14.12.11

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OPE between primary fields

generally:

$$\Phi_1(z)\Phi_2(w) = \sum_{p\in I}\ \sum_{\substack{1\le k_1\le k_2\le\dots\le k_r\\ r\ge 0\\ 1\le \bar k_1\le\dots\le \bar k_s\\ s\ge 0}} C_{12}^{p,\{k,\bar k\}}\ \underbrace{(z-w)^{-h_1-h_2+h_p+\overset{\sum k_j}{\overset{\parallel}{\boxed{K}}}}\ (\bar z-\bar w)^{-\bar h_1-\bar h_2+\bar h_p+\overset{\sum \bar k_j}{\overset{\parallel}{\boxed{\bar K}}}}}_{\text{from scaling } (\overset{w}{L_0}) \text{ symmetry}}\ \Phi_p^{\{-k,-\bar k\}}(w)$$

I ↑ set of primary fields

↑ sum over descendants

$\Phi_p^{\{-k,-\bar k\}}$ ↑ descendant of Φ_p at w

$$C_{12}^{p,\{k,\bar k\}} = C_{12p}\,\beta_{12}^{p\{k\}}\,\beta_{12}^{p\{\bar k\}}$$

↑ ↗ universal rational functions of c, h_1, h_2, h_p, parameterized by partitions $\{k\}$

$\beta_{12}^{p,\emptyset} = 1$ - normalization

Rem: $\Phi_p^{(-k,-\bar k)}$ can only appear in $\Phi_1\Phi_2$ OPE if Φ_p appears.

4-point correlator of primary fields

$$\langle\Phi_1\Phi_2\Phi_3\Phi_4\rangle = \cdots\cdot f(x,\bar x)$$

$(x,\bar x)$ ← cross-ratio of z_1,z_2,z_3,z_4

$$f(x,\bar x) = \langle\Phi_1|\Phi_2(1)\,\underbrace{\Phi_3(x,\bar x)\,|\Phi_4\rangle}$$

$$\sum_p\sum_{\{k,\bar k\}} C_{34p}\,\beta_{34}^{p\{k\}}\,\beta_{34}^{p\{\bar k\}}\,x^{-h_3-h_4+h_p+K}\,\bar x^{-\bar h_3-\bar h_4+\bar h_p+\bar K}\,|\Phi_p;\{-k,-\bar k\}\rangle$$

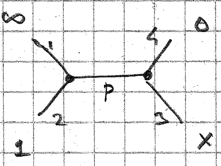
$|\Phi_p;\{-k,-\bar k\}\rangle$ ↑ descendant state of $|\Phi_p\rangle$

$$\rightsquigarrow\ f(x,\bar x) = \sum_{p\in I} C_{12p}C_{34p}\,\mathcal{F}_{34}^{21}(p|x)\,\bar{\mathcal{F}}_{34}^{21}(p|\bar x)$$

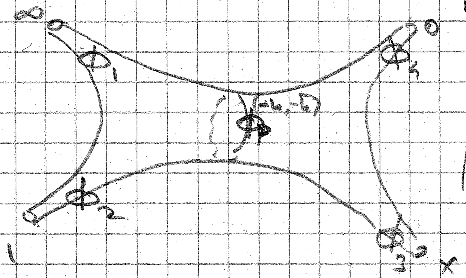
where $\mathcal{F}_{34}^{21}(p|x) = x^{-h_3-h_4+h_p}\sum_{\{k\}} x^K\,\beta_{34}^{p\{k\}}\,\gamma_{12}^{p\{k\}}$ "conformal block for 4-pt function"

where γ's come from $\dfrac{\langle\Phi_1|\Phi_2(1)|\Phi_p;\{-k,-\bar k\}\rangle}{\langle\Phi_1|\Phi_2(1)|\Phi_p\rangle} = \gamma_{12}^{p,\{k\}}\,\gamma_{12}^{p,\{\bar k\}}$

