Modules in this diagram are not connected by non-trivial homs with any other $V_{c,h'}$. In every piece like

$$\begin{tikzcd} A & \\ B \arrow[u] & C \arrow[ul] \\ D \arrow[u] \arrow[ur] & E \arrow[u] \arrow[ul] \end{tikzcd}$$

$\operatorname{im} B$ and $\operatorname{im} C$ in A do not contain each other

$\operatorname{im} D$ and $\operatorname{im} E$ generate $\operatorname{im} B \cap \operatorname{im} C$ in A

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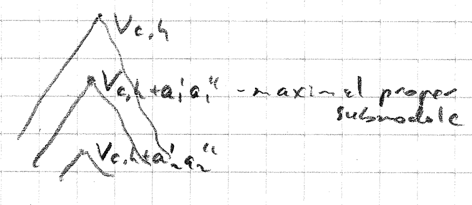
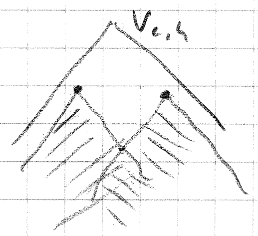
Rem ① in cases $\overline{\underline{III}}_-^{0,0}$, $\overline{\underline{III}}_-^{0}$, $\overline{\underline{III}}_-$, $V_{c,h}$ has infinitely many null-vectors, but can be embedded into finitely many $V_{c,h'}$

$\overline{\underline{III}}_-^{0,0}$, $\overline{\underline{III}}_-^{0}$:



$\overline{\underline{III}}_-$:



$\overline{\underline{III}}_+^{0,0}$, $\overline{\underline{III}}_+^{0}$, $\overline{\underline{III}}_+$ - finitely many null-vectors

② Consider a category $\mathcal{V}$ with objects $V_{c,h}$ and morphisms: classes of proportional $\underset{\text{non-zero}}{\text{homs}}$

Thm $\Rightarrow$ there may be no more than 1 hom between 2 objects

- correspondence $V_{c,h} \longrightarrow V_{26-c,\,1-h}$ extends to iso of categories $\mathcal{V} \sim \mathcal{V}^{op}$

Characters of irreducible modules $M_{c,h}$

Cases $\overline{\underline{II}}$, $\overline{\underline{III}}_{\pm}^{0,0}$, $\overline{\underline{III}}_{\pm}^{0}$: $M_{c,h} \simeq V_{c,h}/V_{c,h+a'a''}$

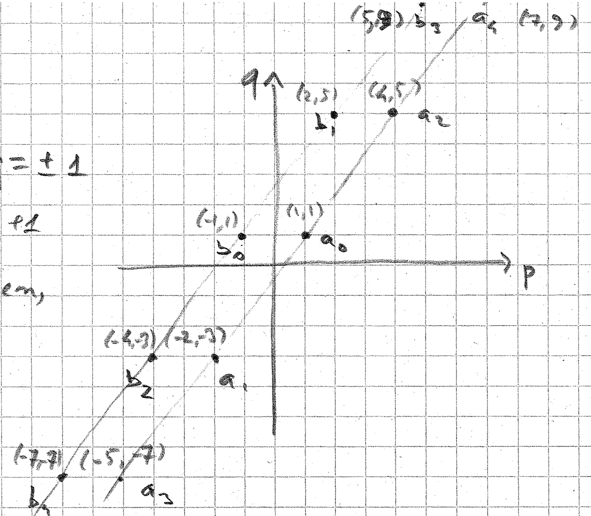
$$\chi_{M_{c,h}}(q) = \chi_{V_{c,h}}(q) - \chi_{V_{c,h+a'a''}}(q) = \frac{q^h(1-q^{a'a''})}{\prod_{n\geq 1}(1-q^n)}$$

$\overline{\underline{III}}_-$: we have an exact sequence

$$0 \leftarrow M_{c,h} \leftarrow V_{c,h} \leftarrow V_{c,h+a_0'a_0''} \oplus V_{c,h+a_1'a_1''} \longleftarrow V_{c,h+b_1'b_1''+a_0'a_0''} \oplus V_{c,h+b_2'b_2''+a_0'a_0''} \longleftarrow \cdots$$

$$\Rightarrow \chi_{M_{c,h}} = \frac{1 - \sum_{n=1}^{\infty} q^{a_n'a_n''} + q^{a_0'a_0''} \sum_{n=1}^{\infty} q^{b_n'b_n''}}{\prod_{n\geq 1}(1-q^n)} \cdot q^h$$

$\overline{\underline{III}}_+$: same, but the sum in numerator becomes finite.



