"bc system" (or "reparametrization ghosts")

$$S = \frac{1}{2\pi}\int d^2x\, b_{\mu\nu}\partial^\mu c^\nu$$

$b_{\mu\nu}$ - odd traceless symmetric (tensor)
c^ν - odd (vector field)

In complex coordinates: $S \propto \int dz\,d\bar z\,(\underset{b_{zz}}{b}\,\bar\partial \underset{c^z}{c} + \bar b\,\partial \bar c)$

Propagator: $\langle b(z)\,c(w)\rangle = \frac{1}{z-w}$, basic OPE: $b(z)\,c(w) \sim \frac{1}{z-w} + \text{reg.}$

Stress-energy: $\hat T(z) = :2\partial\hat c(z)\,\hat b(z) + \hat c(z)\,\partial \hat b(z):$, $\bar T = \dots$

OPE Tb, Tc, TT $\xrightarrow{\text{using Wick's thm}}$
- b, c are primary with conf. weight $(h,\bar h) = (2,0)$ and $(-1,0)$ respectively
- central charge is $(c,\bar c) = (-26,-26)$

- Exercise!

Exercise: describe the space of states

- What does this have to do with (Polyakov) bosonic string theory? (Rough idea)

We want $Z_{string}(\Sigma, \mathbb{R}^D) = \int_{Met(\Sigma)} \mathcal{D}g \int_{Maps(\Sigma,\mathbb{R}^D)} \mathcal{D}X^i\; e^{-S_{Polyakov}(g,X)}$ $\propto$

Z_{string}: Σ as a topological surface

$S_{Polyakov}(g,X) = \int_\Sigma \sqrt{g}\,d^2x\,(g^{-1})^{\mu\nu}(x)\,\partial_\mu X^i\,\partial_\nu X^i$ = action for D free bosons on (Σ, g)

$$\propto \int_{(Met(\Sigma)\times Maps(\Sigma;\mathbb{R}^D))/Diff(\Sigma)} e^{-S_{Polyakov}} \overset{\text{write } g = dz\,d\bar z\cdot e^{2\varphi}}{=} \int_{\left(\begin{pmatrix}\text{conformal}\\ \text{structures}\\ \text{on } \Sigma\end{pmatrix}\times\begin{pmatrix}\text{Weyl}\\ \text{factors}\\ \Omega = e^{2\varphi}\end{pmatrix}\times Maps\right)/Diff(\Sigma)} e^{-S_{Polyakov}} =$$

quotient over $Diff(\Sigma)$

$e^{2\varphi}$: "Liouville field"

$$\overset{\text{Faddeev-Popov gauge-fixing}}{=} \int_{\mathcal{M}_\Sigma} \int_{\text{Liouv. field } \varphi\in C^\infty(\Sigma)} \underbrace{\int \mathcal{D}b\,\mathcal{D}c\; e^{-S_{bc}}}_{\text{Gaussian } \int} \int_{Maps(\Sigma,\mathbb{R}^D)} e^{-S_{Polyakov}} =$$

$\mathcal{M}_\Sigma$ - moduli of conf. str.

Gaussian $\int$ computing FP Jacobian (arising from the choice of section of $\{\text{conf. str.}\}/Diff$)

$$= \int_{\mathcal{M}_\Sigma} d\gamma \int \mathcal{D}\varphi\; Z_{CFT}\left(\begin{matrix}D \text{ bosons}\\ + bc \text{ system}\end{matrix}, \gamma\right)\cdot e^{\frac{D-26}{24} S_{Liouville}(\varphi)}$$

$D-26$: central charge of CFT consisting of D bosons and one bc system

if $D = D_{crit} = 26$, then integrand is independent of φ.