Pictures: $f(x,\bar x) = \sum_p$



or

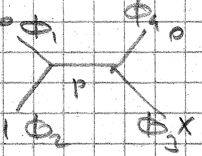


γ_{12}^{p} related to β_{12}^{p} via

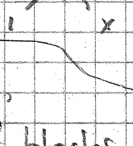
$$\gamma_{12}^{p,\{k\}} = \sum_{\{\ell\}} G^{p}_{\{k\},\{\ell\}}\,\beta_{12}^{p,\{\ell\}}$$

↑ Gram matrix

Crossing symmetry: $f(x,\bar x)$ $\sum_p$



$\sum_p$



i.e.

$$\boxed{\sum_p C_{12p}C_{34p}\,\mathcal{F}_{34}^{21}(p|x)\,\mathcal{F}_{34}^{21}(p,\bar x) = \sum_p C_{23p}C_{14p}\,\mathcal{F}_{32}^{14}(p|1-x)\,\mathcal{F}_{32}^{14}(p|1-\bar x)}$$

- a set of quadratic equations on C_{ijk}

More generally, for $\langle\Phi_1\cdots\Phi_n\rangle$, one constructs a basis of conf. blocks for any 3-valent tree with leaves labelled by $\Phi_1\cdots\Phi_n$

basis elements are assoc. to decorations of edges by primary fields.

transitions between bases for different trees ⟷ "crossing symmetry"

Null-vectors $\longrightarrow$ diff. equations on correlators

If $V_{c,h}$ contains a null-vector $|\chi\rangle$ at level N,

$$V_{c,h} \ni |\chi\rangle = \sum_{\substack{\{k\}\text{ - partitions}\\ \sum k_i = N}} a_{\{k\}} \hat{L}_{-k_s} \cdots \hat{L}_{-k_1} |h\rangle$$

$\downarrow \quad \downarrow$

$M_{c,h} \ni 0$

, then correlators containing $\Phi_{h,\bar{h}}$ (corresponding primary) satisfy PDEs of degree $\leq N$:

coeff's of the null-vector in standard basis in $V_{c,h}$

$$\underbrace{\left(\sum_{\{k\}} a_{\{k\}} L^{(z)}_{-k_s} \cdots L^{(z)}_{-k_1}\right)}_{\nu \in U(Vir^{(z)})} \Phi_{h,\bar{h}} = 0 \quad \rightsquigarrow \quad \boxed{0 = \langle \nu(\Phi_{h,\bar{h}})(z_1,\bar{z}_1) \cdot \Phi_2(z_2,\bar{z}_2) \cdots \Phi_n(z_n,\bar{z}_n) \rangle = \underbrace{\left(\sum_{\{k\}} a_{\{k\}} \mathcal{L}_{-k_s} \cdots \mathcal{L}_{-k_1}\right)}_{\text{a diff. operator in } z_1,\ldots,z_n} \langle \Phi_{h,\bar{h}}(z_1,\bar{z}_1) \Phi_2 \cdots \Phi_n \rangle}$$

any primary fields

where $\mathcal{L}_{-k} = \sum_{\ell=2}^{n} -\frac{1}{(z_\ell - z_1)^{k-1}} \frac{\partial}{\partial z_\ell} + \frac{k-1}{(z_\ell - z_1)^k} h_\ell$

Ex A null-vector at level 2 in $V_{c,h}$ occurs iff ($h = h_{1,2}(c)$ or $h_{2,1}(c)$)

$$\Updownarrow$$

$$h = \frac{5 - c \pm \sqrt{(1-c)(25-c)}}{16}$$

and has the form $|\chi\rangle = \left(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2\right)|h\rangle$

Corresponding diff. eq.:

$$\left(\mathcal{L}_{-2} - \frac{3}{2(2h+1)} \mathcal{L}_{-1}^2\right) \langle \varphi(z_1,\bar{z}_1) \cdot \Phi_2(z_2,\bar{z}_2) \cdots \Phi_n(z_n,\bar{z}_n) \rangle = 0$$

explicitly: $\left(\sum_{\ell=2}^{n} \left(\frac{1}{z - z_\ell} \frac{\partial}{\partial z_\ell} + \frac{h_\ell}{(z - z_\ell)^2} \right) - \frac{3}{2(2h+1)} \frac{\partial^2}{\partial z_1^2} \right) \langle \varphi \Phi_2 \cdots \Phi_n \rangle = 0$