Examples

- $c=\frac{1}{2}$, $h=0$ $\qquad h=h_{p,q} \iff 4p-3q=\pm 1$
  $m=3$ $\qquad$ take the line $4p-3q=\pm 1$
  - this is case $\underline{\text{III}}_-$ of F-F theorem,

$$\begin{tikzcd}[column sep=small]
 & V_{\frac{1}{2},0} & & \arrow[l] h_{1,1} \\
h_{4,3} \arrow[r] & V_{\frac{1}{2},1\cdot 1} \arrow[u] & V_{\frac{1}{2},(-2)(-3)} \arrow[ul] & \arrow[l] h_{5,2} \\
h_{8,3} \arrow[r] & V_{\frac{1}{2},1\cdot 1+2\cdot 5} \arrow[u] \arrow[ur] & V_{\frac{1}{2},1\cdot 1+(-4)(-3)} \arrow[u] \arrow[ul] & \arrow[l] h_{7,1} \\
 & \vdots & & \\
h_{4+6n,3} \arrow[r] & V_{\frac{1}{2},\underbrace{(1+3n)(1+4n)}_{a'_{2n}a''_{2n}}} & V_{\frac{1}{2},\underbrace{(-2-3n)(-3-4n)}_{a'_{2n+1}a''_{2n+1}}} & \arrow[l] h_{5+6n,2} \\
h_{8+6n,3} \arrow[r] & V_{\frac{1}{2},\underbrace{1+(2+3n)(5+4n)}_{a'_0a''_0+b'_{2n+1}b''_{2n+1}}} \arrow[u] \arrow[ur] & V_{\frac{1}{2},\underbrace{1+(-4-3n)(-3-4n)}_{a'_0a''_0+b'_{2n+2}b''_{2n+2}}} \arrow[u] \arrow[ul] & \arrow[l] h_{7+6n,1} \\
 & \vdots & &
\end{tikzcd}$$

Rem symmetries of $h_{p,q}$:

- $h_{p,q} = h_{-p,-q}$
- for $c=\frac{1}{2}$, $h_{p,q} = h_{4+p,3+q}$
- $h_{p,-q} = h_{p,q} + p\cdot q$

character of $M_{\frac{1}{2},0}$: $\quad 0 \leftarrow M_{\frac{1}{2},0} \longleftarrow V_{\frac{1}{2},0} \leftarrow V_{\frac{1}{2},1} \oplus V_{\frac{1}{2},6} \longleftarrow V_{\frac{1}{2},11} \oplus V_{\frac{1}{2},13} \longleftarrow \cdots$

$$\Rightarrow \quad \chi_{M_{\frac{1}{2},0}}(q) = \frac{1 - \sum_{n\in\mathbb{Z}} q^{(1+3n)(1+4n)} + q\sum_{n\neq 0} q^{(-1+3n)(1+4n)}}{\prod_{n=1}^{\infty}(1-q^n)} \qquad (*)$$

Rem From free fermion theory, we know another construction for $M_{\frac{1}{2},0}$

$$M_{\frac{1}{2},0} \simeq \left(\mathrm{Span}\left\{ \dots b_{-\frac{5}{2}}^{k_{5/2}} b_{-\frac{3}{2}}^{k_{3/2}} b_{-\frac{1}{2}}^{k_{\frac{1}{2}}} |vac\rangle \right\}_{k_{1/2},k_{3/2},\dots \in\{0,1\}} \right)^{\text{even}}$$

with character $\quad \chi^{\text{chiral fermion}}_{\frac{1}{2},0} = \frac{1}{2}\left(\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}) + \prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}})\right) \qquad (**)$

