Ref: Kohno "CFT & topology", Gawedzki "CFT: a case study" hep-th/9904145

Affine Lie algebras

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19.12.11

G - compact simple Lie group $\rightsquigarrow$ $LG = \mathrm{Maps}(S^1, G)$ with pointwise multiplication - "loop group"

assoc. Lie algebra; $\mathbb{C}\otimes$ assoc. Lie alg.

$$\mathfrak{g} \rightsquigarrow L\mathfrak{g} = \underset{\text{pointwise bracket}}{\mathrm{Maps}}(S^1, \mathfrak{g}) \simeq \mathfrak{g}\otimes\mathbb{C}[[t,t^{-1}]] \quad - \text{"loop algebra"}$$

$$\underbrace{[X\otimes f, Y\otimes g] = [X,Y]\otimes fg}$$

$L\mathfrak{g}$ admits unique ($H^2(L\mathfrak{g},\mathbb{C}) = \mathbb{C}$) central extension $\hat{\mathfrak{g}}$ - "affine Lie algebra"

$$0 \to \mathbb{C} \to \hat{\mathfrak{g}} \to L\mathfrak{g} \to 0$$

$$[X\otimes f, Y\otimes g]_{\hat{\mathfrak{g}}} = [X,Y]\otimes fg + \underset{\text{central element}}{\mathbb{K}} \underbrace{\langle X,Y\rangle}_{\text{Killing form on } \mathfrak{g}} \underbrace{\mathrm{Res}_{t=0}(df\cdot g)}_{\text{coeff of } t^{-1}}$$

More specifically:

$$[X\otimes t^m, Y\otimes t^n] = [X,Y]\otimes t^{m+n} + \mathbb{K}\langle X,Y\rangle\, m\,\delta_{m,-n}$$

Rem Specialization $\mathbb{K} = k = 1,2,3,\dots$ ("level") corresponds to a central extension of loop group

$$1 \to \mathbb{C}^* \to \widehat{LG}^k_{\mathbb{C}} \to LG_{\mathbb{C}} \to 1$$

Highest weight representations

$$\hat{\mathfrak{g}} = \underbrace{\left(\mathfrak{g}\otimes t\,\mathbb{C}[[t]] \oplus \mathfrak{g}_+\right)}_{N_+} \oplus \underbrace{(\mathbb{C}\mathbb{K}\oplus \mathfrak{h})}_{N_0} \oplus \underbrace{\left(\mathfrak{g}\otimes t^{-1}\mathbb{C}[[t^{-1}]] \oplus \mathfrak{g}_-\right)}_{N_-}$$

($\mathfrak{h}$ - Cartan for $\mathfrak{g}$)

N_+ - positive part; N_0 - Cartan subalgebra; N_- - negative part

Verma modules: $V^{\hat{\mathfrak{g}}}_{k,\lambda} = \mathrm{Ind}^{\hat{\mathfrak{g}}}_{N_0\oplus N_+} \mathbb{C}_{k,\lambda} = U(\hat{\mathfrak{g}}) \otimes_{N_0\oplus N_+} \mathbb{C}$

k - level, λ - highest weight for $\mathfrak{g}$; $\mathbb{C}_{k,\lambda}$ - 1-dim rep. of $N_0\oplus N_+$ where N_+ acts trivially, $\mathbb{K}$ acts as k (mult. by), $\mathfrak{h}$ - as λ (mult. by)

Irreducible h.w. modules:

$$M^{\hat{\mathfrak{g}}}_{k,\lambda} = V^{\hat{\mathfrak{g}}}_{k,\lambda} \Big/ \text{maximal proper submodule}$$

A distinguished set of h.w. modules:

integrable modules - those that induce fin.dim. rep. of $\mathfrak{g} \hookrightarrow \hat{\mathfrak{g}}$, $X \mapsto X\otimes t^0$

(this implies $k\in\mathbb{N}$) • There are finitely many integrable reps at given level k

Case $G = SU(2)$: For $k\in\mathbb{N}$, $0\le\lambda\le k$ (integer), there is an integrable rep. $H_{k,\lambda}$ of $\widehat{su(2)}_k$, it induces $(\lambda+1)$-dimensional ("spin-$\frac{\lambda}{2}$") irrep of $su(2)$, $M^{\mathfrak{g}}_{\lambda}$

