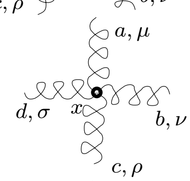
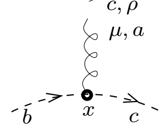
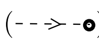
Feynman rules for perturbative calculation of the Faddev-Popov integral for Yang-Mills theory (129) in the case $M = \mathbb{R}^{3,1}$ – the flat Lorentzian space with metric $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ – are as follows.

Hald-edge	field
	$A_\mu^a(x)$
	$c^a(x)$
	$\bar{c}_a(x)$

Here $A_\mu^a(x)$ are the local components of the connection evaluated at a point x, $A = \sum_{a=1}^{\dim \mathfrak{g}} \sum_{\mu=1}^{4} T_a A_\mu^a(x) dx^\mu$, with $\{T_a\}$ the chosen basis in $\mathfrak{g}$ (which we assume to be orthonormal w.r.t. to the Killing form in $\mathfrak{g}$). Likewise, $c^a(x)$ are the components of $c = \sum_{a=1}^{\dim \mathfrak{g}} T_a c^a(x)$ and $\bar{c}_a(x)$ are the components of $\bar{c} = \sum_{a=1}^{\dim \mathfrak{g}} T_a \bar{c}_a(x) d^4x$.

Edge	propagator
	$\int \frac{d^4k}{(2\pi)^4} e^{-i(k,x-y)} \frac{i\delta_{ab}\eta_{\mu\nu}}{k^2+i\epsilon}$
	$\int \frac{d^4k}{(2\pi)^4} e^{-i(k,x-y)} \frac{i\delta_{ab}}{k^2+i\epsilon}$

Here a limit $\epsilon \to +0$ is implied. This provides a regularization for the propagators which, in pseudo-Riemannian case, are singular on the light-cone $(x-y, x-y) = 0$, as opposed to the Riemannian case, where the singularity is just at $x = y$.

Vertex	vertex tensor
	$f^{abc}\eta^{\mu\nu}\left(i\frac{\partial}{\partial x_\rho}\right.$  $- i\frac{\partial}{\partial x_\rho}$  $\left.\right)$ + cycl. perm. of $\{(a,\mu),(b,\nu),(c,\rho)\}$
	$-i\sum_e f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho})$ + cycl. perm. of $\{(a,\mu),(b,\nu),(c,\rho),(d,\sigma)\}$
	$if^{abc}\frac{\partial}{\partial x_\mu}$ ()

These vertices correspond to the cubic and quartic terms $\frac{1}{2}\int \mathrm{tr}\,[A,A]dA$, $\frac{1}{8}\int \mathrm{tr}\,[A,A]\wedge [A,A]$, $\int \langle \bar{c}, d^*[A,c]\rangle$ in the Taylor expansion in the fields of the Faddev-Popov extension of the Yang-Mills action (129).

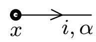
One can also enhance the Yang-Mills theory by adding a *matter term* to the action,

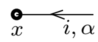
$$S_{YM} \mapsto S_{YM} + \int_M dx\, \langle \bar{\psi}, (i\partial\!\!\!/_A + m)\psi\rangle$$

Here the new matter field ψ is an *odd* complex Dirac fermion field on M – a section of $E \otimes R$ with $E \to M$ the spinor bundle and R a representation of the structure group G. Field ψ has local components $\psi_\alpha^i(x)$ with i the index of spanning the basis of the representation space R and α the spinor index; $\partial\!\!\!/_A = \sum_{\mu,\alpha,\beta,i,j}(\gamma^\mu)_{\alpha\beta}(\delta_{ij}\partial_\mu + (T_a)_{ij}A_\mu^a(x))$ is the Dirac operator, with γ^μ the Dirac gamma-matrices and $(T_a)_{ij}$

the representation matrices of the basis elements T_a of $\mathfrak{g}$; $\langle , \rangle$ is the inner product of Dirac spinors; m is the *mass* of the fermion.

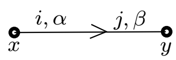
Adjoining the matter field results in the extension of Feynman rules by new half-edges



$\mapsto \psi^i_\alpha(x)$,


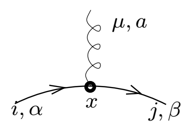
$\mapsto \bar\psi^i_\alpha(x)$

The new edge is:



$$\mapsto \int \frac{d^4k}{(2\pi)^4} e^{-i(k,x-y)} \left(\frac{-i}{\not{k}+m}\right)_{\alpha\beta} \delta_{ij}$$

where the dash in $\not{k} := \sum_\mu k_\mu (\gamma^\mu)_{\alpha\beta}$ stands for contraction with Dirac gamma-matrices. The new vertex is:



$$\mapsto i(\gamma^\mu)_{\alpha\beta}(T_a)_{ij}$$

Remark 4.10. Yang-Mills theory for the group $G = SU(3)$ is the theory of the strong interaction (quantum chromodynamics). The Yang-Mills field A corresponds to the *gluon* – the carrier of the strong interaction and the matter fields ψ correspond to *quarks*. Abelian case $G = U(1)$ corresponds to quantum electrodynamics, with A the photon field and $\psi, \bar\psi$ the electron/positron field. Standard model of particle physics is the Yang-Mills theory with $G = U(1) \times SU(2) \times SU(3)$ (with the factors corresponding to the electromagnetic, weak and strong interactions).

Remark 4.11. Frequently, instead of scaling the Yang-Mills-Faddeev-Popov action in the path integral with $\frac{1}{\hbar}$, as in (129), one sets $\hbar = 1$ but scales the Yang-Mills action as $S_{YM} \mapsto \frac{1}{2g^2}\int \mathrm{tr}\, F_A \wedge *F_A$ (instead of (126)) with g the *coupling constant* of the strong interaction.[40] This normalization can be converted back to ours by setting $\hbar = g^2$ and rescaling the auxiliary fields $\lambda, c, \bar c$ (and the matter fields ψ, $\bar\psi$, if present), by appropriate powers of g. Put another way, with the normalization by the coupling constant g, Feynman graphs are weighed with $g^{-2\chi(\Gamma)}$ instead of $\hbar^{-\chi(\Gamma)}$.

4.2. **Elements of supergeometry.** [41]

4.2.1. *Supermanifolds.*

Definition 4.12. An $(n|m)$-supermanifold $\mathcal{M}$ is a sheaf $\mathcal{O}_\mathcal{M}$, over a smooth n-manifold M (the *body* of $\mathcal{M}$), of supercommutative algebras locally isomorphic to algebras of form $C^\infty(U) \otimes \wedge^\bullet V^*$ with $U \subset M$ open and V a fixed m-dimensional vector space. I.e., there is an atlas on M comprised by open subsets $U_\alpha \subset M$ with chart maps $\phi_\alpha : U_\alpha \to W = \mathbb{R}^n$, with isomorphisms of supercommutative algebras $\Phi_\alpha : \mathcal{O}_\mathcal{M}(U_\alpha) \to C^\infty(\phi_\alpha(U)) \otimes \wedge^\bullet V^* =: \mathcal{A}_\alpha$.

[40] Or equivalently, by rescaling $A \mapsto g \cdot A$, one has $S_{YM} = \frac{1}{2}\int \mathrm{tr}\, dA \wedge *dA + \frac{g}{2}\int \mathrm{tr}\,[A,A] \wedge dA + \frac{g^2}{8}\int \mathrm{tr}\,[A,A]\wedge[A,A]$. In the matter term, if present, the quark-gluon interaction term $\bar\psi A \psi$ also gets rescaled by a factor g.

[41] A reference for the basic definitions on supermanifolds and $\mathbb{Z}$-graded (super)manifolds: Appendix B in [7].

Locally a function on $\mathcal{M}$ is an element of $\mathcal{A}_\alpha$, i.e., has local form

$$f|_{U_\alpha} = \sum_k \sum_{1\leq i_1<\cdots<i_k\leq m} f_{i_1\cdots i_k}(x)\cdot\theta_{i_1}\cdots\theta_{i_k}$$

with $x_1,\ldots,x_n$ the local *even* coordinates on M (pullbacks of the standard coordinates on $\mathbb{R}^n$ by ϕ_α) and $\theta_1,\ldots,\theta_m\in V^*$ the *odd* (anti-commuting) coordinates on V.

Remark 4.13. The augmentation map $\wedge^\bullet V^*\to\mathbb{R}$ induces a globally well-defined augmentation map

$$\mathcal{O}_\mathcal{M}\to C^\infty(M) \tag{131}$$

Example 4.14. Let $\mathcal{V}=V_{\text{even}}\oplus\Pi V_{\text{odd}}$ be a super-vector space. We can define an associated supermanifold, also denoted $\mathcal{V}$, by $\mathcal{O}_\mathcal{V}(U):=C^\infty(U)\otimes\wedge^* V^*_{\text{odd}}$ for any open $U\subset V_{\text{even}}$.

Example 4.15 (Split supermanifolds)**.** Let $E\to M$ be a rank m vector bundle over an n-manifold M. Then we can construct a "split" $(n|m)$-supermanifold ΠE with body M and the structure sheaf $\mathcal{O}_{\Pi E}=\Gamma(M,\wedge^\bullet E^*)$ – the space of smooth sections, over M, of the bundle of supersommutative algebras $\wedge^\bullet E^*$.

E.g., for M an n-manifolds, we have two distinguished $(n|n)$-supermanifolds, ΠTM and ΠT^*M, obtained by applying the construction above to the tangent and cotangent bundle of M, respectively.

Definition 4.16. A morphism of supermanifolds $\phi:\mathcal{M}\to\mathcal{N}$ consist of the data of:

- A smooth map between the bodies $f:M\to N$,
- An extension of f to a morphism of sheaves of supercommutative algebras $\phi^*:\mathcal{O}_\mathcal{N}\to\mathcal{O}_\mathcal{M}$. In particular, for an open $U\subset\mathcal{N}$, we have a morphism $\phi^*_U:\mathcal{O}_\mathcal{N}(U)\to\mathcal{O}_\mathcal{M}(f^{-1}(U))$ commuting with the augmentation maps (131):

$$\begin{array}{ccc} \mathcal{O}_\mathcal{N}(U) & \xrightarrow{\phi^*} & \mathcal{O}_\mathcal{M}(f^{-1}(U)) \\ \downarrow & & \downarrow \\ C^\infty(U) & \xrightarrow{f^*} & C^\infty(f^{-1}(U)) \end{array}$$

Theorem 4.17 (Batchelor)**.** Every smooth supermanifold with body M is (non-canonically) isomorphic to a split-supermanifold ΠE for some vector bundle $E\to M$.

Example 4.18. Let us construct a morphism $\phi:\mathbb{R}^{1|2}\to\mathbb{R}^{1|2}$ where the source $\mathbb{R}^{1|2}$ has even coordinate x and odd coordinates θ_1,θ_2 and the target $\mathbb{R}^{1|2}$ has the even coordinate y and odd coordinates ψ_1,ψ_2. We define ϕ by specifying the pullbacks of the target coordinates:

$$\phi^*:\begin{array}{ccc} y & \mapsto & x+\theta_1\theta_2 \\ \psi_1 & \mapsto & \theta_1 \\ \psi_2 & \mapsto & \theta_2 \end{array}$$

Example 4.19. By Remark 4.13, for $\mathcal{M}$ any supermanifold the inclusion of the body $M\hookrightarrow\mathcal{M}$ is a canonically defined morphism of supermanifolds.

Example 4.20. A morphism of vector bundles

$$\begin{array}{ccc} E & \xrightarrow{\phi_E} & E' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\phi_M} & M' \end{array}$$

induces a map of the corresponding split supermanifolds $\Pi E \to \Pi E'$. **Warning:** the converse is not true – there are morphisms of $\Pi E \to \Pi E'$ not coming from morphisms of vector bundles! (E.g., the morphism constructed in Example 4.18 does not come from a morphism of vector bundles.)

Definition 4.21. A vector field $v \in \mathfrak{X}(\mathcal{M})$ of parity $|v| \in \{0,1\}$ (with the convention 0=even, 1=odd) is a derivation of $\mathcal{O}_\mathcal{M}$ of parity $|v|$, i.e., an $\mathbb{R}$-linear map $v : \mathcal{O}_\mathcal{M} \to \mathcal{O}_\mathcal{M}$ satisfying

$$v(f \cdot g) = v(f) \cdot g + (-1)^{|v|\cdot|f|} f \cdot v(g) \tag{132}$$

$$|v(f)| = |v| + |f| \mod 2 \tag{133}$$

Vector fields on $\mathcal{M}$ form a Lie superalgebra with Lie bracket

$$[v,w] := v \circ w - (-1)^{|v|\cdot|w|} w \circ v \tag{134}$$

4.2.2. *$\mathbb{Z}$-graded (super)manifolds.*

Definition 4.22 ($\mathbb{Z}$-graded supermanifold)**.** Let $\mathcal{M}$ be a supermanifold. Assume that, in terms of Definition 4.12, both $V = \bigoplus_k V_k$ (the odd fiber) and $W = \bigoplus_k W_k$ (the target of even coordinate charts) are $\mathbb{Z}$-graded vector spaces (we assume that only finitely many of V_k, W_k are nonzero). This grading induces a grading on the polynomial subalgebra Sym $W^* \otimes \wedge V^*$ in $\mathcal{A}_\alpha$ where linear functions x^i on V_k are prescribed degree $|x^i| = -k$ and linear functions θ^α on W_k are prescribed degree $|\theta^\alpha| = -k$. If transition maps between the charts $\Phi_\alpha \circ \Phi_\beta^{-1}$ are compatible with this grading, we say that we have a (global) $\mathbb{Z}$-grading on $\mathcal{M}$ or, equivalently, that $\mathcal{M}$ is a *$\mathbb{Z}$-graded supermanifold.*

Using the grading of local coordinates, we can introduce, locally, a vector field

$$\mathbb{E} := \sum_i |x^i| \cdot x^i \frac{\partial}{\partial x^i} + \sum_\alpha |\theta^\alpha| \cdot \theta^\alpha \frac{\partial}{\partial \theta^\alpha} \tag{135}$$

The fact that the grading in local charts is compatible with transitions between charts is equivalent to the local expression (135) gluing to a well-defined vector field $\mathbb{E}$ on $\mathcal{M}$. It has the name *Euler vector field* and has the property that for f a function on $\mathcal{M}$ of well-defined degree $|f|$, we have

$$\mathbb{E} f = |f| \cdot f$$

Unless stated otherwise, we will be making the following simplifying assumption.

Assumption 4.23 (Compatibility of $\mathbb{Z}$-grading and super-structure)**.** We assume that W_k can be nonzero only for k even and V_k can be nonzero only for k odd. Then one says that the $\mathbb{Z}$-grading and the super-structure on $\mathcal{M}$ are compatible, or that the $\mathbb{Z}_2$-grading (responsible for the Koszul sign in the multiplication of functions) is mod 2 reduction of the $\mathbb{Z}$-grading.

Similarly to the Definition 4.21, we can define a vector field of degree k on a $\mathbb{Z}$-graded manifold $\mathcal{M}$. The degree condition (133) gets replaced by $|v(f)| = k + |f|$.

Notation: we denote $C^\infty(\mathcal{M})_k$ or $(\mathcal{O}_\mathcal{M})_k$ the space of functions of degree k on a $\mathbb{Z}$-graded supermanifold.[42] Likewise, we denote $\mathfrak{X}(\mathcal{M})_k$ the space of vector fields of degree k.

In particular, $\mathbb{E} \in \mathfrak{X}(\mathcal{M})_0$ is a vector field of degree 0 and for $v \in \mathfrak{X}(\mathcal{M})_k$ a vector field of degree k, we can probe its degree by looking at its Lie bracket with $\mathbb{E}$:

$$[\mathbb{E}, v] = k \cdot v$$

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Example 4.24. Let $E_\bullet \to M$ be a graded vector bundle with fibers graded by *odd* integers. Then, similarly to the construction of Example 4.15, we can construct a $\mathbb{Z}$-graded manifold $\mathcal{E}$ with body M and with

$$\mathcal{O}_\mathcal{E} := \Gamma(M, \mathrm{Sym}^\bullet_{\mathrm{gr}} E^*) = \Gamma(M, \wedge^\bullet E^*)$$

Here $\mathrm{Sym}^\bullet_{\mathrm{gr}}$ stands for the graded-symmetric algebra of a graded vector bundle (i.e. symmetric algebra of the even part tensored with the exterior algebra of the odd part; the former vanishes in the present example).

Example 4.25. $\mathcal{M} = T[1]M$ – the tangent bundle of M with tangent fiber coordinates assigned grading 1. Locally, we have coordinates x^i in an open $U \subset M$.[43] The corresponding chart on $T[1]M$ has local base coordinates x^i of degree 0 and fiber coordinates $\theta^i =$ " dx^i " of degree 1. An element of $\mathcal{O}_\mathcal{M}$ locally has the form $\sum_{k=0}^n \sum_{1\leq i_1<\cdots<i_k\leq n} f_{i_1\cdots i_k}(x)\theta^{i_1}\cdots\theta^{i_k}$. Globally, we have an identification of functions on $T[1]M$ with forms on M, $\mathcal{O}_\mathcal{M} \cong \Omega^\bullet(M)$ with the form degree providing the $\mathbb{Z}$-grading.

Example 4.26. $\mathcal{M} = T^*[-1]M$ – the cotangent bundle of M with cotangent fiber coordinates assigned degree -1. Locally, we have base coordinates x^i, $\deg x^i = 0$ and fiber coordinates $\psi_i =$ " $\frac{\partial}{\partial x^i}$ ", $\deg \psi_i = -1$. An element of $\mathcal{O}_\mathcal{M}$ locally has the form $\sum_{k=0}^n \sum_{1\leq i_1<\cdots<i_k<n} f^{i_1\cdots i_k}\psi_{i_1}\cdots\psi_{i_k}$. Globally, we have an identification of function on $T[-1]M$ with polyvectors with reversed grading: $(\mathcal{O}_\mathcal{M})_{-k} \cong \mathcal{V}^k(M) = \Gamma(M, \wedge^k TM)$. I.e., a function on $T[-1]M$ of degree $-k$ is the same as a k-vector field on M.