So, we have $(*) = (**)$ - example of a "Dyson-Macdonald identity"

- $c=\frac{1}{2}$, $h=\frac{1}{2}$, $h=\frac{1}{16}$ - similar diagrams of modules

$c=1/2$, $h=1/2$

$$\begin{tikzcd}[column sep=small]
 & V_{1/2,1/2} & \\
V_{1/2,1/2+2\cdot 1} \arrow[ur] & & V_{1/2,1/2+(-2)(-3)} \arrow[ul] \\
V_{\frac{1}{2},\frac{1}{2}+2\cdot 1+1\cdot 5} \arrow[u] \arrow[urr] & & V_{\frac{1}{2},1/2+2\cdot 1+(-5)(-3)} \arrow[u] \arrow[ull] \\
 & \vdots & \\
V_{1/2,1/2+(2+3n)(1+4n)} & & V_{1/2,1/2+(-2-3n)(-3-4n)} \\
V_{1/2,1/2+2\cdot 1+(1+3n)(5+4n)} \arrow[u] \arrow[urr] & & V_{1/2,1/2+2\cdot 1+(-5-3n)(-3-4n)} \arrow[u] \arrow[ull] \\
 & \vdots &
\end{tikzcd}$$

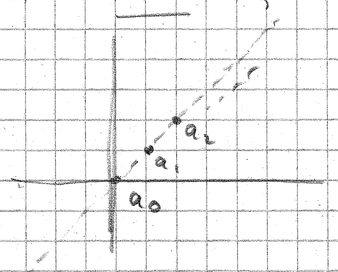
$c=1/2$, $h=1/16$

$$\begin{tikzcd}[column sep=small]
 & V_{1/2,1/16} & \\
V_{\frac{1}{2},\frac{1}{16}+1\cdot 2} \arrow[ur] & & V_{\frac{1}{2},\frac{1}{16}+(-2)(-2)} \arrow[ul] \\
V_{\frac{1}{2},\frac{1}{16}+1\cdot 2+(-4)(-2)} \arrow[u] \arrow[urr] & & V_{\frac{1}{2},\frac{1}{16}+1\cdot 2+2\cdot 6} \arrow[u] \arrow[ull] \\
 & \vdots & \\
V_{\frac{1}{2},\frac{1}{16}+(1+3n)(2+4n)} & & V_{\frac{1}{2},\frac{1}{16}+(-2-3n)(-2-4n)} \\
V_{\frac{1}{2},\frac{1}{16}+1\cdot 2+(-4-3n)(-2-4n)} \arrow[u] \arrow[urr] & & V_{\frac{1}{2},\frac{1}{16}+1\cdot 2+(2+3n)(6+4n)} \arrow[u] \arrow[ull]
\end{tikzcd}$$

[+ similar Dyson-Macdonald identities]

- $\underline{c=1}$, $h=0$, $h_{p,q}=\frac{(p-q)^2}{4}$; $h_{p,q}=h \Leftrightarrow p=q$

  - case $\mathrm{III}^{0,0}_{-}$ of F-F

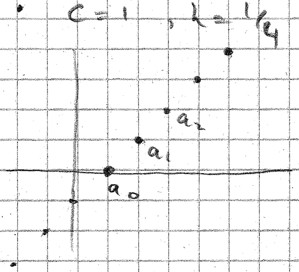


$$V_{1,0} \hookleftarrow V_{1,1\cdot 1} \hookleftarrow V_{1,2\cdot 2} \hookleftarrow \cdots$$

character: $\chi_{M_{1,0}} = \frac{1-q}{\prod_{n=1}^{\infty}(1-q^n)}$

- $c=1$, $h=1/4$

  - $\mathrm{III}^{0,0}_{-}$



$$V_{1,\frac{1}{4}} \hookleftarrow V_{1,\underbrace{\frac{1}{4}+2\cdot 1}_{\frac{3^2}{4}}} \hookleftarrow V_{1,\underbrace{\frac{1}{4}+3\cdot 2}_{\frac{5^2}{4}}} \hookleftarrow \cdots$$

character: $\chi_{M_{1,\frac{1}{4}}} = \frac{q^{1/4}(1-q^2)}{\prod_{n=1}^{\infty}(1-q^n)}$