Sugawara construction

One constructs Virasoro action on $H_{k,\lambda}$ by setting

- $$L_n = \frac{1/2}{k+C^\vee}\sum_{j\in\mathbb{Z}}\sum_{a=1}^{\dim G} :\underbrace{(T^a\otimes t^{-j})}\cdot\underbrace{(T^a\otimes t^{n+j})}:$$

  dual Coxeter number for $\mathfrak{g}$, e.g. $C^\vee=2$ for $su(2)$ — in representation $H_{k,\lambda}$

- $\{L_n\}$ satisfy Virasoro comm. relations with central charge $\boxed{c=\dfrac{k\cdot\dim G}{k+C^\vee}}$
  (e.g. for $su(2)$: $c=\dfrac{3k}{k+2}$)

- For the highest vector $v\in H_{k,\lambda}$, $L_0 v = \underbrace{\dfrac{C_\lambda}{k+C^\vee}}_{=:\Delta_{k,\lambda}}\cdot v$ — value of quadratic Casimir in $M^{\mathfrak{g}}_\lambda$, e.g. for $su(2)$: $C_\lambda = j(j+1)$ with $j=\lambda/2$

$[L_0, T^a\otimes t^{-j}] = j\, T^a\otimes t^{-j}$

$$\Rightarrow \quad H_{k,\lambda} = \bigoplus_{d\geq 0}^{\infty} \underbrace{H_{k,\lambda}(d)}$$

L_0-eigenspace with eigenvalue $\Delta_{k,\lambda}+d$.

spaces $H_{k,\lambda}(d)$ are finite-dimensional

$H_{k,\lambda}(0)\simeq M^{\mathfrak{g}}_\lambda$ — the "multiplet" of Virasoro-primary states

WZW model

(Wess–Zumino–Witten)

fix $G=SU(2)$ [for simplicity; story extends to any compact simple group]

$$\sigma = \frac{1}{24\pi^2}\, tr\left((X^{-1}dX)\wedge(X^{-1}dX)\wedge(X^{-1}dX)\right) \in \Omega^3(SU(2))$$

is a left-invariant volume form on $SU(2)$, representing the generator of $H^3(SU(2),\mathbb{Z})$

for Σ closed Riemannian surface,

$$S_\Sigma(f) = \frac{1}{4\pi}\int_\Sigma tr(f^{-1}\partial f\wedge f^{-1}\bar\partial f) + \underbrace{\frac{1}{12\pi}\int_B tr(\tilde f^{-1}d\tilde f)^3}_{2\pi\int_B \tilde f^*\sigma}$$

wedge cube as in

field $f\in Maps(\Sigma,G)$

B - a compact oriented 3-mfd with boundary Σ, $\tilde f: B\to G$ - a smooth extension of Σ

- for $k\in\mathbb{Z}$, $e^{ikS_\Sigma(f)}$ does not depend on B or choice of $\tilde f$.
  [σ is integral $\rightsquigarrow$ $2\pi\left(\int_B \tilde f^*\sigma - \int_{B'}\tilde f'^*\sigma\right) = 2\pi\int_M F^*\sigma \in 2\pi\mathbb{Z}$]
  $M = B\cup\bar B'$ - closed 3-mfd

- Equation of motion: $\partial(f^{-1}\bar\partial f)=0$

  solution: $f(z,\bar z) = h(z)\cdot\bar h(\bar z)$

• Gauge invariance

$S_\Sigma(f)$ is invariant under $f(z,\bar z) \longmapsto \Omega(z)\, f(z,\bar z)\, \bar\Omega^{-1}(\bar z)$

$\Omega \in \mathrm{Maps}_{Holom}(\Sigma, G)$

$\bar\Omega \in \mathrm{Maps}_{anti\text{-}holom}(\Sigma, G)$

associated Noether currents: $J = f^{-1}\partial f$, $\bar J = f^{-1}\bar\partial f$ conservation: $\bar\partial J, \partial \bar J \sim 0$ mod E-L