4.2.3. *Differential graded manifolds (a.k.a. Q-manifolds).*

Definition 4.27. For $\mathcal{M}$ a $\mathbb{Z}$-graded supermanifold, one calls a vector field Q on $\mathcal{M}$ a *cohomological vector field* if

- Q has degree 1,
- $Q^2 = 0$ (as a derivation of $\mathcal{O}_\mathcal{M}$). Or, equivalently, the Lie bracket of Q with itself vanishes, $[Q, Q] = 0$.[44]

Then we say that the pair $(\mathcal{M}, Q)$ is a *differential graded (dg) manifold* or, equivalently, a *Q-manifold.*

[42]We use notations $C^\infty(\mathcal{M})$ and $\mathcal{O}_\mathcal{M}$ for the algebra of functions on $\mathcal{M}$ interchangeably.

[43]We adopt the following (standard) convention for shifts of homological degree: if $V^\bullet$ is a $\mathbb{Z}$-graded vector space, then the degree-shifted vector space $V[k]$ is defined by $(V[k])^i := V^{k+i}$. In particular, e.g., for V concentrated in degree zero, $V^{\neq 0} = 0$, $V[k]$ is concentrated in degree $-k$.

[44]Note that, by (134), for an odd vector field, we have $[Q, Q] = 2Q^2$. In particular, vanishing of $[Q, Q]$ is not a tautological property, unlike for a bracket of an even vector field with itself.

Remark 4.28. Note that Q defines a differential on the algebra of functions, $Q: C^\infty(\mathcal{M}_k) \to C^\infty(\mathcal{M})_{k+1}$, thus endowing $C^\infty(\mathcal{M})$ with the structure of a commutative differential graded algebra.

Remark 4.29 (Carchedi-Roytenberg?)**.** Vector fields $\mathbb{E}, Q$ satisfy the commutation relations

$$[\mathbb{E},\mathbb{E}] \underset{\text{tautologically}}{=} 0 = [Q,Q], \qquad [\mathbb{E},Q] = Q$$

Thus, the pair of vector fields $\mathbb{E}, Q$ define an action on $\mathcal{M}$ of a Lie superalgebra of automorphisms of the odd line $\mathbb{R}^{0|1}$. This algebra is generated by infinitesimal dilatation $\mathsf{e} = -\theta\frac{\partial}{\partial\theta}$ and an infinitesimal translation $\mathsf{q} = \frac{\partial}{\partial\theta}$ (with θ the odd coordinate on $\mathbb{R}^{0|1}$), satisfying same super Lie algebra relations as above.

Example 4.30. For

$$\mathcal{M} = T[1]M$$

the degree-shifted tangent bundle of M, we have a cohomological vector field Q on $\mathcal{M}$ corresponding to the de Rham operator d_M on M, so that we have

$$\begin{array}{ccc} C^\infty(\mathcal{M})_k & \xrightarrow{Q} & C^\infty(\mathcal{M})_k \\ \| & & \| \\ \Omega^k(M) & \xrightarrow{d_M} & \Omega^{k+1}(M) \end{array}$$

Locally, in terms of local coordinates $(x^i, \theta^i = dx^i)$ (cf. Example 4.25), we have

$$Q = \sum_i \theta^i \frac{\partial}{\partial x^i}$$

This local formula glues, over coordinate charts on $\mathcal{M}$, to a globally well-defined vector field $Q = d_M \in \mathfrak{X}(\mathcal{M})_1$.

Example 4.31. Let $\mathfrak{g}$ be a Lie algebra. Consider a graded manifold

$$\mathcal{M} = \mathfrak{g}[1]$$

with body a point and $C^\infty(\mathcal{M}) = \wedge^\bullet \mathfrak{g}^*$. Note that functions on $\mathcal{M}$ can be identified with Chevalley-Eilenberg cochains on $\mathfrak{g}$, $C^\infty(\mathcal{M}) \cong C^\bullet_{CE}(\mathfrak{g})$. We define the cohomological vector field Q on $\mathcal{M}$ to be the Chevalley-Eilenberg differential $d_{CE}: \wedge^k\mathfrak{g}^* \to \wedge^{k+1}\mathfrak{g}^*$, obtained from the dual of the Lie bracket $[,]^*: \mathfrak{g}^* \to \wedge^2\mathfrak{g}^*$ by extension to $\wedge^\bullet\mathfrak{g}^*$ as a derivation, by Leibniz identity. The property $d_{CE}^2 = 0$ then corresponds to the Jacobi identity in $\mathfrak{g}$. Let $\{T_a\}$ be a basis in $\mathfrak{g}$ and $\{\psi^a\}$ be the corresponding degree 1 coordinates on $\mathcal{M}$ (the dual basis to $\{T_a\}$); let also f^c_{ab} be the structure constants of $\mathfrak{g}$, i.e. $[T_a, T_b] = \sum_c f^c_{ab} T_c$. Then we have

$$Q = d_{CE} = \frac{1}{2}\sum_{a,b,c} f^c_{ab}\psi^a\psi^b\frac{\partial}{\partial\psi^c} \qquad \in \mathfrak{X}(\mathcal{M})_1$$

Definition 4.32. An L_∞ algebra is a graded vector space $\mathfrak{g}^\bullet$ endowed with multilinear, graded skew-symmetric operations $l_k: \wedge^k_{\mathrm{gr}}\mathfrak{g} \to \mathfrak{g}$ for each $k \geq 1$, such that:

- l_k has degree $2-k$,
- the following quadratic relations hold for each $n \geq 1$:

$$\sum_{n=r+s,\ r\geq 0, s\geq 1} \sum_{\sigma\in \mathrm{Sh}(r,s)} \pm\, l_{r+1}(x_{\sigma_1}, \ldots, x_{\sigma_r}, l_s(x_{\sigma_{r+1}}, \ldots, x_{\sigma_n})) = 0 \tag{136}$$

for $x_1, \ldots, x_n \in \mathfrak{g}^\bullet$ any n-tuple of vectors. Here $\mathrm{Sh}(r,s)$ stands for (r,s)-shuffles, i.e., permutations of numbers $1, \ldots, n = r+s$, such that $\sigma_1 < \cdots < \sigma_r$ and $\sigma_{r+1} < \cdots < \sigma_n$.

In particular, for small values of n, relations (136) have the following form:

- $n = 1$: $l_1(l_1(x)) = 0$, i.e. $l_1 =: d$ is a differential on $\mathfrak{g}^\bullet$.
- $n = 2$: $l_1(l_2(x,y)) = l_2(l_1(x)) + (-1)^{|x|} l_2(x, l_1(y))$ – Leibniz identity, i.e. d is a derivation of the binary operation $l_2 =: [,]$.
- $n = 3$: Jacobi identity *up to homotopy* for $l_2 = [,]$, i.e. the Jacobiator equals a commutator (in appropriate sense) of a trinary operation l_3 with $l_1 = d$:

$$[x,[y,z]] - [[x,y],z] - (-1)^{|x|\cdot|y|}[y,[x,z]] = \pm d l_3(x,y,z) \pm l_3(dx,y,z) \pm l_3(x,dy,z) \pm l_3(x,y,dz)$$

An alternative definition of an L_∞ algebra is as follows.

Definition 4.33. An L_∞ algebra is a graded vector space $\mathfrak{g}^\bullet$ together with a *coderivarion*[45] $\mathcal{D}$ of the cofree cocommutative coalgebra generated by $\mathfrak{g}[1]$, $\mathcal{D} : \mathrm{Sym}^\bullet(\mathfrak{g}[1]) \to \mathrm{Sym}^\bullet(\mathfrak{g}[1])$, satisfying the following:

- $\mathcal{D}^2 = 0$,
- $p_0 \circ \mathcal{D} = 0$ where $p_0 : \mathrm{Sym}^\bullet(\mathfrak{g}[1]) \to \mathrm{Sym}^0(\mathfrak{g}[1]) = \mathbb{R}$ is the counit,
- $\mathcal{D}$ has degree $+1$.

Remark 4.34. Coderivation $\mathcal{D}$ is determined by its projection to (co)generators in $\mathfrak{g}[1]$, i.e., by a sequence of maps

$$p \circ \mathcal{D}^{(k)} : \mathrm{Sym}^k(\mathfrak{g}[1]) \to \mathfrak{g}[1] \tag{137}$$

where $p : \mathrm{Sym}^\bullet(\mathfrak{g}[1]) \to \mathrm{Sym}^1(\mathfrak{g}[1]) = \mathfrak{g}[1]$ is the projection to (co)generators. In (137) we restricted the input of $\mathcal{D}$ to k-th symmetric power of $\mathfrak{g}[1]$, with $k \geq 1$. One has a tautological *décalage isomorphism* $\alpha : \mathrm{Sym}^k(\mathfrak{g}[1]) \to (\wedge^k \mathfrak{g})[k]$ which sends $\alpha : s(x_1) \odot \cdots \odot s(x_k) \mapsto \pm s^k(x_1 \wedge \cdots \wedge x_n)$ for $x_1, \ldots, x_k \in \mathfrak{g}$, with s the suspension symbol. The relation of the L_∞ operations l_k from Definition 4.32 with the components of the coderivation (137) is via

$$l_k = p \circ \mathcal{D}^{(k)} \circ \alpha^{-1} : \quad \wedge^k \mathfrak{g} \to \mathfrak{g}$$

The quadratic relations on operations correspond to the equation $\mathcal{D}^2 = 0$.

Example 4.35. Let $(\mathfrak{g}^\bullet, \{l_k\})$ be an L_∞ algebra. Then $(\mathfrak{g}^\bullet[1], Q = \mathcal{D}^*)$ is a dg manifold. I.e., we identify the dual of $\mathrm{Sym}^\bullet(\mathfrak{g}[1])$ with a polynomial subalgebra in $C^\infty(\mathfrak{g}[1])$. The dual of the coderivation $\mathcal{D}$ is a derivation of polynomial functions on $\mathfrak{g}[1]$ and thus yields a vector field on $\mathfrak{g}[1]$. If $\{T_a\}$ is a basis in $\mathfrak{g}$, $\{T^a\}$ the dual basis in $\mathfrak{g}^*$, and θ^a the corresponding coordinates on $\mathfrak{g}[1]$, we have

$$Q = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{a_1, \ldots, a_k, b} \pm \left\langle T^b, l_k(T_{a_1}, \ldots, T_{a_k}) \right\rangle \theta^{a_1} \cdots \theta^{a_k} \frac{\partial}{\partial \theta^b}$$

[45] Recall that a linear map $\mathcal{D} : C \to C$ is a coderivation of a coalgebra C if the co-Leibniz identity holds: $\Delta \circ \mathcal{D} = (\mathcal{D} \otimes \mathrm{id}) \circ \Delta + (\mathrm{id} \otimes \mathcal{D}) \circ \Delta$, with $\Delta : C \to C \otimes C$ the coproduct. In particular, if $\delta : A \to A$ is a derivation of an algebra A, then the dual map $\delta^* : A^* \to A^*$ is a coderivation of the dual coalgebra $C = A^*$.

Introducing a "generating function for coordinates on $\mathfrak{g}[1]$" (or "superfield") $\underline{\theta} = \sum_a \theta^a T_a \in \mathrm{Sym}^1(\mathfrak{g}[1])^* \otimes \mathfrak{g}$, we can write

$$Q = \sum_{k=1}^{\infty} \frac{1}{k!} \left\langle l_k(\underline{\theta}, \ldots, \underline{\theta}), \frac{\partial}{\partial \underline{\theta}} \right\rangle$$

where $\frac{\partial}{\partial \underline{\theta}} := \sum_a T^a \frac{\partial}{\partial \theta^a}$, operations l_k act only on elements of $\mathfrak{g}$ (the T^as) and $\langle , \rangle$ pairs $\mathfrak{g}$ with $\mathfrak{g}^*$.

The property $Q^2 = 0$ is equivalent to the quadratic relations (136) on operations $\{l_k\}$.

Remark 4.36 (From [1])**.** If $(\mathcal{M}, Q)$ is a dg manifold and $x_0 \in M$ a point of the body such that Q vanishes at x_0, then the shifted tangent space $\mathfrak{g} := T_{x_0}[-1]\mathcal{M}$ inherits the structure of L_∞ algebra: Taylor expansion of Q at x_0 produces a sequence of elements

$$Q^{(k)} \in \mathrm{Sym}^k T^*_{x_0}\mathcal{M} \otimes T_{x_0}\mathcal{M} = \mathrm{Sym}^k(\mathfrak{g}[1])^* \otimes \mathfrak{g}[1]$$

which, by the décalage isomorphism (cf. Remark 4.34), yield the L_∞ operations $l_k : \wedge^k \mathfrak{g} \to \mathfrak{g}$.[46]

Definition 4.37. A *Lie algebroid* is a vector bundle $E \to M$ with skew-symmetric Lie bracket on sections $[,] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ satisfying Jacobi identity, endowed additionally with the *anchor map* – a bundle map $\rho : E \to TM$ (covering the identity map on M), such that for $\alpha, \beta \in \Gamma(E)$ and $f \in C^\infty(M)$ the following version of Leibniz identity holds:

$$[\alpha, f \cdot \beta] = f \cdot [\alpha, \beta] + \rho(\alpha)(f) \cdot \beta \tag{138}$$

Example 4.38 (Vaintrob, [33])**.** Let $(E \to M; [,]; \rho)$ be a Lie algebroid. Consider the graded manifold $E[1]$ with body M and functions $C^\infty(E[1]) = \Gamma(M, \wedge^\bullet E^*)$. One can endow $E[1]$ with a cohomological vector field $Q : \Gamma(M, \wedge^k E^*) \to \Gamma(M, \wedge^{k+1} E^*)$ defined as follows: for $\psi \in \Gamma(M, \wedge^k E^*)$ and $\alpha_0, \ldots, \alpha_k \in \Gamma(M, E)$, we set

$$\begin{aligned} Q\psi(\alpha_0, \ldots, \alpha_k) := \sum_{\sigma \in \mathrm{Sh}(2,k)} (-1)^\sigma \rho(\alpha_{\sigma_0}) \left(\psi(\alpha_{\sigma_1}, \ldots, \alpha_{\sigma_k})\right) + \\ + \sum_{\sigma \in \mathrm{Sh}(2,k-1)} (-1)^\sigma \psi\left([\alpha_{\sigma_0}, \alpha_{\sigma_1}], \alpha_{\sigma_2}, \ldots, \alpha_{\sigma_k}\right) \end{aligned} \tag{139}$$

Locally, let $\{x^i\}$ be local coordinates in a neighborhood U on M and $\{e_a\}$ be a basis of sections of E over U. In particular, $[e_a, e_b] = \sum_c f^c_{ab}(x) e_c$ with $f^c_{ab}(x)$ the structure constants of the Lie bracket of sections of E. The anchor maps e_a to a vector field $\sum_i \rho^i_a(x) \frac{\partial}{\partial x^i}$. On $E[1]$ we have local coordinates x^i, $\deg x^i = 0$ and θ^a, $\deg \theta^a = 1$. The cohomological vector field (139) locally takes the form

$$Q = \frac{1}{2} \sum_{a,b,c} f^c_{ab}(x) \theta^a \theta^b \frac{\partial}{\partial \theta^c} + \sum_{a,i} \theta^a \rho^i_a(x) \frac{\partial}{\partial x^i} \tag{140}$$

Equation $Q^2 = 0$ is equivalent to the structure relations of a Lie algebroid:

- the Jacobi identity for sections of E,

[46] The L_∞ structure induced this way on the shifted tangent space depends on the choice of a local chart near x_0. Choosing a different chart induces an isomorphism of L_∞ algebras.

- the condition that the anchor $\rho : \Gamma(M,E) \to \mathfrak{X}(M)$ is a Lie algebra morphism (which follows from (138)).

Example 4.39. A special case of Example 4.38 is as follows. Let G be a group acting on a manifold M with $\gamma : G \times M \to M$ the action. Let $d_{1,x}\gamma : \mathfrak{g} \to T_x M$ be the corresponding infinitesimal action, with $x \in M$. We can construct the *action Lie algebroid*, with $E = \mathfrak{g} \times M$ (as a trivial bundle over M), with the bracket of sections given by pointwise bracket in $\mathfrak{g}$ and with the anchor map $\rho = d_{1,-}\gamma : E \to TM$ given by the Lie algebra action. The corresponding graded manifold is $E[1] = M \times \mathfrak{g}[1]$ with the algebra of functions

$$C^\infty(E[1]) = \wedge^\bullet \mathfrak{g}^* \otimes C^\infty(M) = C^\bullet_{CE}(\mathfrak{g}, C^\infty(M))$$

– Chevalley-Eilenberg cochains of $\mathfrak{g}$ with coefficients in the module $C^\infty(M)$ with module structure given by $T_a \otimes f \mapsto v_a(f)$ with v_a the fundamental vector fields of $\mathfrak{g}$-action and with $f \in C^\infty(M)$ an arbitrary function. The cohomological vector field is the Chevalley-Eilenberg differential twisted by the module $C^\infty(M)$. Locally on M:

$$Q = \sum_{a,b,c} f^c_{ab}\theta^a\theta^b \frac{\partial}{\partial\theta^c} + \sum_{a,i} \theta^a v^i_a(x) \frac{\partial}{\partial x^i}$$

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4.2.4. *Integration on supermanifolds.* Let $p : E \to M$ be a vecor bundle of rank m over an n-manifold M. Let $\mathcal{M} = \Pi E$ be the corresponding split $(n|m)$-supermanifold.