A modification of bc system: $T = :\partial c\cdot b:$, then $h_b = +1$, $h_c = 0$, $c = -2$ "simple ghosts"

More generally: $T = :(1+j)\,\partial c\cdot b + j\,c\cdot\partial b:$, then $h_b = 1+j$, $h_c = -j$, $c = -12j^2 + 12j - 2$

$j\in\mathbb{R}$ - parameter

Verma modules over Virasoro algebra

- $V_{c,h} = \mathrm{Span}_{\mathbb{C}} \{ L_{-n_r} \cdots L_{-n_2} L_{-n_1} |h\rangle \}_{\substack{r\geq 0 \\ 1\leq n_1 \leq n_2 \leq \cdots \leq n_r}}$

  $|h\rangle$ - highest vector,

  $L_0 |h\rangle = h|h\rangle$, $L_{>0} |h\rangle = 0$

N	basis of $V^N_{c,h}$
0	$\|h\rangle$
1	$L_{-1}\|h\rangle$
2	$L_{-1}^2 \|h\rangle, L_{-2}\|h\rangle$
3	$L_{-1}^3 \|h\rangle, L_{-2}L_{-1}\|h\rangle, L_{-3}\|h\rangle$
4	$L_{-1}^4\|h\rangle, L_{-2}L_{-1}^2\|h\rangle, L_{-3}L_{-1}\|h\rangle,$ $L_{-2}^2\|h\rangle, L_{-4}\|h\rangle$

$V_{c,h} = \bigoplus_{N\geq 0} \underset{\uparrow}{V^N_{c,h}}$, $\quad V^N_{c,h} = \mathrm{Span}_{\mathbb{C}} \{ L_{-n_r} \cdots L_{-n_1} |h\rangle \}_{\substack{r\geq 0 \\ 1\leq n_1 \leq \cdots \leq n_r \\ n_1 + \cdots + n_r = N}}$

"level N states" $\quad V^N_{c,h}$ is an eigen-space of L_0 with eigen-value $N+h$

- Character of a Verma module:

$$\chi_{V_{c,h}}(q) := \mathrm{tr}_{V_{c,h}} q^{L_0} = \sum_{N\geq 0} \underbrace{\dim(V^N_{c,h})}_{P(N) \text{ - no of partitions}} q^{N+h} = \frac{q^h}{\prod_{n\geq 1} (1-q^n)}$$

- Inner product on $V_{c,h}$

  is defined by $\langle h|h\rangle = 1$, $(L_n)^\dagger = L_{-n}$ $\forall n \in \mathbb{Z}$

  - is not necessarily positive-definite!

  e.g. $(L_{-n}|h\rangle, L_{-m}|h\rangle) = \langle h| L_n L_{-m} |h\rangle = \begin{cases} 0, \text{ unless } m\geq 0, n=m \\ h^2 \text{ if } n=m=0 \\ \langle h| \underbrace{[L_m, L_{-m}]}_{2mL_0 + \frac{c}{12}(m^3-m)} |h\rangle = 2mh + \frac{c}{12}(m^3-m) \text{ if } m>0, n=m \end{cases}$

  Rem This implies that for $c<0$
  $\| L_{-n}|h\rangle \|^2 < 0$ for n sufficiently large

  Can compute $(L_{-n_r} \cdots L_{-n_1}|h\rangle, L_{-m_s} \cdots L_{-m_1}|h\rangle) = \langle h| L_{n_1} L_{n_2} \cdots L_{n_r} L_{-m_s} \cdots L_{-m_1} |h\rangle$ using comm. relations of Virasoro algebra

- Null-vectors (or "singular vectors")

  Def $|\chi\rangle \in V_{c,h}$, $|\chi\rangle \neq |h\rangle$ is a null-vector if $L_n |\chi\rangle = 0$ $\forall n > 0$

  - null-vector $|\chi\rangle$ is orthogonal to whole $V_{c,h}$:

    $\langle h| L_{n_1} \cdots L_{n_r} |\chi\rangle = 0 \quad (*)$

    in particular $\langle \chi | \chi \rangle = 0$

  - $|\chi\rangle$ generates its own Verma module $V_\chi \simeq V_{c,h+N}$ (level of χ: N) embedded in $V_{c,h}$ as sub module

    $V_\chi = \mathrm{Span}\{ L_{-n_r} \cdots L_{-n_1} |\chi\rangle \}$

  - V_χ is orthogonal to whole $V_{c,h}$:

  [potentially non-zero are: $\langle h| L_{n_1} \cdots L_{n_r} L_{-m_s} \cdots L_{-m_1} |\chi\rangle$ with $n_1 + \cdots + n_r = N + m_1 + \cdots + m_s$,
  But this can be reduced to inner products $(*)$ by using Virasoro comm. relations]

- Null-vectors at levels 1, 2:

① $L_{-1}|h\rangle$ is null iff $\underbrace{L_1 L_{-1}}_{2L_0 + L_{-1}L_1} |h\rangle (= 2h|h\rangle) = 0 \iff h = 0$

② $\underbrace{(\alpha L_{-2} + \beta L_{-1}^2)|h\rangle}_{|\chi\rangle}$ is null iff $\begin{cases} 0 = L_1 |\chi\rangle = (3\alpha + (4h+2)\beta) L_{-1}|h\rangle \\ 0 = L_2 |\chi\rangle = ((4h + \frac{c}{2})\alpha + 6h\beta)|h\rangle \end{cases}$

- is possible for some α, β iff $\begin{vmatrix} 3 & 4h+2 \\ 4h+\frac{c}{2} & 6h \end{vmatrix} = 0$ (Exercise!)

- Gram matrix

$M^{(N)} = \left(\langle i | j \rangle\right)_{\substack{i,j \text{ run over basis} \\ \text{in } V_{c,h}^N}}$ - $P(N) \times P(N)$ matrix

E.g. $M^{(0)} = (1)$

$M^{(1)} = (2h)$

$$M^{(2)} = \begin{pmatrix} 4h(2h+1) & 6h \\ 6h & 4h + \frac{c}{2} \end{pmatrix}$$

$\ldots\ldots$

- <u>Kac determinant formula</u>

$$\det M^{(N)} = \alpha_N \prod_{\substack{p,q \geq 1 \\ pq \leq N}} \left(h - h_{p,q}(c)\right)^{P(N-pq)}$$

Here $\alpha_N = \prod_{\substack{p,q\geq 1 \\ pq \leq N}} \left((2p)^q q!\right)^{P(N-pq) - P(N-p(q+1))}$ - numerical factor (indep. of c, h)

$$\boxed{h_{p,q} = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}}$$ where $m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}}$ $\Updownarrow$ $c = 1 - \frac{6}{m(m+1)}$

for $c=1$: $h_{p,q} = \frac{(p-q)^2}{4}$

<u>Rem</u> $h_{1,1} = 0 \ \forall c$

- $h = h_{p,q}$ for some $p,q \geq 1$ implies that $V_{c,h}$ contains a null-vector at level pq
[Ref: Feigin, Fuchs "Verma modules over Virasoro algebra" '84]

<u>Assertion</u> 1) $V_{c,h}$ is irreducible iff it contains no null-vectors

2) Null-vector $|\chi\rangle$ at level N in $V_{c,h}$ generates a submodule isomorphic to $V_{c,h+N}$

3) Quotient of $V_{c,h}$ over $\underbrace{\text{maximal proper submodule}}_{M_{c,h}}$ is irreducible

4) <u>Every highest weight</u> irr. module is $M_{c,h}$ for some c, h

<u>Thm (Kac '79)</u> 1) $V_{c,h}$ is reducible iff $h = h_{p,q}$ for some $p,q \geq 1$

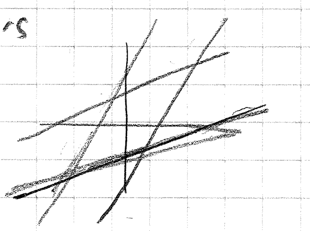
2) $V_{c,h}^N$ contains a null-vector iff $N = pq$ for some $p,q \geq 1$ such that $h = h_{p,q}$