• Polyakov-Wiegmann f-la

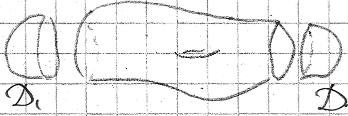
$$e^{ik S'_\Sigma(f\cdot g)} = e^{ik S'_\Sigma(f) + ik S'_\Sigma(g) - \frac{ik}{2\pi}\underbrace{\overbrace{\left(\int_\Sigma tr(f^{-1}\bar\partial f \wedge \partial g \cdot g^{-1})\right)}^{\Gamma_\Sigma(f,g)}}_{\text{2-cocycle for group } \mathrm{Maps}(\Sigma, G)}}$$

pointwise multiplication

• case with boundary

(it is not straightforward to generalize S_Σ to Σ with boundary due to non-local term in action)

construct $\Sigma' \supset \Sigma$ - closed, $\Sigma' \setminus \Sigma = \coprod_{i=1}^{n}$ disks D_i



Idea: set $e^{ikS_\Sigma(f)} := e^{ikS'_{\Sigma'}(f')}$ — some extension to Σ' of f

ambiguity in choice of extension f'

$\rightsquigarrow$ $e^{ikS_\Sigma(f)}$ takes values in the fiber of $\mathcal{L}_k^{(1)} \otimes \cdots \otimes \mathcal{L}_k^{(n)}$ over $f|_{\partial\Sigma} \in \mathrm{Maps}(\partial\Sigma, G)$

$\mathcal{L}_k$ is a complex line bundle over LG, constructed as

$$\{(f_D : \underset{\text{unit disc}}{D} \to G,\ u \in \mathbb{C})\} \Big/ (f_D, u) \sim (g_D, v) \text{ iff } f|_{\partial D} = g|_{\partial D} \text{ and}$$

$$v = u \cdot e^{ikS_{\mathbb{CP}^1}(h) - \frac{ik}{2\pi}\Gamma_D(f_D, h_D)}$$

where h_D is defined by $g_D = f_D \cdot h_D$ and h is the extension by 1_G to $\mathbb{CP}^1 \setminus D$

Rem

$\mathcal{L}_k \cong \mathcal{L}^{\otimes k}$ where $\mathcal{L}$ is the hermitian line bundle over LG with 1st Chern class $c_1 = [\omega]$,

$$\omega = \int_{S^1} \varphi^* \underset{H^3(G,\mathbb{Z})}{\underset{\in}{\sigma}} \in H^2(LG, \mathbb{Z})$$

$$\begin{array}{ccc} LG \times S^1 & \xrightarrow{\varphi} & G \\ \downarrow & & \\ LG & & \end{array}$$

Quantization

Space of states: $\mathcal{H}_k = \bigoplus_{\lambda=0}^{k} H_{k,\lambda} \otimes H^*_{k,\lambda} \hookrightarrow \Gamma(LG, \mathcal{L}_k)$ - space of sections

it carries the action of $\hat{\mathfrak{g}}_k \oplus \hat{\mathfrak{g}}_k$ and therefore (by Sugawara)

$Vir \oplus \overline{Vir}$ - action with $c = \bar c = \frac{3k}{k+2}$

basic OPEs

we chose orthon. basis in $\mathfrak{g}$ w.r.t. Killing form

$$\hat{J}^a(z)\hat{J}^b(w) \sim \frac{k\delta^{ab}\mathbb{1}}{(z-w)^2} + \frac{f^{abc}}{z-w}\hat{J}^c(w) + \text{reg.}$$

components of $\hat{J}(z) = \widehat{g^{-1}\partial g}$ writing $\hat{J}^a(z) = \sum_{n\in\mathbb{Z}} \hat{J}^a_n z^{-n-1}$, we have