We define the Berezin line bundle of the supermanifold $\mathcal{M}$ as the real line bundle $\mathrm{Ber}(\mathcal{M}) = \wedge^n T^*M \otimes \wedge^m E$ over $M = \mathrm{body}(\mathcal{M})$. We call sections of Ber the *Berezinians.*

Given a Berezinian $\mu \in \Gamma(M, \mathrm{Ber}(\mathcal{M}))$, we have an $\mathbb{R}$-linear *integration map*

$$\int_{\mathcal{M}} \mu \cdot \bullet : \quad C^\infty_c(\mathcal{M}) \to \mathbb{R}$$

defined as follows:

$$\int_{\mathcal{M}} \mu\, f = \int_M \langle \mu, (f)_m \rangle \tag{141}$$

where $\langle , \rangle$ is the fiberwise pairing between line bundles $\wedge^m E$ and $\wedge^m E^*$; $(f)_m$ is the component of $f \in C^\infty(\Pi E) = \Gamma(M, \wedge^\bullet E^*)$ in the top exterior power of E^*. Note that the integrand on the r.h.s. $\langle \mu, (f)_m \rangle$ is a section of $\wedge^n T^*M$ over M, i.e., a top degree form, and thus can be integrated. One can understand the definition (141) as doing a standard Berezin integral in odd fibers of ΠE and then integrating the result over the body in the ordinary (measure-theoretic) sense.

In fact, sections of $\mathrm{Ber}(\mathcal{M})$ over M correspond to Berezinians that are *constant in the fiber direction* of $\Pi E \to M$. More generally, we can consider the super-vector bundle $\widetilde{\mathrm{Ber}}(\mathcal{M}) = \mathrm{Ber}(\mathcal{M}) \otimes \wedge^\bullet E^*$ over M, such that $\Gamma(M, \widetilde{\mathrm{Ber}}(\mathcal{M})) = \Gamma(M, \mathrm{Ber}(\mathcal{M})) \otimes_{C^\infty(M)} C^\infty(\mathcal{M})$. We denote the space of sections $\mathrm{BER}(\mathcal{M}) := \Gamma(M, \widetilde{\mathrm{Ber}}(\mathcal{M}))$. Its elements are the (general) Berezinians. By construction, $\mathrm{BER}(\mathcal{M})$ is a module over $C^\infty(\mathcal{M})$. Note that we can alternatively understand $\mathrm{BER}(\mathcal{M})$ as the space of sections of the pullback line bundle $p^*\mathrm{Ber}(\mathcal{M})$ over the whole of $\mathcal{M}$ rather than just the body M (where $p : \Pi E \to M$ is the bundle projection). In the language of general Berezinians, integration (141) is simply a map

$$\int_{\mathcal{M}} : \quad \mathrm{BER}(\mathcal{M}) \to \mathbb{R}$$

Remark 4.40. The notion of a Berezinian constant in the fiber direction depends on the splitting of the supermanifold $\mathcal{M}$, i.e. on a particular identification of it with ΠE for $E \to M$ a vector bundle. On the other hand, the general notion of a Berezian (element of $\mathrm{BER}(\mathcal{M})$) does not depend on the splitting.

Remark 4.41. Parity-shifted tangent bundle $\mathcal{M} = \Pi TM$ carries a distinguished Berezinian $\mu_{\Pi TM}$, characterized as follows. For $f \in C^\infty(\Pi TM) \cong \Omega^\bullet(M)$ denote $\widetilde{f}$ the corresponding differential form on M. Then $\mu_{\Pi TM}$ satisfies

$$\int_{\Pi TM} \mu_{\Pi TM} \cdot f = \int_M \widetilde{f}$$

where on the r.h.s. we have an ordinary integral over M of a differential form. In the local coordinates (cf. Example 4.25), we have $\mu_{\Pi TM} = \prod_i (dx^i D\theta^i) \ \in \mathrm{BER}(\Pi TM)$.

When one considers integration over $\mathbb{Z}$-graded manifolds, only the underlying $\mathbb{Z}_2$-grading (superstructure) plays role for the integration theory.

4.2.5. *Change of variables formula for integration over supermanifolds.*

Definition 4.42. Let S be a supermanifold of parameters and $J \in \mathrm{End}(\mathbb{R}^{n|m}) \otimes C^\infty(S)$ an S-dependent endomorphism of $\mathbb{R}^{n|m}$ of block form

$$J = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

with the blocks

$$A \in [\mathrm{End}(\mathbb{R}^n) \otimes C^\infty(S)]_{\mathrm{even}}, \quad D \in [\mathrm{End}(\mathbb{R}^m) \otimes C^\infty(S)]_{\mathrm{even}},$$
$$B \in [\mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n) \otimes C^\infty(S)]_{\mathrm{odd}}, \quad C \in [\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m) \otimes C^\infty(S)]_{\mathrm{odd}}$$

Assume that D is invertible. Then the *superdeterminant* of J is defined as

$$\mathrm{Sdet}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \det(A - BD^{-1}C) \cdot (\det D)^{-1} \qquad \in C^\infty(S) \tag{142}$$

Remark 4.43. Superdeterminant is characterized by the following two properties:

- Multiplicativity: for $J, K \in \mathrm{End}(\mathbb{R}^{n|m}) \otimes C^\infty(S)$, we have
$$\mathrm{Sdet}(JK) = \mathrm{Sdet}(J) \cdot \mathrm{Sdet}(K)$$
where JK is the composition of J and K as endomorphisms of $\mathbb{R}^{n|m}$.
- For $j = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right)$ an S-dependent endomorphism of $\mathbb{R}^{n|m}$, we have
$$\mathrm{Sdet}\ (\mathrm{id} + \epsilon \cdot j) = 1 + \epsilon \cdot \mathrm{Str}\ j + O(\epsilon^2)$$
Here $\mathrm{Str}\ j = \mathrm{tr}\, a - \mathrm{tr}\, d$ is the *supertrace* of j.

Note that these two properties imply that

$$\mathrm{Sdet}\ e^j = e^{\mathrm{Str}\ j}$$

Theorem 4.44 (Change of variables formula)**.** Let $\mathbb{R}^{n|m}_I, \mathbb{R}^{n|m}_{II}$ be two copies of the $(n|m)$-dimesnional vector superspace, endowed with coordinates x^i, θ^a on the first copy and coordinates y^i, ψ^a on the second copy. Let $\phi : \mathbb{R}^{n|m}_I \to \mathbb{R}^{n|m}_{II}$ be a smooth map of supermanifolds and $f(y, \psi) \in C^\infty_c(\mathbb{R}^{n|m}_{II})$ a compactly supported function. Then the integral of f over $\mathbb{R}^{n|m}_{II}$ against the standard coordinate Berezinian can be expressed as an integral of the pullback of f by ϕ as follows:

(143)
$$\int_{\mathbb{R}^{n|m}_{II}} d^n y \mathcal{D}^m \psi \; f(y,\psi) = \int_{\mathbb{R}^{n|m}_{I}} d^n x \mathcal{D}^m \theta \;\; \mathrm{sign}\det\left(\frac{\partial y^i(x,0)}{\partial x^j}\right)\cdot \mathrm{Sdet}\; \frac{\partial(y,\psi)}{\partial(x,\theta)}\cdot f(y(x,\theta),\psi(x,\theta))$$

Here on the r.h.s.

$$\frac{\partial(y,\psi)}{\partial(x,\theta)} = \left(\begin{array}{c|c} \frac{\partial y^i}{\partial x^j} & \frac{\partial y^i}{\partial \theta^b} \\ \hline \frac{\partial \psi^a}{\partial x^j} & \frac{\partial \psi^a}{\partial \theta^b} \end{array}\right) \qquad \in \mathrm{End}(\mathbb{R}^{n|m})\otimes C^\infty(\mathbb{R}^{n|m}_I)$$

is the super-matrix of first derivatives of ϕ. The sign factor in (143) is the sign of the determinant of the even-even block of the matrix of derivatives.[47]

4.2.6. *Divergence of a vector field.*

Definition 4.45. For $v \in \mathfrak{X}$ a vector field on a supermanifold $\mathcal{M}$ and $\mu \in \mathrm{BER}(\mathcal{M})$ a Berezinian, we define the *divergence* $\mathrm{div}_\mu(v) \in C^\infty(\mathcal{M})$ of v with respect to μ via the property

$$\int_{\mathcal{M}} \mu \; v(f) = -\int_{\mathcal{M}} \mu \; \mathrm{div}_\mu(v)\cdot f \tag{144}$$

for any compactly supported test function $f \in C^\infty_c(\mathcal{M})$.

Example 4.46. For $\mathcal{M} = M$ an ordinary manifold and μ a volume form, by Stokes' theorem we have

$$0 \underset{\text{Stokes}'}{=} \int_M \mathcal{L}_v(\mu\, f) = \int_M \mu\; v(f) + \underbrace{(\mathcal{L}_v \mu)}_{\mu\cdot \mathrm{div}_\mu(v)}\cdot f$$

where $\mathcal{L}_v$ is the Lie derivative along v. Thus, definition (144) is compatible, in the context of ordinary geometry, with the definition of divergence as $\mathrm{div}_\mu(v) = \frac{\mathcal{L}_v \mu}{\mu}$. I.e., roughly speaking, the divergence measures how the flow by v changes volumes of subsets of M, as measured using μ.

The following is a straightforward consequence of the Definition 4.45.

Lemma 4.47. Let μ, μ_0 be two Berezinians on $\mathcal{M}$ with $\mu = \rho\cdot\mu_0$ where $\rho \in C^\infty(\mathcal{M})$ is a nonvanishing function. Then, for $v \in \mathfrak{X}(\mathcal{M})$ a vector field, divergences with respect to μ and μ_0 are related as follows:

$$\mathrm{div}_\mu(v) = \mathrm{div}_{\mu_0}(v) + \underbrace{\frac{1}{\rho}\cdot v(\rho)}_{=v(\log\rho)} \tag{145}$$

On a general supermanifold $\mathcal{M}$, using local coordinates x^i, θ^a, assume first that $\mu = \mu_{\mathrm{coord}} = d^n x \mathcal{D}^m \theta$ – the standard coordinate Berezinian. The vector field can be expressed locally as

$$v = \sum_i v^i(x,\theta)\frac{\partial}{\partial x^i} + \sum_a v^a(x,\theta)\frac{\partial}{\partial \theta^a}$$

Then the divergence of v is given by the local formula:

$$\mathrm{div}_{\mu_{\mathrm{coord}}} v = \sum_i \frac{\partial}{\partial x^i} v^i - (-1)^{|v|}\sum_a \frac{\partial}{\partial \theta^a} v^a \tag{146}$$

[47]It corresponds to the fact that in the change of variables formula for an ordinary integral, the *absolute value* of the Jacobian appears.

Note that, using derivatives acting on the left,[48] we can simplify the signs:

$$\mathrm{div}_{\mu_{\mathrm{coord}}}(v) = \sum_i v^i \frac{\overleftarrow{\partial}}{\partial x^i} - \sum_a v^a \frac{\overleftarrow{\partial}}{\partial \theta^a}$$

In a more general case one, when μ is not the coordinate Berezinian, one obtains the local formula by combining (146) with (145).

4.3. **BRST formalism.** BRST formalism arose in [4, 32] independently as a cohomological formalism for treating gauge symmetry.

4.3.1. *Classical BRST formalism.* We will call a *classical BRST theory* the following supergeometric data:

- A $\mathbb{Z}$-graded supermanifold $\mathcal{F}$ (the "space of fields"),
- A cohomological vector field – a vector field $Q \in \mathfrak{X}(\mathcal{F})_1$ satisfying $\boxed{Q^2 = 0}$ – the "BRST operator" (encoding the data of gauge symmetry),
- A function $S \in C^\infty(\mathcal{F})_0$ – the "action" satisfying $\boxed{Q(S) = 0}$ (gauge-invariance property).

Example 4.48. Starting from Faddeev-Popov data – action of a group G on a manifold X and an invariant function $S \in C^\infty(X)^G$, we construct the BRST package as follows: $\mathcal{F} = X \times \mathfrak{g}[1]$ with

$$Q = \frac{1}{2} \sum_{a,b,c} f^c_{ab} c^a c^b \frac{\partial}{\partial c^c} + \sum_{a,i} c^a v^i_a(x) \frac{\partial}{\partial x^i}$$

with x^i local coordinates on X, c^a the degree 1 coordinates on $\mathfrak{g}[1]$; v_a are the fundamental vector fields of G-action on X.

In other words, the functions of fields $C^\infty(\mathcal{F}) = \wedge^\bullet \mathfrak{g}^* \otimes C^\infty(X) = C^\bullet_{CE}(\mathfrak{g}, C^\infty(X))$ are the Chevalley-Eilenberg cochains of $\mathfrak{g}$ twisted by the module $C^\infty(X)$ with $Q = d_{CE}$ the corresponding Chevalley-Eilenberg differential. Equivalently, $(\mathcal{F}, Q)$ is the dg manifold associated to the action Lie algebroid for the action of G on X (via the construction of Examples 4.38, 4.39).

Note that $Q^2 = 0$ is equivalent to the pair of properties: Jacobi identity for the bracket in $\mathfrak{g}$ and the condition that the infinitesimal action $\mathfrak{g} \to \mathfrak{X}(X)$ is a Lie algebra homomorphism. The equation $Q(S) = 0$ is equivalent to $\mathfrak{g}$-invariance of S (cast as $v_a(S) = 0$ with v_a the fundamental vector fields).

4.3.2. *Quantum BRST formalism.* We define the quantum (finite-dimensional) BRST theory as the data of classical BRST theory $(\mathcal{F}, Q, S)$ with an additional structure adjoined: a Berezinian μ on $\mathcal{F}$ (the finite-dimensional toy model for the functional integral measure), such that the following property holds:

$$\boxed{\mathrm{div}_\mu Q = 0} \tag{147}$$

– compatibility of the integration measure on fields with gauge symmetry.

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[48]For f a function of commuting variables x^i and anti-commuting variables θ^a, let y be one of xs or θs. One denotes the ordinary derivative as $\frac{\overrightarrow{\partial}}{\partial y} f = \frac{\partial}{\partial y} f$ and sets $f \frac{\overleftarrow{\partial}}{\partial y} := (-1)^{|y|\cdot(|f|+1)} \frac{\partial}{\partial y} f$. In particular $y \frac{\overleftarrow{\partial}}{\partial y} = 1$. The idea is that, if f is monomial, in order to calculate $f \frac{\overleftarrow{\partial}}{\partial y}$, if y occurs in f, one commutes y to the right in the monomial, using Koszul sign rule, and then y gets killed by the derivative from the right (acting on the left).

Lemma 4.49. For any $f \in C^\infty_c(\mathcal{F})$

$$\int_\mathcal{F} \mu\, Q(f) = 0 \tag{148}$$

(Follows immediately from divergence-free condition (147) and the Definition 4.45.)

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A *BRST integral* is an integral of Q-cocycle, $I = \int_\mathcal{F} \mu\ f$ with $Q(f) = 0$. By Lemma 4.49, the integral is invariant under shifts of the integrand by a Q-coboundary. I.e., the integrand can be considered modulo shifts $f \sim f + Q(g)$ for any g. In other words, the BRST integral is a map

$$\int_\mathcal{F} \mu :\ H_Q(C^\infty(\mathcal{F})) \to \mathbb{R} \text{ (or } \mathbb{C})$$

assigning numbers to cohomology classes of Q. The relevant case for field theory is when the Q-cocycles are complex-valued, in which case the BRST integral takes values in $\mathbb{C}$.

In particular, we are interested in the oscillatory BRST integral

$$Z = \int_\mathcal{F} \mu\, e^{\frac{i}{\hbar}S} = \int_\mathcal{F} \mu\, e^{\frac{i}{\hbar}(S+Q(\Psi))} \tag{149}$$

Here $\Psi \in C^\infty(\mathcal{F})_{-1}$ is an arbitrary function generating the shift of the integrand by a Q-exact term;[49] in this context Ψ is known as the *gauge-fixing fermion*.

Remark 4.50. *Observables* in BRST formalism are Q-cocycles $\mathcal{O} \in C^\infty(\mathcal{F})$. Given a collection of observables $\mathcal{O}_1, \ldots, \mathcal{O}_N$, one can consider their expectation value (correlation function):

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_N \rangle := \frac{1}{Z} \int_\mathcal{F} \mu\, \mathcal{O}_1 \cdots \mathcal{O}_N\, e^{\frac{i}{\hbar}S} = \frac{1}{Z} \int_\mathcal{F} \mu\, \mathcal{O}_1 \cdots \mathcal{O}_N\, e^{\frac{i}{\hbar}(S+Q(\Psi))}$$

The fact that $\mathcal{O}_i$ are Q-cocycles imply that the entire integrand is a Q-cocycle, and thus one can again shift S by a Q-coboundary.

The idea of gauge-fixing in BRST formalism: L.h.s. of (149) typically *perturbatively ill-defined*, i.e., cannot be evaluated (in the aymptotic regime $\hbar \to 0$) by the stationary phase formula, due to the degeneracy of critical points of S arising from gauge symmetry. On the other hand, the r.h.s. of (149) is *perturbatively well-defined* (i.e. critical points are non-degenerate and the stationary phase formula is applicable), for a good choice of Ψ. So, the r.h.s. of (149) is the gauge-fixed BRST integral which can be evaluated in terms of Feynman diagrams. By Lemma 4.49, the result is independent on the choice of gauge-fixing fermion Ψ (though the particular Feynman rules for calculating the r.h.s. of (149) do depend on Ψ; the result is independent of Ψ once all contributing graphs are summed over).

[49] The fact that the integrand in the l.h.s. and r.h.s. of (149) differs by a Q-exact term, i.e., that $e^{\frac{i}{\hbar}(S+Q(\Psi))} - e^{\frac{i}{\hbar}S} = Q(\cdots)$, follows from a simple computation: $e^{X+Q(Y)} - e^X = e^X \sum_{n=1}^\infty \frac{1}{n!} Q(Y)^n = Q\left(e^X Y \Phi(Q(Y))\right)$. Here X (of degree 0) is assumed to be Q-closed and we denoted $\Phi(x) := \frac{e^x-1}{x} = \sum_{n=0}^\infty \frac{1}{(n+1)!} x^n$. Setting $X = \frac{i}{\hbar}S$, $Y = \frac{i}{\hbar}\Psi$, we obtain the statement above.

