Another form of the Ward identity

(plugging $v = \frac{1}{w-z}\partial_w$ into general one; we assume $\phi_1 \dots \phi_n$ primary)

$$\langle T(z)\,\phi_1(z_1,\bar z_1)\dots\phi_n(z_n,\bar z_n)\rangle = \sum_{k=1}^{n}\left(\frac{h_k}{(z-z_k)^2}+\frac{1}{z-z_k}\frac{\partial}{\partial z_k}\right)\langle\phi_1(z_1,\bar z_1)\dots\phi_n(z_n,\bar z_n)\rangle$$

and likewise for $\langle \bar T \phi_1 \dots \phi_n\rangle$

Note that same identity can be recovered from $T\phi$ OPE

CFT on $\mathbb{CP}^1$ (genus = 0)

Minimal data: ① central charges of $Vir, \overline{Vir}$: $c, \bar c$;

② conf. weights $(h_i, \bar h_i)$ of all primary fields

Space of fields: $\mathcal{H} = \bigoplus_{i\in\{\text{Primaries}\}} \underbrace{V^{c,h_i}\otimes V^{\bar c,\bar h_i}}_{\text{"conf. family" of } \phi_i}$

V^{c,h_i}: Verma module for Vir with central charge c and highest weight h_i

> **Recall:** for a Lie algebra $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$ and for $\chi : \mathfrak{h} \to \mathbb{C}$, the Verma module V_χ is
> $U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{g}_+)} \mathbb{C} \underset{\text{as a vect. sp.}}{\simeq} U(\mathfrak{g}_-)$ – has PBW basis
> ($\mathfrak{h}$ acts by χ, $\mathfrak{g}_+$ acts on $\mathbb{C}$ trivially)

③ Coefficients C_{ijk} in 3-point correlators of primary fields:

$$\langle \phi_i(z_1,\bar z_1)\,\phi_j(z_2,\bar z_2)\,\phi_k(z_3,\bar z_3)\rangle = C_{ijk}\,\underbrace{f_{h_i,\bar h_i;h_j,\bar h_j;h_k,\bar h_k}(z_1,\bar z_1;z_2,\bar z_2,z_3,\bar z_3)}_{\text{a universal function recovered from global conf. invariance}}$$

- Then one can construct all correlators:

all OPE is recovered from C_{ijk} and Ward identities,

this gives an inductive procedure allowing to construct n-point correlators,

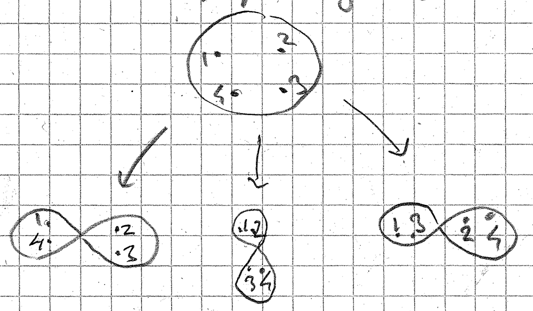
$$\langle\phi_1\dots\phi_n\rangle \in \left((\mathcal{H}^*)^{\otimes n}\otimes \mathrm{Fun}\big(\mathrm{Conf}_n(\mathbb{CP}^1\times\overline{\mathbb{CP}^1})\big)\right)^{\text{Mer. v.f.}\,\oplus\,\overline{\text{Mer. v.f.}}}$$

treat z_k and $\bar z_k$ as independent complex variables

> **Rem:** multi-valued holom. function; becomes single valued when restricted to $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^1\times\overline{\mathbb{CP}^1}$ (real slice)
> $z \mapsto (z,\bar z)$

- "Conformal bootstrap":

there are "associativity" constraints on $\{C_{ijk}\}$ coming from existence of 4-pt function with 3 asympt. regimes:



- "Rational" CFT - with finitely many primary fields
- family with $h = \bar{h} = 0$ is always present, highest vector $= \mathbb{1}$, $T = L_{-2}\mathbb{1}$ a descendant

can split holomorphic and anti-holom. dependence in corr. functions:

- $$\langle \Phi_1 \cdots \Phi_n \rangle = \sum_\alpha F_\alpha(\Phi_1, \dots, \Phi_n; z_1 \dots z_n)\, \bar{F}_\alpha(\Phi_1, \dots, \Phi_n; \bar{z}_1 \dots \bar{z}_n)$$

conformal blocks for n-point function

- a ~~finite~~ sum in case of an RCFT

- Partition function for torus:

$$Z_{torus} = \sum_{\text{primaries}} \chi_{h_i}(q)\, \chi_{\bar{h}_i}(\bar{q}) \qquad q = e^{2\pi i \tau}$$

$$\shortparallel$$

$$tr_{\mathcal{H}}\, q^{L_0} \bar{q}^{\bar{L}_0}$$

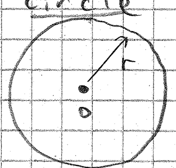
characters of Verma modules $V_{c,h_i}, V_{\bar{c},\bar{h}_i}$

= conf. blocks for torus partition function

$\rangle$———————————————$\langle$

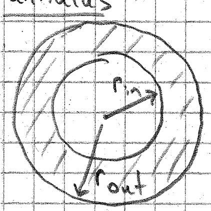
Radial quantization picture ~ Segal's CFT for a subcategory of (punctured) annuli on $\mathbb{C}$

circle



$\longmapsto \mathcal{H}$

annulus

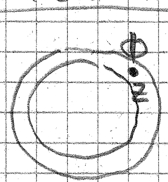


$$\longmapsto Z = \left(\frac{r_{in}}{r_{out}}\right)^{\hat{L}_0 + \hat{\bar{L}}_0} = e^{-(\tau_{out} - \tau_{in})\hat{H}}$$

where $\tau = \log r$

$\hat{H} := \hat{L}_0 + \hat{\bar{L}}_0$ - Hamiltonian

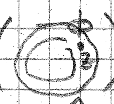
Punctured thin annulus



$$\Phi(z) \longmapsto \hat{\Phi}^{\text{Schrödinger}}(z) \in End(\mathcal{H})$$

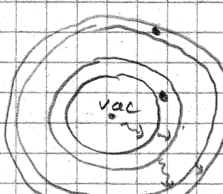
$\Phi(z) \in \mathcal{H}$

$\shortparallel$

$Z\big($  $\big)$

$$\hat{\Phi}^{\text{Heisenberg}}(z) = |z|^{\hat{L}_0 + \hat{\bar{L}}_0}\, \hat{\Phi}^{\text{Schrödinger}}(z)\, |z|^{-\hat{L}_0 - \hat{\bar{L}}_0}$$

n-point correlator



$$\langle \Phi_1(z_1,\bar{z}_1) \cdots \Phi_n(z_n,\bar{z}_n) \rangle = \langle vac | \cancel{e^{-\hat{H}(\infty - \tau_n)}}\, \hat{\Phi}^S(z_n)\, e^{-\hat{H}(\tau_n - \tau_{n-1})} \cdots \hat{\Phi}^S(z_1)\, \cancel{e^{-\hat{H}(\tau_1 + \infty)}} | vac \rangle =$$

$0 < |z_1| < \dots < |z_n| < \infty$

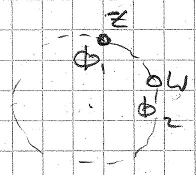
$$= \langle vac | \hat{\Phi}^{Heis}(z_n, \bar{z}_n) \cdots \hat{\Phi}^{Heis}(z_1, \bar{z}_1) | vac \rangle$$

Symmetry data: $\rho_r : Vir \oplus \overline{Vir} \longrightarrow End(\mathcal{H})$

for a circle of radius r

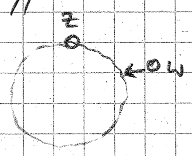
$$L_n \longmapsto \underbrace{r^{-(\hat{L}_0+\hat{\bar{L}}_0)}\, \hat{L}_n\, r^{\hat{L}_0+\hat{\bar{L}}_0}}_{\text{Schrödinger representation}} = r^n \hat{L}_n$$

Rem **Locality:**
For a correlator

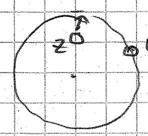


to be well-defined, same-time commutators should vanish $[\hat{\phi}_1(z), \hat{\phi}_2(w)] = 0$ for $|z| = |w|$, $z \neq w$

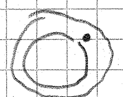
$=$ lim



$\neq$ lim



Rem map $\mathcal{H} \longrightarrow End(\mathcal{H}) \otimes Fun(\mathbb{CP}^1 \times \overline{\mathbb{CP}}^1)$

$\Phi \longmapsto \hat{\Phi}^{Heis}(z,\bar{z})$ (obtained from )

is called "state-field correspondence"

the inverse map (field-state correspondence) is:

$$\hat{\Phi}(z,\bar{z}) \longmapsto \Phi = \lim_{z\to 0} \hat{\Phi}(z,\bar{z})\,|vac\rangle$$