$[\hat{J}^a_m, \hat{J}^b_n] = f^{abc}\hat{J}^c_{m+n} + k\, m\, \delta^{ab}\delta_{m,-n}$ – comm. relations of $\hat{\mathfrak{g}}_k$

in terms of previous notation: $J^a_n = T^a \otimes t^n$

$\hat{\mathfrak{g}}$-primary field (or rather, multiplet): $\varphi_{(\lambda)}$ with values in $M^{\mathfrak{g}}_\lambda$,

$$(*)\quad J^a(z)\varphi_{(\lambda)}(w,\bar{w}) \sim \frac{T^a_{(\lambda)}}{z-w}\varphi_{(\lambda)}(w,\bar{w}) \text{ reg.}$$

matrix of representation $M^{\mathfrak{g}}_\lambda$

stress-energy tensor: $T(z) = \frac{1/2}{k+C^\vee}\sum_{a=1}^{\dim G} :J^a(z)J^a(z):$ – satisfies usual TT OPE with $c = \frac{k\cdot\dim G}{k+C^\vee}$

Ward identity for $\hat{\mathfrak{g}}$-symmetry:

$$(**)\quad \langle J^a(z)\,\varphi_{(\lambda_1)}(z_1,\bar{z}_1)\cdots\varphi_{(\lambda_n)}(z_n,\bar{z}_n)\rangle = \sum_{j=1}^n \frac{T^a_{(\lambda_j)}}{z-z_j}\langle\varphi_{(\lambda_1)}(z_1,\bar{z}_1)\cdots\varphi_{(\lambda_n)}(z_n,\bar{z}_n)\rangle$$

$\hat{\mathfrak{g}}$-primary fields   – derived from $(*)$

by Sugawara, $L_{-1} = \frac{1/2}{k+C^\vee}\sum_{j\in\mathbb{Z}} :J^a_{-j}J^a_{-1+j}:$, hence for a $\hat{\mathfrak{g}}$-primary field $\varphi_{(\lambda)}$:

$$L_{-1}\varphi_{(\lambda)} = \frac{1}{k+C^\vee}J^a_{-1}T^a_{(\lambda)}\varphi_{(\lambda)}$$

local Virasoro generator at z local $\hat{\mathfrak{g}}$ generator

So we have

$$0 = \langle\varphi_{(\lambda_1)}(z_1,\bar{z}_1)\cdots\left(L^{(z_j)}_{-1} - \frac{1}{k+C^\vee}J^{(z_j)a}_{-1}T^a_{(\lambda_j)}\right)\varphi_{(\lambda_j)}(z_j,\bar{z}_j)\cdots\varphi_{(\lambda_n)}(z_n,\bar{z}_n)\rangle =$$

using $(**)$ and fact that $L^{(z_j)}_{-1}$ acts as $\frac{\partial}{\partial z_j}$

$$= \left(\frac{\partial}{\partial z_j} + \frac{1}{k+C^\vee}\sum_{i\neq j}\frac{T^a_{(\lambda_j)}T^a_{(\lambda_i)}}{z_j-z_i}\right)\langle\varphi_{(\lambda_1)}(z_1,\bar{z}_1)\cdots\varphi_{(\lambda_n)}(z_n,\bar{z}_n)\rangle \quad \forall j=1\ldots n$$

– Knizhnik–Zamolodchikov equation

Space of conformal blocks

let $\mathfrak{g}(z_1,\ldots,z_n) = \mathfrak{g}\otimes\left\{\begin{array}{l}\text{merom. functions on } \mathbb{CP}^1\\ \text{with poles allowed at } z_1\ldots z_n\end{array}\right\}$ – Lie algebra

$\mathfrak{g}(z_1,\ldots,z_n)$ acts on $H_{\lambda_1}\otimes\cdots\otimes H_{\lambda_n}$ (expand in Laurent series near z_j, use embedding $\mathfrak{g}\otimes\mathbb{C}((t_j))\hookrightarrow\hat{\mathfrak{g}}$, act with $\hat{\mathfrak{g}}$ on H_{λ_j})

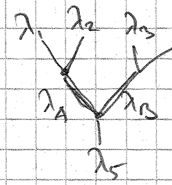
Space of conf. blocks:

$$\mathcal{H}(z_1,\ldots,z_n;\lambda_1,\ldots,\lambda_n) := \mathrm{Hom}_{\mathfrak{g}(z_1\ldots z_n)}(H_{\lambda_1}\otimes\cdots\otimes H_{\lambda_n},\mathbb{C})$$