4.3.3. *Faddeev-Popov via BRST.* We start from Faddeev-Popov data: an n-manifold X acted on by a compact group G, an invariant function $S \in C^\infty(X)^G$, an invariant volume form $\mu_X \in \Omega^n(X)^G$ and a gauge-fixing function $\phi : X \to \mathfrak{g}$ defining a local section of G-orbits $\phi^{-1}(0) \subset X$.

First attempt. Set, as in Example 4.48, $\mathcal{F} = X \times \mathfrak{g}[1]$, with the cohomological vector field locally written as

$$Q = \frac{1}{2} \sum_{a,b,c} f^c_{ab} c^a c^b \frac{\partial}{\partial c^c} + \sum_{a,i} c^a v^i_a(x) \frac{\partial}{\partial x^i} \tag{150}$$

and the Berezinian $\mu = \mu_X \cdot \mathcal{D}^m c$. Here $\mathcal{D}^m c = \overleftarrow{\prod}_a \mathcal{D}c^a$ is the coordinate Berezinian on $\Pi\mathfrak{g}$ (invariantly, it is the element of $\wedge^m \mathfrak{g}$ compatible with the chosen normalization of Haar measure on G).

Note that the divergence of Q is

$$\mathrm{div}_\mu Q = \sum_a c^a \left(\sum_b f^b_{ab} + \mathrm{div}_{\mu_X} v_a \right)$$

The two terms on the r.h.s. vanish individually because:

- μ_X is G-invariant and thus fundamental vector fields v_a are divergence-free,
- the contraction of the structure constants in $\mathfrak{g}$, $\sum_b f^b_{ab} = \mathrm{tr}_{\,\mathfrak{g}}[T_a, -]$ vanishes due to *unimodularity* of $\mathfrak{g}$,[50] which in turn follows from the existence of Haar measure on G.

Problem:

(1) $\mathcal{F}$ has coordinates x^i of degree 0 and c^a of degree 1, in particular $\mathcal{F}$ is non-negatively graded. Thus, there is no non-zero element $\Psi \in C^\infty(\mathcal{F})_{-1}$ which we would need for gauge-fixing (149).

(2) The integral $\int_\mathcal{F} \mu\, e^{\frac{i}{\hbar} S(x)} = 0$ vanishes because of the integral over $\mathfrak{g}[1]$ (note that the integrand is independent of the odd variables c^a, hence the Berezin part of the integral vanishes trivially).

Solution/second attempt. Let us call the quantum BRST package constructed in the first attempt the *minimal* BRST package, $(\mathcal{F}_{\min}, Q_{\min}, \mu_{\min})$. We construct the new, non-minimal BRST package by setting:

- Non-minimal fields: $\mathcal{F} := \mathcal{F}_{\min} \times \mathcal{F}_{\mathrm{aux}}$ where the added auxiliary fields are $\mathcal{F}_{\mathrm{aux}} := \mathfrak{g}^*[-1] \oplus \mathfrak{g}^*$ with degree -1 coordinates $\bar{c}_a$ and degree 0 coordinates λ_a. Thus, we have the following local coordinates on $\mathcal{F} = X \times \mathfrak{g}[1] \times \mathfrak{g}^*[-1] \times \mathfrak{g}^*$:

coordinates	degree
x^i on X	0
c^a on $\mathfrak{g}[1]$	1
$\bar{c}_a$ on $\mathfrak{g}^*[-1]$	-1
λ_a on $\mathfrak{g}^*$	0

- Non-minimal cohomological vector field: $Q := Q_{\min} + Q_{\mathrm{aux}}$ with $Q_{\mathrm{aux}} = \sum_a \lambda_a \frac{\partial}{\partial \bar{c}_a}$. The added term can be regarded as a de Rham vector field on $\mathcal{F}_{\mathrm{aux}} = T[1]\mathfrak{g}^*[-1]$.[51]

[50] Recall that a Lie algebra $\mathfrak{g}$ is called *unimodular* if the matrices of adjoint representation are traceless, $\mathrm{tr}_{\,\mathfrak{g}}[x, -] = 0$ for any $x \in \mathfrak{g}$.

[51] Note that the complex $C^\infty(\mathcal{F}_{\mathrm{aux}}), Q_{\mathrm{aux}}$ has the cohomology of a point. Thus, complexes $C^\infty(\mathcal{F}), Q$ and $C^\infty(\mathcal{F}_{\min}), Q_{\min}$ are quasi-isomorphic.

- The non-minimal Berezinian $\mu = \mu_X \cdot \mathcal{D}^m c \cdot \mathcal{D}^m \bar{c} \cdot d^m \lambda$.

The integral

$$\int_{\mathcal{F}} \mu \, e^{\frac{i}{\hbar} S} \tag{151}$$

contains a $0 \cdot \infty$ indeterminacy: 0 comes from the Berezin integral over the odd variables $c, \bar{c}$ of the integrand independent on them; ∞ comes from the integral over the even variable λ of the integrand independent on λ.

However, let us replace the ill-defined integral $\int_{\mathcal{F}} \mu \, e^{\frac{i}{\hbar} S}$ by a gauge-fixed integral

$$I = \int_{\mathcal{F}} \mu \, e^{\frac{i}{\hbar}(S + Q(\Psi_\phi))} \tag{152}$$

with the gauge-fixing fermion

$$\Psi_\phi := \langle \bar{c}, \phi(x) \rangle \quad \in C^\infty(\mathcal{F})_{-1} \tag{153}$$

Note that this implies

$$Q(\Psi_\phi) = \langle \lambda, \phi(x) \rangle + \langle \bar{c}, FP(x) c \rangle$$

Thus, the gauge-fixed integral (152) is precisely the Faddeev-Popov integral (113). In particular, the integral (152)

(a) exists (converges) and is equal to $\frac{(2\pi i)^m}{\mathrm{Vol}(G)} \int_X \mu_X \, e^{\frac{i}{\hbar} S}$,
(b) is invariant, by Lemma 4.49, under changes of the gauge-fixing fermion Ψ_ϕ, and in particular invariant under changes of $\phi : X \to \mathfrak{g}$.

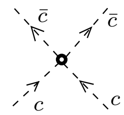
Remark 4.51. Note that the comparison of the ill-defined integral (151) with the gauge-fixed integral (152) is outside of the assumptions of the Lemma 4.49: the difference of the integrands is Q-exact but not compactly supported (in particular, in the direction of the Lagrange multiplier variables λ_a). This is why in this case the gauge-fixing (151)→(152) is simultaneously a regularization of the ill-defined integral (151), rather than being an equality of two well-defined integrals as in (149). Change of the gauge-fixing in (b) also leads, generally, to a non-compactly supported Q-exact shift of the integrand. However, as long as the integrals converge, (149) still applies (in particular, we can deform the gauge-fixing $\phi : X \to \mathfrak{g}$ in a 1-parameter family ϕ_t, $t \in [0,1]$, as long as $\phi_t^{-1}(0) \subset X$ is a local transversal section of G-orbits for all t).

Remark 4.52. One can employ more general gauge-fixing fermions than (153). E.g.

- $\Psi = \langle \bar{c}, \phi(x) \rangle + \frac{\varkappa}{2}(\bar{c}, \lambda)$ with $(,)$ a non-degenerate pairing on $\mathfrak{g}^*$ (e.g. the dual Killing form) and $\varkappa \in \mathbb{R}$ a parameter of gauge-fixing. Then $Q(\Psi) = Q(\Psi_\phi) + \frac{\varkappa}{2}(\lambda, \lambda)$. Then one can take the Gaussian integral over λ in $\int_{\mathcal{F}} \mu \, e^{\frac{i}{\hbar}(S + Q(\Psi))}$. The result is a perturbatively well-defined integral over $X \times \mathfrak{g}[1] \times \mathfrak{g}^*[-1]$.[52]

[52] E.g. in the case of Yang-Mills theory in Lorentz gauge, we have (writing only the quadratic in the fields part of the gauge-fixed action $S + Q(\Psi)$; we do not write the ghost term): $\int_M \mathrm{tr}\, \frac{1}{2} dA \wedge * dA + \lambda \wedge d^* A + \frac{\varkappa}{2} \lambda \wedge * \lambda$. After integrating out the field λ, we obtain $\int_M \mathrm{tr}\, \frac{1}{2} A(d^* d - \frac{1}{\varkappa} d d^*) A$. In particular, taking $\varkappa = -1$, we obtain the standard Hodge-de Rham Laplacian as the $A - A$ part of the Hessian.

- We can allow Ψ to contain monomials of higher degree in c and $\bar{c}$. This leads to new vertices in the Feynman rules for the gauge-fixed integral. E.g., if Ψ contains a term $\propto \bar{c}\bar{c}c$ and thus $Q(\Psi)$ contains a term $\propto \bar{c}\bar{c}cc$ leading to the new vertex



Remark 4.53. Due to freeness of the G-action on X, the BRST cohomology $H^\bullet_Q \cong H^\bullet_{Q_{\min}}$ is concentrated in degree zero and $H^0_Q \cong C^\infty(X)^G \cong C^\infty(X/G)$. In this sense, one may say that $(\mathcal{F}, Q)$ is a *resolution* of the quotient X/G.

Remark 4.54. In the construction of Faddeev-Popov setup cast withing the BRST framework, one can replace the symmetry given by a group action on X by a symmetry given by an (injective) Lie algebroid $E \to TX$. In this case the infinitesimal symmetry is given by an integrable distribution $\mathrm{im}(E) \subset TX$ on X and gauge orbits are replaced by the leaves of the foliation on X induced by this distribution via Frobenius theorem. In this case $\mathcal{F}_{\min} = E[1]$ with the corresponding cohomological vector field (see Example 4.38). The full space of BRST fields is $\mathcal{F} = E[1] \oplus E^*[-1] \oplus E^*$ (as a Whitney sum of graded vector bundles over X) with the homologically trivial extension of Q to the auxiliary fields. If the foliation is globally well-behaved (induces a fibration of X over a smooth quotient X/E), then, similarly to (a) above, one has a comparison theorem [26] asserting that the BRST integral equals

$$(2\pi i)^{\mathrm{rk}(E)}) \int_X \frac{\mu_X}{\mathrm{Vol}(\lambda_x)} e^{\frac{i}{\hbar}S}$$

where $\mathrm{Vol}(\lambda_x)$ is the volume of the leaf of the foliation passing through the point of integration $x \in X$.

4.3.4. *Remark: reducible symmetries and higher ghosts.* BRST formalism can be applied to the case when the $G = G^1$ acts on X with a stabilizer G^2 – in this case, in addition to the ghosts of degree (ghost number) 1 associated to the Lie algebra $\mathfrak{g} = \mathfrak{g}^1$, one introduces *higher ghosts* for the Lie algebra $\mathfrak{g}^2$. It may happen that it is convenient (in order to be compatible with locality structure on the underlying spacetime manifold) to have G^2 over-parameterizing the stabilizer of the $G = G^1$-action (i.e. different elements of G^2 may correspond to the same element in the stabilizer), then one introduces a second stabilizer G^3 and, respectively, new higher ghosts of degree 3. This process can be iterated further. An example of this situation is the "p-form electrodynamics" – a field theory on a Riemannian manifold M with classical fields $X = \Omega^p(M) \ni \alpha$ and action $S = \frac{1}{2}\int_M d\alpha \wedge *d\alpha$. We have gauge-symmetry $\alpha \mapsto \alpha + d\beta$ with $\beta \in \Omega^{p-1}(M) =: G = G^1$. Clearly, G^1 acts on X non-freely. In particular $\beta \sim \beta + d\gamma$, with $\gamma \in \Omega^{p-2} =: G^2$, correspond to the same gauge transformation[53] etc. We have a tower of (infinitesimal) symmetry

$$\underbrace{\Omega^0(M)}_{\mathfrak{g}^p} \to \cdots \to \underbrace{\Omega^{p-2}(M)}_{\mathfrak{g}^2} \to \underbrace{\Omega^{p-1}(M)}_{\mathfrak{g}^1} \to \underbrace{\Omega^p(M)}_{T_\alpha X}$$

[53]Note that G^2 fails to parameterize the entire stabilizer of the G-action, if de Rham cohomology $H^{d-1}(M) \neq 0$. One solution is to twist the differential forms on M by an acyclic local system. Another way is to allow this discrepancy. It will result in BRST cohomology not being concentrated just in degree zero (however, the degree nonzero cohomology will have *finite rank*).

– a truncation of de Rham complex. The minimal BRST resolution in this case is $\mathcal{F} = \bigoplus_{k=0}^{p} \Omega^{p-k}(M)[k] \ni (\alpha, c^{(1)}, \ldots, c^{(p)})$ where field $c^{(k)} \in \Omega^{p-k}(M)$ is the k-th ghost (and has ghost number k).

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4.4. **Odd-symplectic manifolds.** (Main reference: [29].)

4.4.1. *Differential forms on super (graded) manifolds.* Let $E \to M$ be a vector bundle and ΠE the corresponding split supermanifold. Then one defines the space of p-forms on ΠE as

$$\Omega^p(\Pi E) = \Gamma(M, \bigoplus_{j=0}^{p} \wedge^\bullet E^* \otimes \wedge^j T^* M \otimes \mathrm{Sym}^{p-j} E^*)$$

Here the bundle of p-forms on the r.h.s. is split according to the base/fiber bi-degree $(j, p-j)$.

More generally, for $\mathcal{M}$ a supermanifold, one can define $\Omega^\bullet(\mathcal{M})$ as functions on the parity-shifted tangent bundle $\Pi T\mathcal{M}$. If (x^i, θ^a) are local even and odd coordinates on $\mathcal{M}$, then $\Pi T\mathcal{M}$ has local coordinates x^i, θ^a (on the base of the tangent bundle) and $dx^i, d\theta^a$ (on the fiber). Here $x^i, d\theta^a$ are even and θ^a, dx^i are odd. Also, one prescribes form degree (or de Rham degree) 0 to the base coordinates and form degree 1 to the fiber coordinates. Transition maps between charts on $\Pi T\mathcal{M}$ are written naturally in terms of transition maps between the underlying charts on $\mathcal{M}$.

For $\mathcal{M}$ a $\mathbb{Z}$-graded supermanifold, differential forms $\Omega^\bullet(\mathcal{M})$ have the following three natural gradings:

(1) Form (de Rham) degree $\deg_{\mathrm{dR}}$.
(2) Internal degree (also called "grade") gr, coming from $\mathbb{Z}$-grading of coordinates on $\mathcal{M}$. In particular, grades of x^i and dx^i are the same, and similarly for θ^a and $d\theta^a$.
(3) Total degree $\deg_{\mathrm{dR}} + \mathrm{gr}$.

By convention, it is the parity of the total degree that governs the signs.

In $\mathbb{Z}$-graded context, we will use notation $\Omega^p(\mathcal{M})_k$ for p-forms of grade k.

Example 4.55. p-forms on the odd line $\mathbb{R}^{0|1}$ are functions $f(\theta, x)$ of an odd variable θ (the coordinate on $\mathbb{R}^{0|1}$) and even variable $x = d\theta$, which are of degree p in x. I.e., $\Omega^p(\mathbb{R}^{0|1}) = \{x^p(a + b \cdot \theta) \mid a, b \in \mathbb{R}\}$. In particular, unlike forms on an ordinary n-manifold, whose degree is bounded above by n, there are differential forms of arbitrarily large degree $p \geq 0$ on $\mathbb{R}^{0|1}$!

4.4.2. *Odd-symplectic supermanifolds.* Let $\mathcal{M}$ be a supermanifold.

Definition 4.56. An odd-symplectic structure on $\mathcal{M}$ is a 2-form ω on $\mathcal{M}$ which is:

- closed, $d\omega = 0$;
- odd, i.e., in local coordinates x^i, θ^a on $\mathcal{M}$ (with x^i even and θ^a odd) has the form $\sum_{i,a} \omega_{ia}(x, \theta) dx^i \wedge d\theta^a$, with $(\omega_{ia}(x, \theta))$ a matrix of local functions on $\mathcal{M}$;
- is *non-degenerate*, i.e., the matrix of coefficients $(\omega_{ia}(x, \theta))$ is invertible.

A supermanifold $\mathcal{M}$ endowed with an odd-symplectic structure ω is called an *odd-symplectic supermanifold*.

Note that it follows from non-degeneracy of ω that the dimension of $\mathcal{M}$ is $(n|n)$ for some n.

We survey the main properties of odd-symplectic supermanifolds and Lagrangian submanifolds in them without giving proofs. For proofs and details, see [29].

Theorem 4.57 (Schwarz, [29])**.** *Let $(\mathcal{M},\omega)$ be an odd-symplectic manifold with body M.*

- (i) *In the neighborhood of any point of M, one can find local coordinates (x^i,ξ_i) on $\mathcal{M}$, such that $\omega=\sum_i dx^i\wedge d\xi_i$.*
- (ii) *There exists a (global) symplectomorphism*[54] *$\phi:(\mathcal{M},\omega)\to(\Pi T^*M,\omega_{\mathrm{stand}})$ where ω_{stand} is the standard (odd-)symplectic structure on the (odd) cotangent bundle, locally written as $\omega_{\mathrm{stand}}=\sum_i dx^i\wedge d\xi_i$.*

Here (i) is the analog of Darboux theorem in odd-symplectic case. As in the ordinary symplectic geometry, one calls local coordinates (x^i,ξ_i) such that $\omega=\sum_i dx^i\wedge d\xi_i$ the *Darboux coordinates.* The global statement (ii) is very much unlike the situation of ordinary symplectic geometry: it says that, up to symplectomorphism, all odd-symplectic manifolds are (odd) cotangent bundles.