- Generally, for $c=1$,

  if $h \neq \frac{N^2}{4}$ for some $N\in\mathbb{Z}$ then $V_{1,h}$ is irreducible and has no non-trivial maps to/from other Verma modules and $\chi_{M_{1,h}} = \frac{q^h}{\prod_{n\geq 1}(1-q^n)}$ $(*)$

  if $h=\frac{N^2}{4}$ then $V_{1,h}$ fits into one of two sequences:

$$V_{1,0} \leftarrow V_{1,1} \leftarrow V_{1,2^2} \leftarrow \cdots$$
$$V_{1,\left(\frac{1}{2}\right)^2} \leftarrow V_{1,\left(\frac{3}{2}\right)^2} \leftarrow V_{1,\left(\frac{5}{2}\right)^2} \leftarrow \cdots$$

and $\chi_{M_{1,h}} = \frac{q^{\frac{N^2}{4}} - q^{\frac{(N+2)^2}{4}}}{\prod_{n\geq 1}(1-q^n)}$ $(**)$

- These character f-las allow us to understand $\mathcal{H}^{boson}$ in terms of $M_{1,h}$:

$\mathcal{H}_{boson} = \bigoplus_{\alpha\in\mathbb{R}} V_\alpha^{Heis} \otimes \overline{V_\alpha^{Heis}}$, want to decompose V_α^{Heis} into $M_{1,h}$

character: $\mathrm{tr}_{V_\alpha^{Heis}} q^{L_0} = \frac{q^{\alpha^2/2}}{\prod_{n\geq 1}(1-q^n)}$

- if $\alpha \notin \frac{1}{\sqrt{2}}\mathbb{Z}$, then $V_\alpha^{Heis} \cong M_{1,\frac{\alpha^2}{2}}$
- if $\alpha\in\frac{1}{\sqrt{2}}\mathbb{Z}$, $|\alpha| = N/\sqrt{2}$, then

$$\chi_{V_\alpha^{Heis}} = \frac{q^{N^2/4}}{\prod_{n\geq 1}(1-q^n)} = \underbrace{\frac{q^{\frac{N^2}{4}} - q^{\frac{(N+2)^2}{4}}}{\prod_{n\geq 1}(1-q^n)}}_{\chi_{M_{1,\frac{\alpha^2}{2}}}} + \underbrace{\frac{q^{\frac{(N+2)^2}{4}} - q^{\frac{(N+4)^2}{4}}}{\prod_{n\geq 1}(1-q^n)}}_{\chi_{M_{1,\frac{(\alpha+\sqrt{2})^2}{2}}}} + \cdots$$

So, $V_\alpha^{Heis} \cong M_{1,\frac{N^2}{4}} \oplus M_{1,\frac{(N+2)^2}{4}} \oplus M_{1,\frac{(N+4)^2}{4}} \oplus \cdots$

E.g. V_0^{Heis} contains infinitely many Virasoro primary states.

"Category of Verma modules"

- $c=1$ for $h \neq \frac{N^2}{4}$, objects are not mapped to/from anything

for $h=\frac{N^2}{4}$, objects organize into two sequences

$$V_{1,0} \leftarrow V_{1,1} \leftarrow V_{1,4} \leftarrow V_{1,9} \leftarrow \cdots \quad , \quad V_{1,\frac{1}{4}} \leftarrow V_{1,\frac{9}{4}} \leftarrow V_{1,\frac{25}{4}} \leftarrow \cdots$$

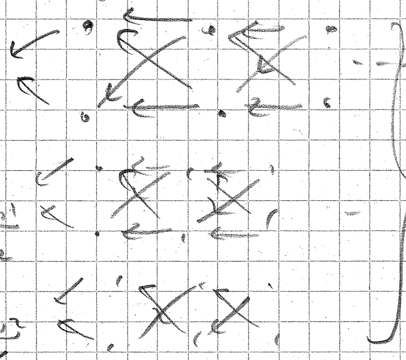
- $c=\frac{1}{2}$ for $h \neq \frac{N^2-1}{48}$, objects are isolated (i.e. $h \neq h_{p,q}$)

for $h=\frac{N^2-1}{48}$, objects organize into one of the following pieces (depending on N mod 24)