$n=2 \rightsquigarrow$ a tautologically satisfied equation

$n=3 \rightarrow$ a quadratic relation between $h, h_2, h_3 \rightarrow$ selection rule for allowed conf. families in $\varphi \cdot \Phi_2$ OPE ("fusion rules")

$n=4 \rightsquigarrow$ hypergeometric ODE on $f(x,\bar{x})$ - non-triv. part of $\langle \varphi \Phi_2 \Phi_3 \Phi_4 \rangle$, depending on cross-ratio

Ex: 4-spin correlator in Ising model

$$G^{(4)} = \langle \sigma(z_1,\bar{z}_1) \cdots \sigma(z_4,\bar{z}_4) \rangle = \left(\frac{z_{13} z_{24}}{z_{12} z_{23} z_{34} z_{41}}\right)^{1/8} \overline{\left(\frac{z_{13} z_{24}}{z_{12} z_{23} z_{34} z_{41}}\right)^{1/8}} F(x,\bar{x})$$

$\left(\frac{1}{16},\frac{1}{16}\right)$-field, $c = 1/2 = \bar{c}$

$x = \frac{z_{12} z_{34}}{z_{13} z_{24}}$

from global conf. invariance

null-vector at level 2 for $V_{1/2,\, h_{1,2} = \frac{1}{16}} \Rightarrow$

$$\Rightarrow \left(x(1-x) \frac{\partial^2}{\partial x^2} + \left(\frac{1}{2} - x\right) \frac{\partial}{\partial x} + \frac{1}{16} \right) F(x,\bar{x}) = 0$$

solutions: $f_{1,2}(x) = \left(1 \pm \sqrt{1-x}\right)^{1/2}$ $\leftarrow$ conformal blocks for 4-spin function

$$\Rightarrow G^{(4)} = \left| \frac{z_{13} z_{24}}{z_{12} z_{23} z_{34} z_{41}} \right|^{1/4} \underbrace{\sum_{i,j=1}^{2} a_{ij} f_i(x) f_j(\bar{x})}_{a(|f_1(x)|^2 + |f_2(x)|^2)}$$

due to single-valuedness of $G^{(4)}$ when $\bar{x} = x^*$

coefficient $a = \frac{1}{2}$

- determined from OPE $\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2) \sim \frac{1}{z_{12}^{1/8} \bar{z}_{12}^{1/8}} + \ldots$ (from normalization of σ w.r.t. 2-pt function)

Fusion rules

if $h = h_{p,q}$, then there is an algebraic equation on conf. weights h', h'' which can appear in $\Phi \cdot \Phi' = \sum_{\Phi''} \sum_{\{k,\bar{k}\}} \Phi''^{\{k,\bar{k}\}}$ OPE

– from degree pq diff. equation on 3-pt. function

(notation: $\Phi \times \Phi' = \sum_{\Phi'' \in \underset{\uparrow}{I''} \subset I} \Phi''$)

allowed conf. families

all primaries

E.g. 1) if $h = h_{1,1} = 0$ then equation is $h' = h''$ (indeed, $\mathbb{1} \times \Phi' = \Phi'$)

2) if $h = h_{2,1}$ then (quadratic equation on h', h'') $\Leftrightarrow$ ($\alpha'' = \alpha' \pm \alpha_+$)

with substitution $h = \frac{\alpha^2 - 1}{4m(m+1)}$, $\alpha_+ = m+1$, $m = m(c)$:

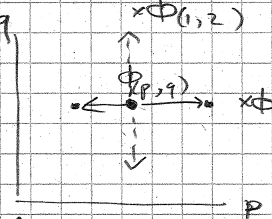
if $h = h_{1,2}$ - - - - - - - - - - $\alpha'' = \alpha' \pm \alpha_-$, $\alpha_- = m$