Definition 4.58. A submanifold $\mathcal{L}$ of an odd-symplectic manifold $(\mathcal{M},\omega)$ is called *Lagrangian* if it *maximally isotropic* in $\mathcal{M}$, i.e., if

- $\mathcal{L}$ is isotropic: $\omega|_{\mathcal{L}}=0$,
- $\mathcal{L}$ is not a proper submanifold of another isotropic submanifold of $\mathcal{M}$.

A Lagrangian $\mathcal{L}$ in an $(n|n)$-dimensional odd-symplectic manifold $\mathcal{M}$ has dimension $(k|n-k)$ for some $0\leq k\leq n$.

Example 4.59 ("Conormal Lagrangian")**.** Given a k-dimensional submanifold C in an (ordinary) n-manifold M, we can construct a $(k|n-k)$-dimensional Lagrangian $\mathcal{L}_C\subset\Pi T^*M$ (with ΠT^*M equipped with the standard symplectic structure of the cotangent bundle). The Lagrangian $\mathcal{L}_C$ is constructed as the *odd conormal bundle* (conormal bundle[55] with reversed parity of conormal fibers) of C:

$$\mathcal{L}_C=\Pi N^*C\quad\subset\Pi T^*M$$

The following theorem is a direct analog, in odd-symplectic context, of Weinstein's tubular neighborhood theorem in the context of ordinary symplectic manifolds.

Theorem 4.60 (Tubular neighborhood theorem in odd-symplectic context, [29])**.** *Given a Lagrangian $\mathcal{L}$ in an odd-symplectic manifold $\mathcal{M}$, there exists*

- *a tubular neighborhood $U\subset\mathcal{M}$ (with projection $p:U\to\mathcal{L}$) of the Lagrangian $\mathcal{L}\subset\mathcal{M}$,*
- *a tubular neighborhood $U_0\subset\Pi T^*\mathcal{L}$ (with projection $p_0:U_0\to\mathcal{L}$) of the zero-section $\mathcal{L}_0\simeq\mathcal{L}$ of the parity-shifted cotangent bundle $\Pi T^*\mathcal{L}$ (endowed with the standard odd-symplectic structure of the cotangent bundle),*
- *a symplectomorphism $\phi:U\xrightarrow{\sim}U_0$,*

[54] I.e. an invertible map of supermanifolds, such that the pullback along it intertwines the symplectic forms.

[55] Recall that, for $C\subset M$, the conormal bundle $N^*C\subset(T^*M)|_C$ has the fiber $N^*_xC:=\mathrm{Ann}(T_xC)=\{\alpha\in T^*_xM\ \text{s.t.}\ \langle\alpha,v\rangle=0\ \forall v\in T_xC\}$ over a point $x\in C$. Here Ann stands for *annihilator* (of the subspace $T_xC\subset T_xM$).

such that ϕ sends the Lagrangian $\mathcal{L} \subset \mathcal{M}$ to the zero-section of $\Pi T^*\mathcal{L}$ and intertwines the projections p, p_0.

The tubular neighborhood theorem above states, essentially, that in the neighborhood of a Lagrangian, the ambient odd-symplectic manifold always looks like (is locally symplectomorphic to) the odd cotangent bundle of the Lagrangian.

Example 4.61 ("Graph Lagrangian")**.** Let $\mathcal{N}$ be a $(k|n-k)$-supermanifold and $\Psi \in C^\infty(\mathcal{N})_{\text{odd}}$ an odd function. One has a Lagrangian

$$\mathcal{N}_\Psi := \text{graph}(d\Psi) \subset \Pi T^*\mathcal{N} \tag{154}$$

Note that $\Pi T^*\mathcal{N}$ has dimension $(n|n)$. If X^α are the local coordinates on $\mathcal{N}$ (some of X^αs are even and some are odd), we have coordinates (X^α, Ξ_α) on $\Pi T^*\mathcal{N}$ with parity of the cotangent fiber coordinate Ξ_α opposite to the parity of X^α. Then the submanifold $\mathcal{N}_\Psi$ is given by

$$\Xi_\alpha(X) = \frac{\partial}{\partial X^\alpha}\Psi(X)$$

For $\Psi = 0$, $\mathcal{N}_\Psi$ is the zero-section of $\Pi T^*\mathcal{N}$. For Ψ nonzero, we get a deformation of the zero-section in the cotangent bundle, given as a graph of the exact 1-form $d\Psi$ on the base.

Theorem 4.62 (Classification of Lagrangians, [29])**.** (a) Given a Lagrangian $\mathcal{L}$ in an odd-symplectic manifold $\mathcal{M}$ with body M, there exists a submanifold $C \subset M$ and a symplectomorphism $\phi : \mathcal{M} \xrightarrow{\sim} \Pi T^*M$ such that ϕ maps $\mathcal{L} \subset \mathcal{M}$ to $\mathcal{L}_C = \Pi N^*C \subset \Pi T^*M$ (cf. Example 4.59).

(b) A Lagrangian $\mathcal{L}$ in ΠT^*M can be obtained from a Lagrangian of the standard form $\mathcal{L}_C = \Pi N^*C$ for some $C \subset M$, as a graph of $d\Psi$ for some $\Psi \in C^\infty(\mathcal{L}_C)_{\text{odd}}$ (cf. Example 4.61). Here we use the tubular neighborhood theorem to identify ΠT^*M in the neighborhood of $\mathcal{L}_C$ with $\Pi T^*\mathcal{L}_C$.

4.4.3. *Odd-symplectic manifolds with a compatible Berezinian. BV Laplacian.*

Definition 4.63. For $(\mathcal{M}, \omega)$ an $(n|n)$-dimensional odd-symplectic manifold, a Berezinian μ on $\mathcal{M}$ is called *compatible* with ω, if there exists an atlas of Darboux charts (x^i, ξ_i) on $\mathcal{M}$ such that locally $\mu = d^n x \mathcal{D}^n \xi$ is the coordinate Berezinian in all charts of the atlas. (We will call the Darboux charts with this property the *special* Darboux charts.) Note that, in particular, this implies that the transition functions between charts are *unimodular*: Sdet $\frac{\partial(x_\beta, \xi_\beta)}{\partial(x_\alpha, \xi_\alpha)} = 1$. In the terminology of Schwarz [29], an odd-symplectic manifold $(\mathcal{M}, \omega)$ endowed with a compatible Berezinian μ is called an *SP-manifold* (where "P-structure" refers to the odd-symplectic form and "S-structure" refers to the Berezinian).

For $(\mathcal{M}, \omega, \mu)$ an odd-symplectic manifold with a compatible Berezinian, one introduces the odd second order operator $\Delta : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, the *Batalin-Vilkovisky Laplacian*, defined locally, in the special Darboux charts of the Definition 4.63, as

$$\Delta = \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i} \tag{155}$$

Unimodularity of transition functions implies that Δ is a globally well-defined operator. Also, the BV Laplacian squares to zero,

$$\Delta^2 = 0 \tag{156}$$

This follows from a local computation $\Delta^2 = \sum_{ij} \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\frac{\partial}{\partial \xi_i}\frac{\partial}{\partial \xi_j}$. Note that the summand changes sign under the transposition $(i,j) \mapsto (j,i)$, therefore the sum over i,j vanishes.

Another way to define the BV Laplacian is as follows. Let $(\mathcal{M},\omega)$ be an odd-symplectic manifold and μ – *any* Berezinian on $\mathcal{M}$. Define the operator $\Delta_\mu : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ by setting

$$\Delta_\mu(f) := \frac{1}{2}\mathrm{div}_\mu X_f \tag{157}$$

where $X_f \in \mathfrak{X}(\mathcal{M})$ is the *Hamiltonian vector field* generated by the Hamiltonian f, defined by the equation

$$\iota_{X_f}\omega = df$$

For $f,g \in C^\infty(\mathcal{M})$, one defines the *odd Poisson bracket* (also known as the *anti-bracket*), similarly to Poisson bracket in ordinary symplectic geometry, as

$$\{f,g\} := X_f(g) \tag{158}$$

Locally, in a Darboux chart (x^i,ξ_i) on $\mathcal{M}$, assuming that the Berezinian has local form $\mu = \rho(x,\xi)\, d^n x\, \mathcal{D}^n\xi$ with ρ a local density function, we have:

$$\Delta_\mu(f) = \sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i} f + \frac{1}{2}\{\log\rho, f\} \tag{159}$$

And the local form of the odd Poisson bracket is:

$$\{f,g\} = \sum_i f\left(\frac{\overleftarrow{\partial}}{\partial x^i}\frac{\overrightarrow{\partial}}{\partial \xi_i} - \frac{\overleftarrow{\partial}}{\partial \xi_i}\frac{\overrightarrow{\partial}}{\partial x^i}\right) g \tag{160}$$

The BV Laplacian Δ_μ, as defined by (157), does not automatically square to zero. Rather, it squares to zero, $\Delta_\mu^2 = 0$ if and only if the Berezinian μ is compatible with $(\mathcal{M},\omega)$, in the sense of Definition 4.63. And in this case, Δ_μ coincides with the BV operator (155) defined in the special Darboux charts.

Remark 4.64. A straightforward local computation shows that the operator (157) squares to zero iff $\sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i}\sqrt{\rho} = 0$. This is also turns out to be the necessary and sufficient condition for a special local Darboux chart to exist.

4.4.4. *BV integrals. Stokes' theorem for BV integrals.* Note that, for M an n-manifold, the Berezin line bundle of the odd cotangent bundle $\mathrm{Ber}(\Pi T^*M)$, as a line bundle over M, is canonically identified with the tensor square of the bundle of volume forms on M, i.e.,

$$\mathrm{Ber}(\Pi T^*M) \cong (\wedge^n T^*M)^{\otimes 2} \tag{161}$$

Similarly, for $\mathcal{N}$ a supermanifold, one has

$$\mathrm{Ber}(\Pi T^*\mathcal{N})|_{\mathcal{N}} \cong \mathrm{Ber}(\mathcal{N})^{\otimes 2} \tag{162}$$

Here we understand $\mathrm{Ber}(\mathcal{N})$ as a line bundle over $\mathcal{N}$ and the l.h.s. is a pullback of a line bundle over $\Pi T^*\mathcal{N}$ to $\mathcal{N}$.

In particular, (162) implies that there is a canonical map sending Berezinians μ on $\Pi T^*\mathcal{N}$ to Berezinians "$\sqrt{\mu|_{\mathcal{N}}}$" on $\mathcal{N}$. Locally, if X^α are local coordinates on $\mathcal{N}$ (of even and odd parity), and (X^i,Ξ_i) the respective Darboux coordinates on $\Pi T^*\mathcal{N}$, a Berezinian $\mu = \rho(X,\Xi)\, \mathcal{D}X\, \mathcal{D}\Xi$ on $\Pi T^*\mathcal{N}$ is mapped to a Berezinian $\sqrt{\mu|_{\mathcal{N}}} := \sqrt{\rho(X,0)}\, \mathcal{D}X$ on $\mathcal{N}$.

For $(\mathcal{M},\omega,\mu)$ an odd-symplectic manifold with a compatible Berezinian, a *BV integral* is an integral of the form

$$\int_{\mathcal{L}\subset\mathcal{M}} f\,\sqrt{\mu|_{\mathcal{L}}} \tag{163}$$

with $\mathcal{L}$ a Lagrangian submanifold of $\mathcal{M}$ and $f\in C^\infty(\mathcal{M})$ a function satisfying $\Delta_\mu f=0$.

Theorem 4.65 (Stokes'theorem for BV integrals, Batalin-Vilkovisky-Schwarz, [29])**.** Let $(\mathcal{M},\omega,\mu)$ be an odd-symplectic manifold with compact[56] body endowed with a compatible Berezinian.

(i) For any $g\in C^\infty(\mathcal{M})$ and $\mathcal{L}\subset\mathcal{M}$ a Lagrangian submanifold, one has

$$\int_{\mathcal{L}} \Delta_\mu g\,\sqrt{\mu|_{\mathcal{L}}}=0 \tag{164}$$

(ii) Let $\mathcal{L}$ and $\mathcal{L}'$ be two Lagrangian submanifolds whose bodies are homologous cycles in the body of $\mathcal{M}$ and let $f\in C^\infty(\mathcal{M})$ be a function satisfying $\Delta_\mu f=0$. Then the following holds:

$$\int_{\mathcal{L}} f\,\sqrt{\mu|_{\mathcal{L}}}=\int_{\mathcal{L}'} f\,\sqrt{\mu|_{\mathcal{L}'}} \tag{165}$$

Idea of proof. By (ii) of Theorem 4.57, without loss of generality we can assume $\mathcal{M}=\Pi T^*M$ for M an ordinary n-manifold. One introduces the *odd (fiberwise) Fourier transform*

$$OFT: C^\infty(\Pi T^*M)\xrightarrow{\sim} C^\infty(\Pi TM)$$

In local coordinates (x^i,ξ_i) on ΠT^*M and (x^i,θ^i) on ΠTM, assuming that the Berezinian μ has form $\mu=\rho(x,\xi)\,d^nx\,\mathcal{D}^n\xi$, the odd Fourier transform acts as follows:

$$f(x,\xi)\mapsto\widetilde{f}(x,\theta):=\int_{\Pi T^*_xM}\sqrt{\rho(x,\xi)}\,\mathcal{D}^n\xi\;e^{\langle\theta,\xi\rangle}f(x,\xi)$$

For example, in the special case when $\mu=\nu^{\otimes 2}$ for $\nu\in\Omega^n(M)$ a top form, the odd Fourier transform simply maps polyvectors $\alpha\in\mathcal{V}^\bullet(M)\cong C^\infty(\Pi T^*M)$ to differential forms $\iota_\alpha\nu\in\Omega^{n-\bullet}(M)\cong C^\infty(\Pi TM)$ via contraction with the top form ν.

The odd Fourier transform maps the BV Laplacian Δ_μ on $C^\infty(\Pi T^*M)$ to the de Rham operator d on $\Omega^\bullet(M)\cong C^\infty(\Pi TM)$, i.e., $OFT\circ\Delta_\mu=d\circ OFT$.

Consider $\mathcal{L}=\mathcal{L}_C=\Pi N^*C$ the Lagrangian of Example 4.59, for $C\subset M$ a closed submanifold. Then the BV integral is

$$\int_{\mathcal{L}_C} f\,\sqrt{\mu|_{\mathcal{L}}}=\int_C\widetilde{f}$$

where on the r.h.s. we have an integral of a differential form $\widetilde{f}=OFT(f)$ on M over the submanifold $C\subset M$.

Restricting to Lagrangians of form $\mathcal{L}_C$, (164) and (165) follow from the usual Stokes' theorem on M:

$$\int_{\mathcal{L}_C}\Delta_\mu g\,\sqrt{\mu|_{\mathcal{L}}}=\int_C d\widetilde{g}=0$$

[56]Compactness condition can be dropped, but then one has to request that the integrals converge.

and
$$\int_{\mathcal{L}_{C'}} f\,\sqrt{\mu|_{\mathcal{L}_{C'}}} - \int_{\mathcal{L}_C} f\,\sqrt{\mu|_{\mathcal{L}_C}} = \left(\int_{C'} - \int_C\right)\widetilde{f} = \int_D \widetilde{f}$$
where $D \subset M$ is a submanifold with boundary $\partial D = C' - C$; to apply Stokes' theorem here, we used that $\widetilde{f}$ is a closed form on M which follows from the assumption $\Delta_\mu f = 0$.

For general Lagrangians in ΠT^*M, one can reduce to the case of Lagrangians of form $\mathcal{L}_C$ using (a) of Theorem 4.62 for (164). For (165), one reduces to Lagrangians of form $\mathcal{L}_C$ using (b) of Theorem 4.62 together with the following calculation. Let $\mathcal{L}_t$ be a smooth family of Lagrangians in $\mathcal{M}$ with $t \in [0,1]$ a parameter, such that $\mathcal{L}_{t+\epsilon} = \mathrm{graph}(\epsilon \cdot d\Psi_t + O(\epsilon^2))$ (cf. Example 4.61) for $\Psi_t \in C^\infty(\mathcal{L}_t)$. Then, for $f \in C^\infty(\mathcal{M})$ satisfying $\Delta_\mu f = 0$, one has
$$\frac{d}{dt}\int_{\mathcal{L}_t} f\,\sqrt{\mu|_{\mathcal{L}_t}} = \int_{\mathcal{L}_t} \Delta(f\cdot\Psi_t)\,\sqrt{\mu|_{\mathcal{L}_t}} \qquad = 0$$
which vanishes by (164). Thus, we can take $\mathcal{L}_t$ to be a family connecting a given Lagrangian $\mathcal{L} \subset \Pi T^*M$ with a Lagrangian of form $\mathcal{L}_C$. Such a family exists by (b) of Theorem 4.62 and the value of the BV integral is constant along this family by the calculation above.

□

Remark 4.66. In this Subsection we were focusing on the case of supermanifolds. In the setting of $\mathbb{Z}$-graded supermanifolds, the convention is that an odd-symplectic form ω has internal degree (grade) -1, so that the odd Poisson bracket and the BV Laplacian Δ have degree $+1$.

Definition 4.67. We say that two Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ in an odd-symplectic manifold $(\mathcal{F},\omega)$ are *homotopic as Lagrangians* (or *Lagrangian-homotopic*) if there exists a smooth family $\mathcal{L}_t$, $t \in [0,1]$, of Lagrangians in $(\mathcal{F},\omega)$ (the *Lagrangian homotopy*) connecting $\mathcal{L}$ and $\mathcal{L}'$. Then we denote $\mathcal{L} \sim \mathcal{L}'$.

Remark 4.68. More generally, since the main reason we are interested in homotopic Lagrangians is because they yield same values for the BV integral of a Δ-cocycle, one can replace notion of homotopy of Lagrangians above by the (weaker) equivalence relation of (ii) of Theorem 4.65 – the condition of having homologous bodies.

Lecture 22, 11/14/2016.