<u>Example:</u> $V_{1,h}$ is reducible iff $h = n^2/4$, $n \in \mathbb{Z}$

$V_{0,h}$ is reducible iff $h = \frac{n^2-1}{24}$, $n \in \mathbb{Z}$

<u>Thm</u> Any submodule of $V_{c,h}$ is generated by null-vectors

- Equation $h_{p,q}(c) = h$ defines a diamond of 4 lines on (p,q)-plane



$c < 1$ lines are real and have positive slope

$c > 25$ — " — negative —

$1 < c < 25$ lines are complex

$c = 1$: 2 lines parallel to $p = q$

$c = 25$ — " — $p = -q$

Denote $\ell_{c,h}$ one of the lines

<u>Cases</u> (I) $\ell_{c,h}$ has no integer points

(II) $\ell_{c,h}$ has single integer point $(p,q) = (a', a'') \in \mathbb{Z}^2$

$\mathrm{II}_+)$ $a'a'' > 0$ $\quad$ $\mathrm{II}_0)$ $a'a'' = 0$ $\quad$ $\mathrm{II}_-)$ $a'a'' < 0$

(III) $\ell_{c,h}$ has infinitely many integer points ($\Leftrightarrow m \in \mathbb{Q}$, either $c \leq 1 \to$ subcase III_- or $c \geq 25 \to \mathrm{III}_+$)

$\mathrm{III}^{0,0}_{\pm})$ $\ell_{c,h}$ intersects $q=0$ and $p=0$ axes at integer points

denote P the middle point of interval connecting these two points, enumerate integer points of upper half of $\ell_{c,h}$

as $\underbrace{\ldots (a'_{-1}, a''_{-1})}_{\substack{\mathrm{III}_-:\ \text{finite} \\ \mathrm{III}_+:\ \text{infinite}}}, (a'_0, a''_0), \underbrace{(a'_1, a''_1), \ldots}_{\substack{\text{infinite} \\ \text{finite}}}$ $(*)$ s.t. $\ldots < a'_{-1}a''_{-1} < a'_0a''_0 = 0 < a'_1a''_1 < \ldots$ $(**)$

$\mathrm{III}^{0}_{\pm})$ $\ell_{c,h}$ intersects one of the coord axes at integer point
enumerate $\mathbb{Z}^2$-points of $\ell_{c,h}$ (not just half) as $(*)$ s.t. $(**)$

$\mathrm{III}_{\pm})$ $\ell_{c,h}$ intersects both coord. axes at non-integer points
enumerate $\mathbb{Z}^2$-points of $\ell_{c,h}$ as $(*)$ s.t. $\ldots < a'_{-1}a''_{-1} < 0 < a'_0a''_0 < a'_1a''_1 < \ldots$

• draw a line $\ell'_{c,h}$ $\parallel$ to $\ell_{c,h}$ through $(-a'_0, a''_0)$ and enum. its $\mathbb{Z}^2$-points as
$\ldots, (b'_{-1}, b''_{-1}), (b'_0, b''_0) = (-a'_0, a''_0), (b'_1, b''_1), \ldots$

Then

$\ldots < a'_{-2}a''_{-2} < a'_{-1}a''_{-1} < b'_{-1}b''_{-1} + a'_0a''_0 < b'_0b''_0 + a'_0a''_0 = 0 < a'_0a''_0 < a'_1a''_1 < b'_1b''_1 + a'_0a''_0 < b'_2b''_2 + a'_0a''_0 < a'_2a''_2 < \ldots$

<u>Thm</u> (Feigin-Fuchs) Fix c, h (and a line $\ell_{c,h}$)

Case I, II_0 : $V_{c,h}$ is irreducible and is not a proper submodule of any Verma module

II_+ : $V_{c,h}$ has a single submodule $\simeq V_{c,h+a'a''}$ (generated by $|\chi\rangle \in V_{c,h}^{a'a''}$)

V_χ and $M_{c,h} = V_{c,h}/V_\chi$ are irreducible ; $V_{c,h}$ is not embedded into any $V_{c,h'}$

II_- : $V_{c,h}$ is irred. , but is embedded into $V_{c,h+a'a''}$ [and is generated there by null-vector at level $N = -a'a''$]

$V_{c,h}$ is not embedded into any other $V_{c,h'}$

$\mathrm{III}^{0,0}_{\pm}$, $\mathrm{III}^{0}_{\pm}$: there is a sequence of embeddings

$$\ldots \to V_{c,h+a'_1a''_1} \to V_{c,h} \to V_{c,h+a'_{-1}a''_{-1}} \to \ldots$$

modules in this sequence are not related by any non-trivial homomorphisms with any $V_{c,h'}$ not from this sequence