- $\mathcal{H}$ is finite-dimensional
- for $n=3$, $\dim\mathcal{H}=1$ iff $\begin{cases}\lambda_1+\lambda_2+\lambda_3\in 2\mathbb{Z}\\ |\lambda_1-\lambda_2|\le\lambda_3\le\lambda_1+\lambda_2\\ \lambda_1+\lambda_2+\lambda_3\le 2k\end{cases}$ ← fusion rules ("quantum Clebsch-Gordan condition")

  $\dim\mathcal{H}=0$ otherwise

• For general n, one can associate a basis in $\mathcal{H}$ to a 3-valent tree with n leaves; basis elements are in $1 \longleftrightarrow 1$ correspondence with "admissible" decorations of the tree: – decorate leaves by $\lambda_1 \ldots \lambda_n$ (fixed)
– decorate edges by weights $\lambda_e \in \{0, 1, \ldots, k\}$.
– allow only decorations where fusion rule holds in every 3-valent vertex



Rem: higher genus: 3-valent graphs with g loops instead of trees

Verlinde formula:

$$\dim \mathcal{H}(z_1, \ldots, z_n; \lambda_1, \ldots, \lambda_n) = \sum_{0 \le \lambda \le k} \frac{S_{\lambda_1 \lambda} \cdots S_{\lambda_n \lambda}}{(S_{0\lambda})^{n-2}} \quad \text{where } S_{\lambda\mu} = \sqrt{\frac{2}{k+2}} \sin \frac{(\lambda+1)(\mu+1)}{k+2}$$

Conformal block bundle

$$\begin{array}{ccc} \mathcal{H}(z_1, \ldots z_n; \lambda_1, \ldots, \lambda_n) & \longrightarrow & \mathcal{E}_{\lambda_1 \ldots \lambda_n} \\ & & \downarrow \\ & & \mathrm{Conf}_n(\mathbb{CP}^1) \end{array}$$

$\leftarrow$ $\mathbb{C}$-vector bundle of conf. blocks

$\mathrm{Conf}_n(\mathbb{CP}^1)$ $\leftarrow$ conf. space of n points on $\mathbb{CP}^1$

$\left\{ \frac{\partial}{\partial z_j} - L_{-1}^{(j)} \right\}$ – a (holom.) flat connection on $\mathcal{E}_{\lambda_1 \ldots \lambda_n}$

$L_{-1}^{(j)}$: Sugawara operator on H_{λ_j}

if $\Psi \in \Gamma(\mathcal{E}_{\lambda_1 \ldots \lambda_n})$ is a horizontal section, then Ψ_0 = restriction of Ψ to $M^g_{\lambda_1} \otimes \ldots \otimes M^g_{\lambda_n}$ satisfies KZ equation:

$$\frac{\partial \Psi_0}{\partial z_i} = \frac{1}{k+2} \sum_{j \neq i} \frac{T^{a(i)} T^{a(j)} \Psi_0}{z_i - z_j}$$

Rem Space $\mathcal{H}_\Sigma(z_1, \ldots, z_n; \lambda_1, \ldots, \lambda_n)$ also arises in 3D Chern-Simons theory (on 3-mflds with boundary, with observables assoc. to tangles with components coloured by $M^g_{\lambda_j}$) as the space of states assoc. to $\Sigma = \partial M$; $\{z_j\}$ correspond to end-points of a tangle

Rem One can construct more CFTs starting from WZW_{G_k} by

• products $WZW_{G_k \times H_\ell} = WZW_{G_k} \times WZW_{G_\ell}$, central charge adds up: $c_{G_k \times G_\ell} = c_{G_k} + c_{G_\ell}$ (as does stress-energy tensor)

• "coset construction" (Goddard-Kent-Olive), for $H \subset G$ – normal Lie subgroup, one constructs "G/H" theory with $T_{G/H} = T_G - T_H$, $c_{G/H} = c_G - c_H$

• combining the two constructions, one can e.g. recover unitary minimal models:
$SU(2)_k \times SU(2)_1 / SU(2)_{k+1}$ – theory has $c = \frac{3k}{k+2} + 1 - \frac{3(k+1)}{k+3} = 1 - \frac{6}{(k+2)(k+3)}$
– this is a CFT equivalent to $\mathcal{M}(k+2, k+3)$