4.5. **Algebraic picture: BV algebras. Master equation and canonical transformations of its solutions.**

4.5.1. *BV algebras.* BV algebras are an algebraic counterpart of odd-symplectic manifolds with a compatible Berezinian. (And Gerstenhaber algebras are the counterpart of odd-symplectic manifolds without a distinguished Berezinian.)

Definition 4.69. A *BV algebra* is a unital commutative graded algebra $\mathcal{A}^\bullet, \cdot$ over $\mathbb{R}$ (the dot stands for the graded-commutative product) endowed with

- A degree $+1$ Poisson bracket $\{-,-\} : \mathcal{A}^j \otimes \mathcal{A}^k \to \mathcal{A}^{j+k+1}$ satisfying
  - skew-symmetry:[57] $\{x,y\} = -(-1)^{(|x|+1)\,(|y|+1)}\{y,x\}$,

[57] The mnemonic rule for signs is that the *comma* in $\{-,-\}$ carries degree $+1$, and one accounts for that via the Koszul sign rule when pulling graded objects to the left/right slot of the Poisson bracket.

– Leibniz identity (in first and second slot):

(166)
$$\{x,yz\}=\{x,y\}z+(-1)^{(|x|+1)\,|y|}y\{x,z\},\qquad \{xy,z\}=x\{y,z\}+(-1)^{|y|\,(|z|+1)}\{x,z\}y$$

– Jacobi identity:

$$\{x,\{y,z\}\}=\{\{x,y\},z\}+(-1)^{(|x|+1)\,(|y|+1)}\{y,\{x,z\}\}$$

- In addition, $\mathcal{A}^\bullet$ should carry a BV Laplacian – an $\mathbb{R}$-linear map $\Delta:\mathcal{A}^j\to\mathcal{A}^{j+1}$ satisfying
  - $\Delta^2=0$,
  - $\Delta(1)=0$ (with 1 the unit in $\mathcal{A}^\bullet$),
  - second order Leibniz identity

(167) $$\Delta(xyz)\pm\Delta(xy)z\pm\Delta(xz)y\pm\Delta(yz)x\pm\Delta(x)yz\pm\Delta(y)xz\pm\Delta(z)xy=0$$

  - Poisson bracket arises as the "defect" of the first order Leibniz identity for Δ:

(168) $$\Delta(xy)=\Delta x\cdot y+(-1)^{|x|}x\cdot\Delta y+(-1)^{|x|}\{x,y\}$$

Remark 4.70. (1) The defining relations of a BV algebra given above are interdependent. E.g., the second order Leibniz identity (167) for Δ follows from (168) and the fact that $\{,\}$ is a bi-derivation of the commutative product (166).

(2) Forgetting Δ, the structure $(\mathcal{A},\cdot,\{,\})$ is the structure of a *Gerstenhaber algebra* (or "degree +1 Poisson algebra", or "$\mathcal{P}_0$ algebra").

(3) Forgetting the commutative product and shifting the grading on $\mathcal{A}$ by 1, we get a differential graded Lie algebra $\mathcal{A}[1],\{,\},\Delta$. The fact that Δ is a derivation of $\{,\}$, i.e. that

(169) $$\Delta\{x,y\}=\{\Delta x,y\}+(-1)^{|x|+1}\{x,\Delta y\}$$

is a consequence of the relations of a BV algebra.

Example 4.71. Let M be an n-manifold and ν a volume form on M. We construct

- $\mathcal{A}^{-j}:=\mathcal{V}^j(M)=\Gamma(M,\wedge^jTM)$ – polyvector fields on M with reverse grading. The graded-commutative product on $\mathcal{A}$ is the wedge product of polyvectors.
- The Poisson bracket $\{,\}:=[,]_{NS}:\mathcal{V}^k\otimes\mathcal{V}^k\to\mathcal{V}^{j+k-1}$ is the Nijenhuis-Schouten bracket of polyvectors (the Lie bracket of vector fields extended to polyvector fields via Leibniz identity).
- The BV Laplacian is the divergence w.r.t. top form ν, $\Delta=\mathrm{div}_\nu:\mathcal{V}^j\to\mathcal{V}^{j-1}$. For $j=1$ this is the ordinary divergence of a vector field, and one extends to polyvectors ($j>1$) by imposing the relation (168).

Example 4.72 (Main example)**.** Let $(\mathcal{M},\omega,\mu)$ be an odd-symplectic $\mathbb{Z}$-graded supermanifold with a compatible Berezinian.

- We set $\mathcal{A}^\bullet:=C^\infty(\mathcal{M})_\bullet$ as a commutative graded algebra.
- We set $\{,\}$ to be the degree +1 Poisson bracket (158,160) induced by the odd-symplectic form ω, $\{f,g\}=X_f(g)$.
- The BV Laplacian is defined to be the standard BV Laplacian (157) on an odd-symplectic manifold with a compatible Berezinian, $\Delta_\mu f=\frac{1}{2}\mathrm{div}_\mu X_f$.

Note that Example 4.71 is a special case of the Example 4.72, corresponding to $\mathcal{M} = T^*[-1]M$ with the standard symplectic structure of the cotangent bundle, and with $\mu = \nu^{\otimes 2}$, cf. (161).

4.5.2. *Classical and quantum master equation.* Given a BV algebra $(\mathcal{A}^\bullet, \cdot, \{,\}, \Delta)$, we say that an element $S \in \mathcal{A}^0$ satisfies *classical master equation* (CME) if

$$\{S, S\} = 0 \tag{170}$$

Note that, unlike in an ordinary Poisson algebra, this equation is not tautological: $\{S, S\}$ does not vanish automatically by skew-symmetry of the Poisson bracket $\{,\}$.

Given a solution S of classical master equation, one can construct $Q := \{S, \bullet\} \in \mathrm{Der}_1 \mathcal{A}^\bullet$ – a degree 1 derivation which, as a consequence of (170), satisfies $Q^2 = 0$. In the case of $\mathcal{A}^\bullet$ being the algebra of functions on an odd-symplectic manifold $\mathcal{M}$ (Example 4.72), the derivation $Q \in \mathfrak{X}(\mathcal{M})_1$ is a cohomological vector field on $\mathcal{M}$ arising as the Hamiltonian vector field with Hamiltonian $S \in C^\infty(\mathcal{M})_0$ solving the classical master equation.

An element $S = S^{(0)} + (-i\hbar)S^{(1)} + (-i\hbar)^2 S^{(2)} + \cdots \in \mathcal{A}^0[[-i\hbar]]$, with $\hbar$ a formal parameter, is said to satisfy the *quantum master equation* (QME) if the following holds

$$\frac{1}{2}\{S, S\} - i\hbar \Delta S = 0 \tag{171}$$

In the case when $\hbar$ can be inverted (e.g. if S as a series in $\hbar$ has nonzero convergence radius and thus $\hbar$ can be taken to be finite), quantum master equation (171) can be equivalently written[58] as

$$\Delta\, e^{\frac{i}{\hbar}S} = 0 \tag{172}$$

In terms of the coefficients $S^{(0)}, S^{(1)}, \ldots$ of the expansion of S in powers of $-i\hbar$, the quantum master equation (171) is equivalent to a sequence of equations:

$$\{S^{(0)}, S^{(0)}\} = 0 \tag{173}$$

$$\{S^{(0)}, S^{(1)}\} + \Delta S^{(0)} = 0 \tag{174}$$

$$\{S^{(0)}, S^{(2)}\} + \frac{1}{2}\{S^{(1)}, S^{(1)}\} + \Delta S^{(1)} = 0 \tag{175}$$

etc. In particular, the leading term $S^{(0)}$ of the $\hbar$-expansion of a solution of QME satisfies the classical master equation.

Given a solution $S^{(0)}$ of CME one may ask whether it can be extended to a solution of QME by adding terms proportional to powers of $\hbar$. Then, to find the first $\hbar$-correction, we need to solve (174). It is solvable iff the class of $\Delta S^{(0)}$ in degree 1 cohomology of $Q = \{S^{(0)}, \bullet\}$ vanishes (note that $\Delta S^{(0)}$ is automatically Q-closed, as follows from CME for $S^{(0)}$ and from (169)). If $\Delta S^{(0)}$ is indeed Q-exact, we can choose the primitive $-S^{(1)}$ which solves (174) and gives the first $\hbar$-correction. Next, we look for the second correction, quadratic in $\hbar$. Equation (175) is solvable for $S^{(2)}$ iff the class of $\frac{1}{2}\{S^{(1)}, S^{(1)}\} + \Delta S^{(1)}$ in H^1_Q vanishes (again, this element is automatically Q-closed). And this process goes on: at each step we have a possible

[58] This can be seen, e.g., from the following calculation. For $x \in \mathcal{A}^0$, we have $\Delta\, x^n = n x^{n-1} \Delta x + \frac{n(n-1)}{2} x^{n-2}\{x, x\}$ (proven by induction in n using (168)). Therefore, $\Delta\, e^x = \Delta\left(\sum_{n=0}^\infty \frac{x^n}{n!}\right) = (\Delta x + \frac{1}{2}\{x, x\}) e^x$. Substituting $x = \frac{i}{\hbar}S$, we obtain $\Delta\, e^{\frac{i}{\hbar}S} = (-i\hbar)^{-2}\left(\frac{1}{2}\{S, S\} - i\hbar \Delta\, S\right) e^{\frac{i}{\hbar}S}$. This proves equivalence of (171) and (172).

obstruction in H^1_Q; if the obstruction vanishes, we can construct the next term in $\hbar$-expansion of the solution of QME. If the obstructions at all steps vanish, we can construct the full extension of $S^{(0)}$ to a solution of QME by incorporating the appropriate corrections in powers of $\hbar$.

4.5.3. *Canonical transformations.*

Definition 4.73. Given two solutions of QME, $S, S' \in \mathcal{A}^0[[-i\hbar]]$, we say that S and S' are *equivalent* (notation: $S \sim S'$) if there exists a *canonical BV transformation* – a family $S_t \in \mathcal{A}^0[[-i\hbar]]$, $R_t \in \mathcal{A}^{-1}[[-i\hbar]]$ parameterized by $t \in [0,1]$, such that $S_0 = S$ and $S_1 = S'$, and the following equation holds:

$$\frac{d}{dt} S_t = \{S_t, R_t\} - i\hbar\Delta R_t \tag{176}$$

R_t is called the *generator* of the canonical BV transformation.

Remark 4.74. Equation (176) together with the fact $S = S_0$ satisfies QME implies that S_t satisfies QME

$$\frac{1}{2}\{S_t, S_t\} - i\hbar\Delta S_t = 0 \tag{177}$$

Indeed, taking the derivative in t of (177), we get $\delta_t(\frac{d}{dt}S_t) = 0$ where the t-dependent differential $\delta_t := \{S_t, \bullet\} - i\hbar\Delta$ squares to zero due to the QME on S_t. On the other hand, (176) reads $\frac{d}{dt}S_t = \delta_t(R_t)$ (i.e. improves δ_t-closedness of $\frac{d}{dt}S_t$ to δ_t-exactness). In particular, (176) implies that $\frac{d}{dt}$ of the QME (177) vanishes at time t if QME is known to hold at time t. Therefore QME for S_t implies that QME for $S_{t+\epsilon}$ is satisfied up to $O(\epsilon^2)$. And this implies (via subdividing shift $t \to t+\epsilon$ into N shifts of length ϵ/N and taking the limit $N \to \infty$) that, in fact, if (177) holds at any time t and (176) holds for all times, then (177) holds for all times.

Remark 4.75. Equations (176,177) together imply that

$$\frac{d}{dt}\, e^{\frac{i}{\hbar}S_t} = \Delta\left(-i\hbar\, e^{\frac{i}{\hbar}S_t} R_t\right)$$

Thus, in particular, if $S \sim S'$, the difference of the exponentials is Δ-exact:

$$e^{\frac{i}{\hbar}S'} - e^{\frac{i}{\hbar}S} = \Delta(\cdots)$$

where $\cdots = -i\hbar\int_0^1 dt\; R_t\, e^{\frac{i}{\hbar}S_t}$.

Remark 4.76. Equation (176,177) together can be packaged as a single "extended quantum master equation"

$$(dt \wedge \frac{d}{dt} - i\hbar\,\Delta)\, e^{\frac{i}{\hbar}\sigma} = 0$$

on an element of total degree zero $\sigma = S_t + dt\, R_t \;\; \in \Omega^\bullet([0,1]) \otimes \mathcal{A}^\bullet[[-i\hbar]]$ in non-homogeneous forms on the interval $[0,1]$ with coefficients in $\mathcal{A}^\bullet[[-i\hbar]]$.

4.6. **Half-densities on odd-symplectic manifolds. Canonical BV Laplacian. Integral forms.**

4.6.1. *Half-densities on odd-symplectic manifolds.*

Definition 4.77. A *density ρ of weight $\varkappa \in \mathbb{R}$* (or a *$\varkappa$-density*) on a supermanifold $\mathcal{M}$, covered by an atlas of coordinate charts U_α with local coordinates $(x^i_{(\alpha)}, \theta^a_\alpha)$, is a collection of locally defined functions $\rho_{(\alpha)}(x_{(\alpha)}, \theta_{(\alpha)})$ satisfying the following transformation rule on the overlaps $U_\alpha \cap U_\beta$:

$$\rho_{(\alpha)}(x_{(\alpha)}, \theta_{(\alpha)}) = \rho_{(\beta)}(x_{(\beta)}, \theta_{(\beta)}) \cdot \left| \mathrm{Sdet} \frac{\partial(x_{(\alpha)}, \theta_{(\alpha)})}{\partial(x_{(\beta)}, \theta_{(\beta)})} \right|^{\varkappa} \tag{178}$$

We denote the space of (smooth) $\varkappa$-densities on $\mathcal{M}$ by $\mathrm{Dens}^{\varkappa}(\mathcal{M})$.

We are interested in the case of densities of weight $\varkappa = 1/2$ (or *half-densities*) on an odd-symplectic manifold $(\mathcal{M}, \omega)$. We assume that the body M of $\mathcal{M}$ is oriented (and thus the odd fiber of $\Pi T^* M \simeq \mathcal{M}$ is also oriented) and the atlas agrees with the orientation, and hence the Jacobians of the transition functions are positive.

We write a half-density on $(\mathcal{M}, \omega)$ locally, in a Darboux chart (x^i, ξ_i) as

$$\rho = \rho(x, \xi) \cdot d^{\frac{1}{2}} x\, \mathcal{D}^{\frac{1}{2}} \xi$$

where $d^{\frac{1}{2}} x\, \mathcal{D}^{\frac{1}{2}} \xi$ is a locally defined symbol (coordinate half-density associated to the local coordinates (x^i, ξ_i)) satisfying the transformation property

$$d^{\frac{1}{2}} x\, \mathcal{D}^{\frac{1}{2}} \xi = \left(\mathrm{Sdet}\, \frac{\partial(x, \xi)}{\partial(x', \xi')} \right)^{\frac{1}{2}} d^{\frac{1}{2}} x'\, \mathcal{D}^{\frac{1}{2}} \xi'$$

This rule is equivalent to the transformation rule (178) with $\kappa = \frac{1}{2}$ for the coefficient functions: $\rho(x, \xi) \mapsto \rho(x', \xi') = \rho(x, \xi) \cdot \left(\mathrm{Sdet}\, \frac{\partial(x,\xi)}{\partial(x', \xi')} \right)^{-\frac{1}{2}}$.

One can view half-densities as sections of the (tensor) square root of the Berezin line bundle:

$$\mathrm{Dens}^{\frac{1}{2}} \mathcal{M} = \Gamma(\mathcal{M}, \mathrm{Ber}(\mathcal{M})^{\otimes \frac{1}{2}})$$

Remark 4.78 (Manin, [23])**.** For $\mathcal{V}$ an $(k|n-k)$-dimensional vector superspace, one can consider the space of constant (coordinate-independent) Berezinians, $\mathrm{BER}_{\mathrm{const}}(\mathcal{V}) = \mathrm{Det}\, \Pi \mathcal{V} = \wedge^n \mathcal{V}^*_{\mathrm{even}} \otimes \wedge^m \mathcal{V}_{\mathrm{odd}}$. For $(\mathcal{W}, \omega)$ an odd-symplectic $(n|n)$-dimensional vector superspace, and $\mathcal{V} = \mathcal{L} \subset \mathcal{W}$ a Lagrangian subspace, the space of constant half-densities on $\mathcal{W}$ is canonically isomorphic to the space of constant Berezinians on $\mathcal{L}$,

$$\mathrm{Dens}^{\frac{1}{2}}_{\mathrm{const}}(\mathcal{W}) \cong \mathrm{BER}_{\mathrm{const}}(\mathcal{L}) \tag{179}$$

via the map

$$\mathrm{BER}_{\mathrm{const}}(\mathcal{L}) \ni \quad \nu \;\mapsto\; (\nu^{\otimes 2})^{\otimes \frac{1}{2}} \quad \in \mathrm{Dens}^{\frac{1}{2}}_{\mathrm{const}}(\mathcal{W}) \cong \mathrm{BER}^{\otimes \frac{1}{2}}_{\mathrm{const}}(\mathcal{W})$$

where $\nu^{\otimes 2} \in \mathrm{BER}_{\mathrm{const}}(\mathcal{W}) \cong \mathrm{BER}_{\mathrm{const}}(\Pi T^* \mathcal{L}) \cong \mathrm{BER}_{\mathrm{const}}(\mathcal{L})^{\otimes 2}$.[59] Thus, one can understand constant a half-densities on an odd-symplectic space $(\mathcal{W}, \omega)$ as a Berezinian on any Lagrangian subspace $\mathcal{L} \subset \mathcal{W}$, or, since one has isomorphisms (179), as a coherent system of Berezinians on all Lagrangian subspaces of $(\mathcal{W}, \omega)$. Or, equivalently, as an equivalence class of pairs $(\mathcal{L}, \mu_{\mathcal{L}})$ of a Lagrangian $\mathcal{L} \subset \mathcal{W}$ and a constant Berezinian on $\mathcal{L}$.