(1,1) $V_{\frac{1}{2},\, \underbrace{h_{1,1}}_{0}}$

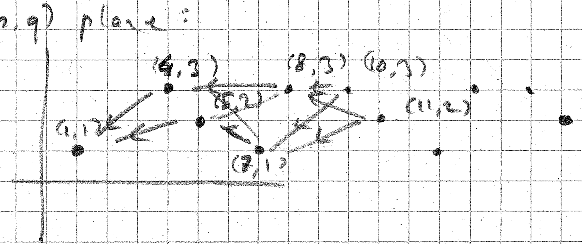
(2,1) $V_{\frac{1}{2},\, \underbrace{h_{2,1}}_{1/2}}$

(1,2) $V_{\frac{1}{2},\, \underbrace{h_{1,2}}_{1/16}}$



$\Big\}$ III_- cases

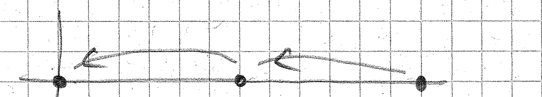
On (p,q) plane:



(0,0) $V_{\frac{1}{2},\, \underbrace{h_{0,0}}_{-1/48}} \leftarrow \cdot \leftarrow \cdot \leftarrow \cdot$

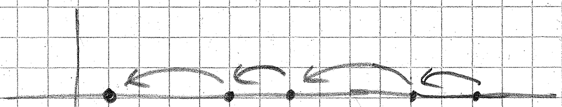
(3,0) $V_{\frac{1}{2},\, \underbrace{h_{3,0}}_{143/48}} \leftarrow \cdot \leftarrow \cdot \leftarrow$

$\Big\}$ $\text{III}^{0,0}_-$ cases

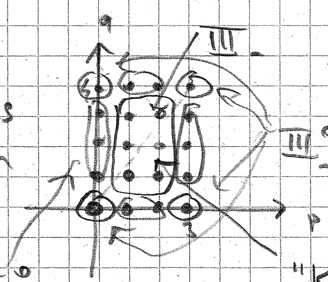


(1,0)
(2,0)
(0,1)
(0,2)
(0,3)

$V_{\frac{1}{2}, h_{p,q}} \leftarrow \cdot \leftarrow \cdot \leftarrow$ $\Big\}$ III^{0}_- cases



- Maximal reducible modules are assoc. to (p,q) in



"Kac table"

modulo $(p,q) \sim (3-p, 4-q)$

- Category of Verma modules is organized similarly for $c = 1 - \frac{6}{m(m+1)}$ with $m \in \mathbb{Q}$,

if $\frac{m+1}{m} = \frac{Q}{P}$, $Q \,\&\, P$ - coprime, $\in \mathbb{Z}$ then max. red. modules are $V_{c, h_{p,q}}$

for $0 \leq p \leq P$, $0 \leq q \leq Q$ [strict inequalities - III cases]

$\rightsquigarrow$ Space of states of minimal model $\mathcal{M}(P,Q)$:

$$\mathcal{H}_{\mathcal{M}(P,Q)} = \bigoplus_{\substack{0<p<P \\ 0<q<Q \\ \frac{p}{q} > \frac{P}{Q}}} M_{c, h_{p,q}}$$

- In particular $\mathcal{M}(m, m+1)$ for $m = 3, 4, \ldots$ are unitary: $\mathcal{H}$ has positive definite inner product

- if $m \notin \mathbb{Q}$ then cat. of Verma modules is a collection of isolated objects for $h \neq h_{p,q}$ and diagrams $\underset{\bullet}{V_{c,h_{p,q}}} \longleftarrow \underset{\bullet}{V_{c,h_{p,-q}}}$, $h_{p,-q} = h_{p,q} + pq$ for $p, q > 0$