In minimal models:

denote $\Phi_{(p,q)}$ the primary field with $h = h_{p,q}$

$$\Phi_{(1,2)} \times \Phi_{(p,q)} = \Phi_{(p,q-1)} + \Phi_{(p,q+1)}$$

$$\Phi_{(2,1)} \times \Phi_{(p,q)} = \Phi_{(p-1,q)} + \Phi_{(p+1,q)}$$



- combining these two, we may arrive to more restrictive fusion rules:

E.g. $\left.\begin{array}{l} \Phi_{(1,2)} \times \Phi_{(2,1)} = \Phi_{(2,0)} + \Phi_{(2,2)} \\ \Phi_{(2,1)} \times \Phi_{(1,2)} = \Phi_{(0,2)} + \Phi_{(2,2)} \end{array}\right\} \Rightarrow \Phi_{(1,2)} \times \Phi_{(2,1)} = \Phi_{(2,2)}$

Generally: for degenerate primary fields

$$\Phi_{(p,q)} \times \Phi_{(p',q')} = \sum_{\substack{p''=1+|p-p'| \\ p+p'+p''=1 \bmod 2}}^{p+p'-1} \; \sum_{\substack{q''=1+|q-q'| \\ q+q'+q''=1 \bmod 2}}^{q+q'-1} \Phi_{(p'',q'')} \qquad (*)$$

- only degenerate fields with $p'', q'' \geq 1$ occur in r.h.s. "truncation of operator algebra"

in minimal model $\mathcal{M}(P,Q)$, we additionally have $h_{p,q} = h_{P-p,Q-q}$

which results in fusion rule:

$$\Phi_{(p,q)} \times \Phi_{(p',q')} = \sum_{p''=1+|p-p'|}^{\min(p+p'-1,\,2P-p-p'-1)} \; \sum_{q''=1+|q-q'|}^{\min(q+q'-1,\,2Q-q-q'-1)} \Phi_{(p'',q'')}$$

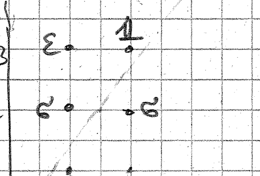
$\rightsquigarrow$ fields in Kac table form a closed algebra under OPE

E.g. for $\mathcal{M}(3,4)$:

$\mathbb{1} := \Phi_{(1,1)} = \Phi_{(2,3)}$ $h = 0$

$\sigma := \Phi_{(1,2)} = \Phi_{(2,2)}$ $h = 1/16$

$\varepsilon := \Phi_{(2,1)} = \Phi_{(1,3)}$ $h = 1/2$



fusion rules:

×	$\mathbb{1}$	σ	ε
$\mathbb{1}$	$\mathbb{1}$	σ	ε
σ	σ	$\mathbb{1}$ + ε	σ
ε	ε	σ	$\mathbb{1}$

Vertex algebras

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def a Vertex algebra is the following data:
- a vector space V (space of states)
- "vacuum vector" $|0\rangle \in V$
- "translation operator" $T \in \mathrm{End}(V)$
- state-field correspondence $Y(\cdot, z) : V \longmapsto \underset{\text{formal Laurent series}}{\mathrm{End}(V)[[z^{\pm 1}]]}$

$$A \longmapsto Y(A,z) = \sum_{n\in\mathbb{Z}} \underbrace{A_{(n)}}_{\in\,\mathrm{End}(V)} z^{-n-1}$$

Axioms:
- (vacuum axiom)

$$Y(|0\rangle, z) = \mathrm{Id}_V$$

$$\forall A \in V \quad Y(A,z)|0\rangle \in V[[z]] \quad , \quad Y(A,z)|0\rangle\big|_{z=0} = A$$

-no negative powers

- (translation axiom) $\forall A \in V$
  - : $[T, Y(A,z)] = \partial_z Y(A,z)$
  - : $T|0\rangle = 0$
- (locality) $\forall A,B \in V$, $Y(A,z)$ and $Y(B,w)$ are "mutually local" distributions, i.e.