[59] The crucial linear algebra observation here, formulated in terms of determinant lines of vector superspaces, is that $\mathrm{Det}(\mathcal{V} \oplus \Pi \mathcal{V}^*) \cong \mathrm{Det}(\mathcal{V})^{\otimes 2}$, cf. (161,162).

Example 4.79. Consider odd-symplectic (1|1)-superspace $\mathcal{W} = \Pi T^*\mathbb{R} = \mathbb{R}^{1|1}$ with Darboux coordinates x, ξ. The constant half-density $\rho = d^{\frac{1}{2}}x\,\mathcal{D}^{\frac{1}{2}}\xi$ on $\mathbb{R}^{1|1}$ induces the Berezinian (volume form) dx on the Lagrangian $\mathbb{R}^1 \subset \mathbb{R}^{1|1}$ and the Berezinian $\mathcal{D}\xi$ on the Lagrangian $\mathbb{R}^{0|1} \subset \mathbb{R}^{1|1}$.

Remark 4.80 (Ševera, [30])**.** Given an odd-symplectic $(n|n)$-supermanifold $(\mathcal{M}, \omega)$, one can consider the operator $\omega\wedge : \Omega^p(\mathcal{M})_k \to \Omega^{p+2}(\mathcal{M})_{k-1}$ as a differential on the space of forms on $\mathcal{M}$ (note that it does indeed square to zero since $\omega \wedge \omega = 0$). Then the cohomology $H^\bullet_{\omega\wedge}(\Omega(\mathcal{M}))$ is canonically isomorphic to the space of half-densities on $\mathcal{M}$. Locally, in Darboux coordinates (x^i, ξ_i) on $\mathcal{M}$, cohomology classes in $H^\bullet_{\omega\wedge}(\Omega(\mathcal{M}))$ have canonical representatives of form

$$\rho(x,\xi)\, dx^1 \wedge \cdots \wedge dx^n \in \Omega^n(\mathcal{M}) \tag{180}$$

which correspond to the half-densities $\rho(x,\xi) \prod_{i=1}^n d^{\frac{1}{2}}x^i\, \mathcal{D}^{\frac{1}{2}}\xi_i$ (with the same coefficient $\rho(x,\xi)$) via Remark 4.78.

4.6.2. *Canonical BV Laplacian on half-densities.* Let $(\mathcal{M}, \omega)$ be an odd-symplectic manifold. One can define (Khudaverdian, [19]) the *canonical BV Laplacian* on half-densities, $\Delta : \mathrm{Dens}^{\frac{1}{2}}\mathcal{M} \to \mathrm{Dens}^{\frac{1}{2}}\mathcal{M}$, locally given in a Darboux chart by

$$\Delta : \quad \rho(x,\xi)\, d^{\frac{1}{2}}x\, \mathcal{D}^{\frac{1}{2}}\xi \;\mapsto\; \left(\sum_i \frac{\partial}{\partial x^i}\frac{\partial}{\partial \xi_i}\rho(x,\xi)\right) d^{\frac{1}{2}}x\,\mathcal{D}^{\frac{1}{2}}\xi \tag{181}$$

The nontrivial check [19] is that the formula above defines a globally well-defined operator on half-densities.

Note that the operator Δ on half-densities does not rely on a choice of a Berezinian on $\mathcal{M}$, unlike the Schwarz's BV Laplacian (157) Δ_μ on functions on $\mathcal{M}$.

Given a compatible Berezinian μ on $(\mathcal{M}, \omega)$, one has the associated "reference" half-density $\sqrt{\mu} \in \mathrm{Dens}^{\frac{1}{2}}(\mathcal{M})$, multiplication by which induces an isomorphism

$$C^\infty(\mathcal{M}) \xrightarrow{\cdot\sqrt{\mu}} \mathrm{Dens}^{\frac{1}{2}}(\mathcal{M})$$

This isomorphism intertwines the operators Δ_μ and Δ. I.e., for $f \in C^\infty(\mathcal{M})$ one has

$$\Delta(\sqrt{\mu}\cdot f) = \sqrt{\mu}\cdot \Delta_\mu(f)$$

Remark 4.81. Note that, for μ an incompatible Berezinian, one can also introduce an operator $\widetilde{\Delta}_\mu : f \mapsto \frac{1}{\sqrt{\mu}}\Delta(f\sqrt{\mu})$ which will be, generally, different from Schwarz's BV Laplacian Δ_μ as defined by (157). More precisely, $\widetilde{\Delta}_\mu = \Delta_\mu + \frac{1}{\sqrt{\mu}}\Delta(\sqrt{\mu})\cdot$ (the last term is a multiplication operator). Operator $\widetilde{\Delta}_\mu$ always squares to zero, but $\widetilde{\Delta}_\mu(1) \neq 0$ for an incompatible Berezinian, whereas one always has $\Delta_\mu(1) = 0$ but $\Delta_\mu^2 \neq 0$ for an incompatible Berezinian. For a compatible Berezinian, we have $\widetilde{\Delta}_\mu = \Delta_\mu$. Indeed, a Berezinian is compatible iff $\Delta\sqrt{\mu} = 0$, cf. Remark 4.64.

Remark 4.82 (Ševera, [30])**.** One can also construct the canonical BV Laplacian Δ on half-densities by considering the spectral sequence calculating the cohomology of the total differential $d + \omega\wedge$ of the bi-complex $\Omega^\bullet(\mathcal{M})$ with differentials $\omega\wedge$ and d (de Rham operator on $\mathcal{M}$). Cohomology of $\omega\wedge$ yields the space of half-densities on $\mathcal{M}$ (cf. Remark 4.80). BV Laplacian arises on the third sheet E_3 of the spectral sequence as the induced differential $\Delta = d(\omega\wedge)^{-1}d$ on $H^\bullet_{\omega\wedge}(\Omega(\mathcal{M})) \cong$

$\mathrm{Dens}^{\frac12}\mathcal{M}$.[60] (First sheet E_1 is $\Omega^\bullet(\mathcal{M}),\omega\wedge$ and second sheet E_2 is $H^\bullet_{\omega\wedge}(\Omega(\mathcal{M}))$ with zero differential.)

For $(\mathcal{M},\omega)$ and $\mathcal{L}\subset\mathcal{M}$ a Lagrangian submanifold, there is a well-defined restriction operation

$$\mathrm{Dens}^{\frac12}\mathcal{M}\to \mathrm{BER}(\mathcal{L})$$

cf. (162) and Remark 4.78. If a (X^α,Ξ_α) is a Darboux chart on $\mathcal{M}$ in which $\mathcal{L}$ is given by $\Xi=0$, the map above sends $\rho(X,\Xi)\,\mathcal{D}^{\frac12}X\,\mathcal{D}^{\frac12}\Xi\mapsto\rho(X,0)\,\mathcal{D}X$.

Thus, in terms of half-densities, a BV integral is an integral of form

$$\int_{\mathcal{L}\subset\mathcal{M}}\alpha:=\int_{\mathcal{L}\subset\mathcal{M}}\alpha|_{\mathcal{L}}$$

with $\mathcal{L}$ a Lagrangian submanifold and α a Δ-closed half-density. The BV-Stokes' theorem (Theorem 4.65) in this language states that:

(i) $\int_{\mathcal{L}}\Delta\beta=0$, for any $\beta\in\mathrm{Dens}^{\frac12}(\mathcal{M})$
(ii) $\int_{\mathcal{L}}\alpha=\int_{\mathcal{L}'}\alpha$ for $\alpha\in\mathrm{Dens}^{\frac12}(\mathcal{M})$ satisfying $\Delta\alpha=0$ and $\mathcal{L}\sim\mathcal{L}'$ two Langrangians with homologous bodies.

Remark 4.83 (Canonical transformation as an action of a symplectic flow on half–densities)**.** In the setting of half-densities, a canonical transformation of solutions of quantum master equation (Definition 4.73) admits the following interpretation. Let $(\mathcal{M},\mu,\omega)$ be an odd-symplectic manifold with a compatible Berezinian. A canonical transformation (176,177) can be viewed as a family of Δ-closed half-densities on $\mathcal{M}$ of form $\rho_t=\mu^{\frac12}e^{\frac{i}{\hbar}S_t}$ ($\Delta\rho_t=0$ is equivalent to the quantum master equation (177)), such that for any $0\le t_0<t_1\le 1$, one has $\rho_{t_1}=(\Phi_{t_0,t_1})_*\rho_{t_0}$. Here $\Phi_{t_0,t_1}:\mathcal{M}\xrightarrow{\sim}\mathcal{M}$ is the symplectomorphism arising as the flow, from time t_0 to time t_1, of the t-dependent Hamiltonian vector field $\{R_t,\bullet\}\in\mathfrak{X}(\mathcal{M})_0$; $(\Phi_{t_0,t_1})_*$ stands for the pushforward of a half-density by the symplectomorphism. In this sense, the first term on the r.h.s. of (176) corresponds to the transformation of the function S_t by the Hamiltonian vector field $\{R_t,\bullet\}$, whereas the second term compensates for the transformation of the reference half-density $\mu^{\frac12}$ under the infinitesimal flow by $\{R_t,\bullet\}$.

4.6.3. *Integral forms.*

Definition 4.84 (Manin, [23])**.** An *integral form* on a supermanifold $\mathcal{N}$ is a a half-density on $\Pi T^*\mathcal{N}$ (with the standard symplectic structure of the cotangent bundle). We denote the space of integral forms on $\mathcal{N}$ by $\mathrm{Int}(\mathcal{N}):=\mathrm{Dens}^{\frac12}(\Pi T^*\mathcal{N})$. Given an integral form α on $\mathcal{N}$, its integral over a submanifold $C\subset\mathcal{N}$ is defined as

$$\int_{C\subset\mathcal{N}}\alpha:=\int_{\Pi N^*C\subset\Pi T^*\mathcal{N}}\widetilde{\alpha}\tag{182}$$

– the integral of the corresponding half-density $\widetilde{\alpha}$ over the conormal Lagrangian $\mathcal{L}_C=\Pi N^*C$ (Example 4.59) in the odd cotangent bundle $\Pi T^*\mathcal{N}$.

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[60] In particular, consider the action of the operator $d(\omega\wedge)^{-1}d$ on the cocycle of form (180): $\rho(x,\xi)dx^1\cdots dx^n\ \xrightarrow{d}\ \sum_i\frac{\partial}{\partial\xi_i}\rho(x,\xi)\ \ d\xi_i\,dx^1\cdots dx^n\ \xrightarrow{(\omega\wedge)^{-1}}$ $(-1)^{|\rho|+1}\sum_i(-1)^{i-1}\frac{\partial}{\partial\xi_i}\rho(x,\xi)\ dx^1\cdots\widehat{dx^i}\cdots dx^n\xrightarrow{d}\left(\sum_i\frac{\partial}{\partial x^i}\frac{\partial}{\partial\xi_i}\rho(x,\xi)\right)\ dx^1\cdots dx^n$.

Integral forms on $\mathcal{N}$ generalize the notion of Berezinians on $\mathcal{N}$ (in particular, $\mathrm{BER}(\mathcal{N}) \subset \mathrm{Int}(\mathcal{N})$). Whereas a Berezinian can be integrated over whole of $\mathcal{N}$, an integral form can be integrated over an arbitrary submanifold $C \subset \mathcal{N}$ (integrating a full Berezinian over a proper submanifold yields zero). Whereas $\mathrm{BER}(\mathcal{N})$ is a torsor over functions $C^\infty(\mathcal{N})$, $\mathrm{Int}(\mathcal{N})$ is a torsor over polyvectors $\mathcal{V}^\bullet(\mathcal{N}) = C^\infty(\Pi T^*\mathcal{N})$. Put another way, one has

$$\mathrm{Int}(\mathcal{N}) = \mathcal{V}^\bullet(\mathcal{N}) \otimes_{C^\infty(\mathcal{N})} \mathrm{BER}(\mathcal{N})$$

Example 4.85. For $\mathcal{N} = M$ an ordinary n-manifold,

$$\mathrm{Int}(M) = \mathcal{V}^\bullet(M) \otimes_{C^\infty(M)} \Omega^n(M) = \Omega^{n-\bullet}(M) \tag{183}$$

is the space of differential forms on M, where non-top degree forms arise as contractions of a top form with a polyvector. Integration of integral forms over submanifolds (182) over submanifolds yields in this case an integral of a differential form over a submanifold $C \subset M$. Canonical BV Laplacian Δ on integral forms (viewed as half-densities on ΠT^*M) under the identification (183) with differential forms becomes the de Rham operator on M.

Example 4.86 (Integral forms on the odd line)**.** Consider integral forms on the odd line $\mathcal{N} = \mathbb{R}^{0|1}$. Let θ be the odd coordinate on $\mathbb{R}^{0|1}$ and Y the even fiber coordinate on $\Pi T^*\mathbb{R}^{0|1}$. Then we the general integral forms on $\mathbb{R}^{0|1}$ have the following form:

$$\mathrm{Int}(\mathbb{R}^{0|1}) \quad \ni \alpha = f(Y,\theta)\cdot \mu^{\frac12} = (f_0(Y) + f_1(Y)\,\theta)\cdot \mu^{\frac12} \tag{184}$$

with f_0, f_1 functions of Y. Here $\mu^{\frac12} = d^{\frac12}Y\,\mathcal{D}^{\frac12}\theta$ is the standard coordinate half-density. By Remark 4.78, $\mu^{\frac12}$ is a class represented by equivalent pairs ($\mathbb{R}^{0|1} \subset \Pi T^*\mathbb{R}^{0|1}, \mathcal{D}\theta$) and ($\mathbb{R}^1 \subset \Pi T^*\mathbb{R}^{0|1}, dY$). Berezinians or $\mathbb{R}^{0|1}$ correspond to the case $f_0(Y) = 0$. An integral form (184) is Δ-closed iff $f_1(Y)$ is a constant function of Y. An integral form (184) is Δ-exact iff $f_1 = 0$ and $\int_{\mathbb{R}} f_0(Y)\,dY = 0$. Supermanifold $\mathbb{R}^{0|1}$ has two nonempty submanifolds: $\{0\} \subset \mathbb{R}^{0|1}$ and $\mathbb{R}^{0|1} \subset \mathbb{R}^{0|1}$. Integral of an integral form α over these two submanifolds is, according to the definition (182), respectively,

$$\int_{\{0\}\subset\mathbb{R}^{0|1}} \alpha = \int_{\mathbb{R}^1\subset\Pi T^*\mathbb{R}^{0|1}} f_0(Y)\,dY, \qquad \int_{\mathbb{R}^{0|1}} \alpha = \int_{\mathbb{R}^{0|1}\subset\Pi T^*\mathbb{R}^{0|1}} f_1(0)\theta\,\mathcal{D}\theta = f_1(0)$$

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4.7. **Fiber BV integrals.** [61]

Let $(\mathcal{F}',\omega')$, $(\mathcal{F}'',\omega'')$ be two odd-symplectic manifolds and

$$\mathcal{F} = \mathcal{F}' \times \mathcal{F}'' \tag{185}$$

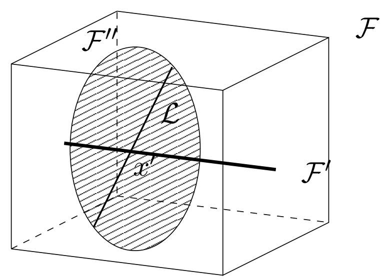
their direct product with the direct sum symplectic structure $\omega = \omega' \oplus \omega''$ (or, more pedantically, $\omega = \omega' \otimes 1 + 1 \otimes \omega'' \in \Omega(\mathcal{F}') \otimes \Omega(\mathcal{F}'') \subset \Omega(\mathcal{F})$). Denote $P : \mathcal{F} \to \mathcal{F}'$ the projection to the first factor in (185).

For $\mathcal{L} \subset \mathcal{F}''$ a Lagrangian submanifold, we denote

$$P_*^{(\mathcal{L})} = \int_{\mathcal{L}\subset\mathcal{F}''} \quad : \quad \mathrm{Dens}^{\frac12}\mathcal{F} \to \mathrm{Dens}^{\frac12}\mathcal{F}' \tag{186}$$

[61] References: [25, 6, 10].

the *fiber BV integral* – the fiber integral, parameterized by points x' of $\mathcal{F}'$, over a Lagrangian $\mathcal{L}_{x'} \subset P^{-1}(x')$ – a copy of $\mathcal{L} \subset \mathcal{F}''$ placed over x'.



In particular, (186) is an $\mathbb{R}$-linear map which sends $\phi = \phi' \otimes \phi'' \in \mathrm{Dens}^{\frac{1}{2}}\mathcal{F}' \otimes \mathrm{Dens}^{\frac{1}{2}}\mathcal{F}'' \subset \mathrm{Dens}^{\frac{1}{2}}\mathcal{F}$ to $P_*^{(\mathcal{L})}\phi = \phi' \cdot \int_{\mathcal{L}\subset\mathcal{F}''}\phi''$. In other words, the map (186) is the full (ordinary) BV integral on $\mathcal{F}''$ tensored with identity on $\mathcal{F}'$:

$$\mathcal{P}_*^{(\mathcal{L})}: \quad \mathrm{Dens}^{\frac{1}{2}}\mathcal{F} \cong \mathrm{Dens}^{\frac{1}{2}}\mathcal{F}'\widehat{\otimes}\mathrm{Dens}^{\frac{1}{2}}\mathcal{F}'' \xrightarrow{\mathrm{id}_{\mathcal{F}'}\otimes\int_{\mathcal{L}\subset\mathcal{F}''}} \mathrm{Dens}^{\frac{1}{2}}\mathcal{F}'$$

We also call the map $P_*^{(\mathcal{L})}$ the *BV pushforward* (of half-densities, along the odd-symplectic fibration $P: \mathcal{F} \to \mathcal{F}'$).