$\mathrm{III}_{\pm}$ there is a comm. diagram of embeddings

$$\begin{tikzcd}
\vdots & \vdots \\
V_{c,h+a'_{-1}a''_{-1}} \arrow[u] \arrow[ur] & V_{c,h+a'_{-2}a''_{-2}} \arrow[u] \arrow[ul] \\
V_{c,h} \arrow[u] \arrow[ur] & V_{c,h+b'_{-1}b''_{-1}+a'_0a''_0} \arrow[u] \arrow[ul] \\
V_{c,h+a'_1a''_1} \arrow[u] \arrow[ur] & V_{c,h+a'_0a''_0} \arrow[u] \arrow[ul] \\
V_{c,h+b'_2b''_2+a'_0a''_0} \arrow[u] \arrow[ur] & V_{c,h+b'_1b''_1+a'_0a''_0} \arrow[u] \arrow[ul]
\end{tikzcd}$$

Modules in this diagram are not connected by non-trivial homs with any other $V_{c,h'}$. In every piece like

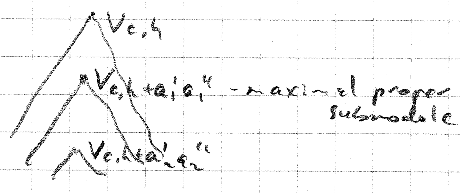
$$\begin{tikzcd} A & \\ B \arrow[u] & C \arrow[ul] \\ D \arrow[u] \arrow[ur] & E \arrow[u] \arrow[ul] \end{tikzcd}$$

im B and im C in A do not contain each other

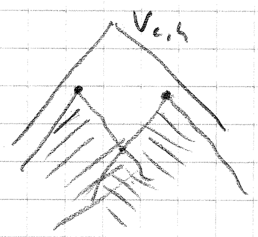
im D and im E generate im $B \cap$ im C in A

Rem ① in cases $\text{III}_-^{0,0}$, III_-^{0}, III_-, $V_{c,h}$ has infinitely many null-vectors, but can be embedded into finitely many $V_{c,h'}$

$\text{III}_-^{0,0}$, III_-^{0} :



III_- :



$\text{III}_+^{0,0}$, III_+^{0}, III_+ - finitely many null-vectors

② Consider a category $\mathcal{V}$ with objects $V_{c,h}$ and morphisms: classes of proportional $\underset{\text{non-zero}}{\text{homs}}$

Thm ⇒ there may be no more than 1 hom between 2 objects

- correspondence $V_{c,h} \longrightarrow V_{26-c,\,1-h}$ extends to iso of categories $\mathcal{V} \sim \mathcal{V}^{op}$

Characters of irreducible modules $M_{c,h}$

Cases II, $\text{III}_\pm^{0,0}$, $\text{III}_\pm^{0}$: $M_{c,h} \simeq V_{c,h}/V_{c,h+a'a''}$

$$\chi_{M_{c,h}}(q) = \chi_{V_{c,h}}(q) - \chi_{V_{c,h+a'a''}}(q) = \frac{q^h(1-q^{a'a''})}{\prod_{n\geq 1}(1-q^n)}$$

III_- : we have an exact sequence

$$0 \leftarrow M_{c,h} \leftarrow V_{c,h} \leftarrow V_{c,h+a_0'a_0''} \oplus V_{c,h+a_1'a_1''} \longleftarrow V_{c,h+b_1'b_1''+a_0'a_0''} \oplus V_{c,h+b_2'b_2''+a_0'a_0''} \leftarrow \cdots$$

$$\Rightarrow \quad \chi_{M_{c,h}} = \frac{1 - \sum_{n=1}^{\infty} q^{a_n'a_n''} + q^{a_0'a_0''} \sum_{n=1}^{\infty} q^{b_n'b_n''}}{\prod_{n\geq 1}(1-q^n)} \cdot q^h$$

III_+ : same, but the sum in numerator becomes finite.