($\forall u \in V^*$, $v \in V$, $\langle u, Y(A,z)Y(B,w)v\rangle$ and $\langle v|Y(B,w)Y(A,z)|u\rangle$ are expansions of the same element $F_{u,A,B,w} \in \mathbb{C}[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$ in $\mathbb{C}((z))((w))$ and $\mathbb{C}((w))((z))$ resp.; order of pole at $z=w$ is uniformly bounded $\forall u,v$)

or equivalently: $\exists N \in \mathbb{N}$ s.t. $(z-w)^N Y(A,z)Y(B,w) = (z-w)^N Y(B,w)Y(A,z)$ (*)

Rem: it follows from def. that
- $T(A) = A_{(-2)}|0\rangle$
- $\frac{T^n}{n!}(A) = A_{(-n-1)}|0\rangle \rightsquigarrow Y(A,z)|0\rangle = e^{zT}(A)$

(Trivial) example: Holomorphic (commutative) V.A.:

$(N=0 \text{ in } (*)\ \forall A,B) \Leftrightarrow \big(Y(A,z)\cdot B = Y(A,z)Y(B,w)|0\rangle\big|_{w=0} = Y(B,w)Y(A,z)|0\rangle\big|_{w=0} \in V[[z]]\big)$

-all vertex operators $Y(A,z)$ are regular at $z=0$

(Holomorphic V.A.) $\longleftrightarrow$ (commutative unital algebras with derivation)

$(V, |0\rangle, T, Y) \longmapsto V$, $A \circ B = \underbrace{Y(A,0)}_{=A_{-1}} \cdot B$ - commutative, associative

$|0\rangle$ - unit, T - derivation

$Y(A,z) := e^{zT}(A) \circ \bullet \longleftarrow$

def V.A. is called $\mathbb{Z}_{\geq 0}$-graded if $V = \bigoplus_{n\geq 0} V_n$, $A \in V_\alpha, B \in V_\beta \Rightarrow A_n(B) \in V_{\alpha+\beta-n-1}$

def conformal V.A. = $\mathbb{Z}_{\geq 0}$-graded V.A. equipped with a "conformal vector" $\omega \in V_2$ s.t. coefficients in $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ s.t.
- $\forall A \in V_n$, $L_0 A = nA$
- $L_{-1} = T$
- $[L_m, L_n] =$ Virasoro comm. relation

$Y(\omega, z)$ - is called "energy-momentum" field

Example: vertex algebra associated to Heisenberg Lie algebra:

$V = V_0^{\text{Heis}}$ - Fock space for Heisenberg Lie algebra $[a_n, a_m] = n\delta_{n,-m}\mathbb{1}$

$a_0|0\rangle = 0$

$$\boxed{\underset{k_1,\ldots,k_n\geq 1}{Y(a_{-k_1}\ldots a_{-k_n}|0\rangle, z)} = \frac{1}{(k_1-1)!\ldots(k_n-1)!} :\partial^{k_1}\varphi(z)\ldots\partial^{k_n}\varphi(z):}$$

where $\varphi(z) = -\sum_{n\in\mathbb{Z}} \frac{a_n z^{-n}}{n}$

$\omega = (a_{-1})^2|0\rangle$

$Y(\omega, z) = \frac{1}{2} :\partial\varphi(z)^2:$ $\quad T = L_{-1} = \frac{1}{2}\sum_{n\in\mathbb{Z}} a_n a_{-1-n}$ $\quad$ e.g. $Y(a_{-1}|0\rangle) = \partial\varphi(z)$

Example: Virasoro vertex algebra

$V = \text{Span}(L_{-k_1}\ldots L_{-k_n}|0\rangle)_{k_i \geq 2} \overset{!}{=} V_{c,0}/V_{c,1}$

$$\underset{k_1,\ldots,k_n\geq 2}{Y(L_{-k_1}\ldots L_{-k_n}|0\rangle)} = \frac{1}{(k_1-2)!\ldots(k_n-2)!} :\partial^{k_1-2}L(z)\ldots\partial^{k_n-2}L(z):$$

where $L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = Y(L_{-2}|0\rangle, z)$

- stress-energy tensor