Theorem 4.87 (Stokes' theorem for fiber BV integrals)**.** (i) $P_*^{(\mathcal{L})}$ is a chain map intertwining the canonical BV Laplacians Δ on $\mathcal{F}$ and Δ' on $\mathcal{F}'$:

$$\Delta' P_*^{(\mathcal{L})} = P_*^{(\mathcal{L})}\Delta \tag{187}$$

(ii) Let $\mathcal{L} \sim \widetilde{\mathcal{L}}$ be two homotopic Lagrangians (cf. Definition 4.67) in $\mathcal{F}''$, and let $\phi \in \mathrm{Dens}^{\frac{1}{2}}\mathcal{F}$ be a half-density such that $\Delta\phi = 0$. Then

$$P_*^{(\widetilde{\mathcal{L}})}\phi - P_*^{(\mathcal{L})}\phi = \Delta'(\cdots) \tag{188}$$

More precisely, if $\widetilde{\mathcal{L}} = \mathrm{graph}(\epsilon \cdot d\Psi)$ is an infinitesimal Lagrangian homotopy with generator $\Psi \in C^\infty(\mathcal{L})_{-1}$ (cf. Example 4.61), then one can write the primitive on the r.h.s. of (188) explicitly in terms of the generator Ψ:

$$(\cdots) = \epsilon \cdot P_*^{(\mathcal{L})}(\Psi \cdot \phi) \tag{189}$$

Next, assume that odd-symplectic manifolds $(\mathcal{F}', \omega')$, $(\mathcal{F}'', \omega'')$ are equipped with compatible Berezinians μ', μ''. Then $\mu = \mu' \cdot \mu''$ is a compatible Berezinian on the direct product $\mathcal{F} = \mathcal{F}' \times \mathcal{F}''$.

Definition 4.88. Let $S \in C^\infty(\mathcal{F})_0[[\hbar]]$ be a solution of quantum master equation on $\mathcal{F}$, i.e. $\Delta_\mu e^{\frac{i}{\hbar}S} = 0 \Leftrightarrow \frac{1}{2}\{S,S\} - i\hbar\Delta_\mu S = 0$. Then we call $S' \in C^\infty(\mathcal{F}')_0[[\hbar]]$ the *effective BV action* for S induced on $\mathcal{F}'$, if

$$\mu'^{\frac{1}{2}}\, e^{\frac{i}{\hbar}S'} = P_*^{(\mathcal{L})}\left(\mu^{\frac{1}{2}}\, e^{\frac{i}{\hbar}S}\right) \tag{190}$$

By an abuse of notations, we will write $S' = P_*S$ for the effective BV action. Or, if we want to emphasize the role of the Lagrangian, $S' = P_*^{(\mathcal{L})}S$.

Definition above is a realization, in the context of BV formalism, of the idea Wilson's effective action (99) of Section 3.11.4.

The following is a corollary of Theorem 4.87.

Corollary 4.89. (i) If S is a solution of QME on $\mathcal{F}$ then the effective action S' induced on $\mathcal{F}'$ via the fiber BV integral (190) satisfies QME on $\mathcal{F}'$.

(ii) If S is a solution of QME on $\mathcal{F}$ and $\mathcal{L} \sim \widetilde{\mathcal{L}}$ are two homotopic Lagrangians in $\mathcal{F}''$, the corresponding effective actions $S' = P_*^{(\mathcal{L})} S$ and $\widetilde{S}' = P_*^{(\widetilde{\mathcal{L}})} S$ are related by a canonical transformation, $S' \sim \widetilde{S}'$.

(iii) Assume that $S \sim \widetilde{S}$ are two solutions of QME on $\mathcal{F}$ are related by a canonical transformation. Then the respective effective actions (defined using the same Lagrangian $\mathcal{L} \subset \mathcal{F}''$) are related by a canonical transformation of $\mathcal{F}'$.

Therefore, the BV pushforward P_* defines a map

$$\mathrm{SolQME}(\mathcal{F})/\sim \quad \xrightarrow{P_*^{[\mathcal{L}]}} \quad \mathrm{SolQME}(\mathcal{F}')/\sim$$

sending classes of solutions of QME on $\mathcal{F}$ modulo canonical tranformations to classes of solutions of QME on $\mathcal{F}'$ modulo canonical tranformations, and the map depends on a class $[\mathcal{L}]$ of Lagrangians in $\mathcal{F}''$ modulo Lagrangian homotopy.

4.8. Batalin-Vilkovisky formalism.

4.8.1. *Classical BV formalism.* We call a *classical BV theory* the following package of supergeometric data:

- A $\mathbb{Z}$-graded supermanifold $\mathcal{F}$ (the space of BV fields),
- an odd-symplectic structure $\omega \in \Omega^2(\mathcal{F})_{-1}$ (the BV 2-form),
- a function $S \in C^\infty(\mathcal{F})_0$ (the BV action, or *master action*) satisfying the classical master equation $\{S, S\} = 0$.

Note that the Hamiltonian vector field on $\mathcal{F}$ generated by S,

$$Q := X_S = \{S, \bullet\} \in \mathfrak{X}(\mathcal{F})_1$$

(the BRST operator), squares to zero due to the CME.

Also, note that Q is compatible with the odd-symplectic form:

$$L_Q \omega = 0$$

(with L_Q the Lie derivative along Q), which follows from $\iota_Q \omega = dS$ (the condition that Q is a Hamiltonian vector field generated by S).

Definition 4.90. A *Hamiltonian dg manifold* of degree k is:

- a dg manifold $(\mathcal{M}, Q)$,
- a symplectic form of grade (internal degree) k, $\omega \in \Omega^2(\mathcal{M})_k$,
- a Hamiltonian $H \in C^\infty(\mathcal{M})_{k+1}$ satisfying $\{H, H\}_\omega = 0$ with $\{-,-\}$ the Poisson bracket of degree $-k$ on $C^\infty(\mathcal{M})$ associated to ω.

In particular, the Hamiltonian vector field $Q = X_H \in \mathfrak{X}(\mathcal{M})_1$ is cohomological.

Case $k = -1$ of the definition above corresponds to a classical BV theory. Case $k = 0$ emerges in the BFV (Batalin-Fradkin-Vilkovisky) formalism – the Hamiltonian counterpart of the BV formalism (and also plays an important role in symplectic geometry, in the problem of describing coisotropic reductions, see [28]). Case $k = n - 1$ for $n \geq 0$ arises as the target structure for n-dimensional AKSZ sigma models [1].

Example 4.91 (A BRST system in BV formalism, classically)**.** Given a classical BRST package $(\mathcal{F}_{\mathrm{BRST}}, Q_{\mathrm{BRST}}, S_{\mathrm{BRST}})$, we construct the following BV package:

- The space of BV fields is constructed as a (shifted) cotangent bundle
$$\mathcal{F}_{\mathrm{BV}} = T^*[-1]\mathcal{F}_{\mathrm{BRST}}$$
with ω_{BV} the standard symplectic structure of the cotangent bundle.
- The BV action is
$$S_{\mathrm{BV}} = p^* S_{\mathrm{BRST}} + \widetilde{Q_{\mathrm{BRST}}} \tag{191}$$
Here $p : \mathcal{F}_{\mathrm{BV}} \to \mathcal{F}_{\mathrm{BRST}}$ is the projection to the base of the cotangent bundle and $\widetilde{Q_{\mathrm{BRST}}}$ is the lifting of the vector field Q_{BRST} on the base of the cotangent bundle to a function on the total space linear in the fibers.[62]
- The cohomological vector field on the total space has the form
$$Q_{\mathrm{BV}} = X_{p^* S_{\mathrm{BRST}}} + Q_{\mathrm{BRST}}^{\text{cot. lift}}$$
where the first term is the Hamiltonian vector field generated by the first term in (191) and $Q_{\mathrm{BRST}}^{\text{cot. lift}}$ is the cotangent lift of a vector field Q_{BRST} on the base of the cotangent bundle to a vector field on the total space.

If Φ^α are local coordinates on $\mathcal{F}_{\mathrm{BRST}}$, then $\mathcal{F}_{\mathrm{BV}}$ has corresponding Darboux coordinates $(\Phi^\alpha, \Phi^+_\alpha)$, where the fiber coordinates Φ^+_α are called *anti-fields* (as opposed to Φ^α which are called *fields*). The odd-symplectic structure is:
$$\omega_{\mathrm{BV}} = \sum_\alpha d\Phi^\alpha \wedge d\Phi^+_\alpha$$
The BV action is:
$$S_{\mathrm{BV}}(\Phi, \Phi^+) = S_{\mathrm{BRST}}(\Phi) + \sum_\alpha Q^\alpha_{\mathrm{BRST}}(\Phi)\cdot \Phi^+_\alpha$$
where $Q^\alpha_{\mathrm{BRST}} = L_{Q_{\mathrm{BRST}}}\Phi^\alpha$ are the components of Q_{BRST} (i.e., $Q_{\mathrm{BRST}} = \sum_\alpha Q^\alpha_{\mathrm{BRST}}(\Phi)\frac{\partial}{\partial \Phi^\alpha}$). The BRST operator on the BV fields (the cohomological vector field) is:
$$Q_{\mathrm{BV}} = \sum_\alpha \left(S_{\mathrm{BRST}}(\Phi) \frac{\overleftarrow{\partial}}{\partial \Phi^\alpha}\right) \frac{\partial}{\partial \Phi^+_\alpha} + \\ + \sum_\alpha Q^\alpha(\Phi)\frac{\partial}{\partial\Phi^\alpha} + \sum_{\alpha,\beta} \pm \Phi^+_\alpha \left(\frac{\partial}{\partial \Phi^\beta} Q^\alpha(\Phi)\right)\frac{\partial}{\partial \Phi^+_\beta}$$

4.8.2. *Quantum BV formalism.* We define a quantum (finite-dimensional) BV theory as the following package of data.

- A $\mathbb{Z}$-graded manifold $\mathcal{F}$ of BV fields,
- an odd-symplectic structure $\omega \in \Omega^2(\mathcal{F})_{-1}$ (the BV 2-form),
- a Berezinian $\mu \in \mathrm{BER}(\mathcal{F})$ compatible with ω (the integration measure on BV fields),
- a master action $S = S^{(0)} - i\hbar S^{(1)} + (-i\hbar)^2 S^{(2)} + \cdots \in C^\infty(\mathcal{F})_0[[-i\hbar]]$ satisfying the quantum master equation
$$\frac{1}{2}\{S,S\} - i\hbar \Delta_\mu S = 0 \quad \Leftrightarrow \quad \Delta_\mu e^{\frac{i}{\hbar}S} = 0$$

[62]Note that, generally, to $\alpha \in \mathcal{V}^p(M)$ a p-polyvector field, one can associate a function $\widetilde{\alpha} \in C^\infty(T^*[-1]M)$ of degree p in fiber coordinates. Here one can replace M by a general $\mathbb{Z}$-graded manifold, and in particular by $\mathcal{F}_{\mathrm{BRST}}$.

Remark 4.92. Unlike in the classical case, the vector field X_S does not automatically square to zero (since S satisfies QME rather than CME). However, one can define the second order operator

$$\delta_S = \{S, \bullet\} - i\hbar\, \Delta = -i\hbar e^{-\frac{i}{\hbar}S} \Delta \left(e^{\frac{i}{\hbar}S} \cdot \bullet \right)$$

which squares to zero due to QME (also note that the second equality above uses QME) and serves as a quantum replacement for the BRST operator in BV formalism. (We have encountered this operator before, in Remark 4.74.) Note also that $\delta_S \bmod \hbar = X_{S^{(0)}} =: Q$ is the classical BRST operator associated to the classical part of the master action S, and it does square to zero.

Idea of gauge-fixing in BV formalism. The partition function, as defined by a BV integral over a Lagrangian $\mathcal{L} \subset \mathcal{F}$

$$Z = \int_{\mathcal{L} \subset \mathcal{F}} \sqrt{\mu}\, e^{\frac{i}{\hbar}S}$$

does not change under the Lagrangian homotopy $\mathcal{L}_0 \sim \mathcal{L}_1$ (smooth deformation staying in the class of Lagrangians, cf. Definition 4.67) by Theorem 4.65, since the integrand is Δ-closed. If it happens that S has degenerate critical points on a Lagrangian $\mathcal{L}_0$, we use the freedom to deform $\mathcal{L}_0$ to another Lagrangian $\mathcal{L}_1$ in such a way that S has non-degenerate critical points on $\mathcal{L}_1$ and the integral can be calculated by the stationary phase formula. Thus, the gauge-fixing in BV formalism is the choice of the Lagrangian submanifold in $\mathcal{F}$.

One can also study observables in BV formalism. One says that $\mathcal{O} \in C^\infty(\mathcal{F})[[\hbar]]$ is a (quantum) BV observable, if $\delta_S \mathcal{O} = 0$ is satisfied. The expectation value of an observable is the BV integral of form

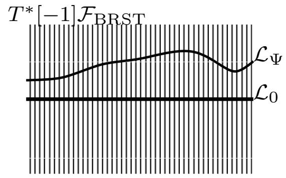
$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int_{\mathcal{L} \subset \mathcal{F}} \sqrt{\mu}\, e^{\frac{i}{\hbar}S} \mathcal{O}$$

Equation $\delta_S \mathcal{O} = 0$ is a way to express gauge-invariance of the observable in BV formalism, and guarantees that the integrand above is Δ-closed and hence one can deform $\mathcal{L}$ in the class of Lagrangians, thereby applying the gauge-fixing strategy as above and converting the integral to the form where it can be calculated by the stationary phase formula.

Remark 4.93. Note that, since δ_S is not a derivation, a product of observables in BV formalism is not necessarily an observable. (Though, one can correct the naive product to a δ_S-cocycle using homological perturbation theory.) However, in the context of local field theory, a product of observables with disjoint support is indeed an observable (e.g. the product of Wilson loop observables in Chern-Simons theory for several non-intersecting loops is an observable).

Example 4.94 (A quantum BRST system in BV formalism)**.** Let $(\mathcal{F}_{\text{BRST}}, Q_{\text{BRST}}, S_{\text{BRST}}, \mu_{\text{BRST}})$ be a quantum BRST package (cf. Section 4.3.2). We define $\mathcal{F}_{\text{BV}}, \omega_{\text{BV}}, S_{\text{BV}}$ as in the Example 4.91. For the Berezinian on the cotangent bundle we set $\mu_{\text{BV}} = \mu_{\text{BRST}}^{\otimes 2}$ (using (161)). Note that, since the BV action (191) does not depend on $\hbar$, the quantum master equations splits into two equations: $\{S_{\text{BV}}, S_{\text{BV}}\} = 0$ (the CME) and $\Delta_{\mu_{\text{BV}}} S_{\text{BV}} = 0$. The CME is satisfied due to the classical BRST relations $Q_{\text{BRST}}^2 = 0$, $Q_{\text{BRST}}(S_{\text{BRST}}) = 0$, while equation $\Delta_{\mu_{\text{BV}}} S_{\text{BV}} = 0$ follows from the relation $\text{div} Q_{\text{BRST}} = 0$ for the quantum BRST package.

Consider the gauge-fixing, within BV framework, for such a system coming from a BRST package. Denote $\mathcal{L}_0$ the zero-section of $\mathcal{F}_{\mathrm{BV}} = T^*[-1]\mathcal{F}_{\mathrm{BRST}}$ and let $\mathcal{L}_\Psi = \mathrm{graph}(d\Psi) \subset T^*[-1]\mathcal{F}_{\mathrm{BRST}}$ be the graph Lagrangian, for $\Psi = \Psi(\Phi) \in C^\infty(\mathcal{F}_{\mathrm{BRST}})_{-1}$.



We use Φ^α for local coordinates on $\mathcal{F}_{\mathrm{BRST}}$ (and we assume for simplicity that $\mu_{\mathrm{BRST}} = \mathcal{D}\Phi$ locally) and Φ^+_α for the corresponding fiber coordinates on $T^*[-1]\mathcal{F}_{\mathrm{BRST}}$. Then gauge-fixing consists in the replacement

(192)
$$\int_{\mathcal{L}_0 \subset T^*[-1]\mathcal{F}_{\mathrm{BRST}}} \sqrt{\mu_{\mathrm{BV}}}\; e^{\frac{i}{\hbar} S_{\mathrm{BV}}(\Phi,\Phi^+)} \quad \mapsto \quad \int_{\mathcal{L}_\Psi \subset T^*[-1]\mathcal{F}_{\mathrm{BRST}}} \sqrt{\mu_{\mathrm{BV}}}\; e^{\frac{i}{\hbar} S_{\mathrm{BV}}(\Phi,\Phi^+)}$$

Since S_{BV} on the zero-section reduces to S_{BRST}, the l.h.s. of (192) reduces to $\int_{\mathcal{F}_{\mathrm{BRST}}} \mathcal{D}\Phi\; e^{\frac{i}{\hbar} S_{\mathrm{BRST}}(\Phi)}$. On the other hand, to evaluate the r.h.s. of (192), we note that S_{BV} restricted to the Lagrangian $\mathcal{L}_\Psi$ is $S_{\mathrm{BV}}(\Phi^\alpha, \Phi^+_\alpha = \frac{\partial}{\partial \Phi^\alpha}\Psi) = S_{\mathrm{BRST}} + Q_{\mathrm{BRST}}(\Psi)$. Therefore, r.h.s. of (192) reads $\int_{\mathcal{F}_{\mathrm{BRST}}} \mathcal{D}\Phi\;\; e^{\frac{i}{\hbar}(S_{\mathrm{BRST}}(\Phi) + Q_{\mathrm{BRST}}(\Psi))}$. Thus, BV gauge-fixing, performing the Lagrangian homotopy $\mathcal{L}_0 \mapsto \mathcal{L}_\Psi$ precisely corresponds to the gauge-fixing procedure of BRST formalism (149), shifting the BRST action by a Q_{BRST}-coboundary.

Lecture 24, 11/21/2016.

4.8.3. *BV for gauge symmetry given by a non-integrable distribution.*

4.8.4. *Felder-Kazhdan existence-uniqueness result for solutions of the classical master equation.*

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