OPE in radial quantization

is really an equality of operators:

$$\underset{\text{radial ordering}}{R}\, \hat{\Phi}_1(z_1,\bar{z}_1)\, \hat{\Phi}_2(z_2,\bar{z}_2) = \sum_{\Phi_3} f^{\Phi_3}_{\Phi_1\Phi_2}(z_1,\bar{z}_1; z_2,\bar{z}_2)\, \hat{\Phi}_3(z_2,\bar{z}_2)$$

$$= \begin{cases} \hat{\Phi}_1(z_1,\bar{z}_1) \circ \hat{\Phi}_2(z_2,\bar{z}_2) & \text{if } |z_1| > |z_2| \\ \hat{\Phi}_2(z_2,\bar{z}_2) \circ \hat{\Phi}_1(z_1,\bar{z}_1) & \text{if } |z_1| < |z_2| \end{cases}$$

Free boson (= free massless scalar field)

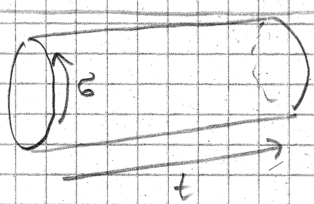
Underlying classical Lagrangian field theory

- Action: $S_{\Sigma,g}[\varphi] = \frac{\varkappa}{2}\int_\Sigma \sqrt{\det g(x)}\, d^D x \cdot g^{\mu\nu}(x)\cdot\partial_\mu\varphi(x)\cdot\partial_\nu\varphi(x)$

  normalization

- Fields: $\varphi \in C^\infty(\Sigma)$
- Euler-Lagrange equation: $\underbrace{\Delta}_{\frac{1}{\sqrt{g}}\partial_\mu \sqrt{g}\cdot g^{\mu\nu}\partial_\nu}\varphi = 0$
- Stress-energy tensor: $T_{\mu\nu} = \varkappa\left(\partial_\mu\varphi\cdot\partial_\nu\varphi - \frac{1}{2} g_{\mu\nu}\, g^{\alpha\beta}\cdot\partial_\alpha\varphi\cdot\partial_\beta\varphi\right)$
- In $D=2$, theory is conformal, i.e. $S_{\Sigma,g} = S_{\Sigma,\Omega\cdot g} \iff T^\mu{}_\mu = 0$

Free boson on a Minkowski cylinder



$\sigma \in \mathbb{R}/2\pi\mathbb{Z}$, $t \in \mathbb{R}$

$$g_{\mu\nu} = \begin{array}{cc} t & \sigma \\ \end{array} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S = \frac{\varkappa}{2}\int_\Sigma dt\, d\sigma\left(\dot\varphi^2 - (\partial_\sigma\varphi)^2\right)$$

– class. mechanics with configuration space $X = C^\infty(S^1) \ni \varphi(\sigma)$

and Lagrangian $L = \frac{\varkappa}{2}\oint d\sigma\left(\dot\varphi^2 - (\partial_\sigma\varphi)^2\right)$

Hamiltonian formalism: phase space $\Phi = T^*X$

coordinates on Φ: $\varphi(\sigma), \pi(\sigma)$; $\{\varphi(\sigma), \pi(\sigma')\} = \delta_{per}(\sigma-\sigma')$

Legendre transform: $TX \longrightarrow T^*X$: $\pi(\sigma) = \frac{\delta L}{\delta\dot\varphi(\sigma)} = \varkappa\dot\varphi(\sigma)$

Hamiltonian: $H = \oint d\sigma\left(\frac{\pi(\sigma)^2}{2\varkappa} + \frac{\varkappa}{2}(\partial_\sigma\varphi)^2\right)$ Ham. eq.: $\begin{cases}\dot\varphi = \frac{\pi}{\varkappa} \\ \dot\pi = \varkappa\,\partial_\sigma^2\varphi\end{cases}$

- Rem In terms of $T_{\mu\nu}$: $T_{00} = T_{11} = \frac{\varkappa}{2}\left(\dot\varphi^2 + (\partial_\sigma\varphi)^2\right)$ $H = \oint d\sigma\, T_{00}$ - total energy

  $T_{01} = T_{10} = \varkappa\,\dot\varphi\,\partial_\sigma\varphi$ $P = \oint d\sigma\, T_{01}$ - total (source-) momentum

Fourier modes: $\varphi(\sigma,t) = \sum_{n\in\mathbb{Z}} \varphi_n(t)\, e^{in\sigma}$, $\pi(\sigma,t) = \sum_{n\in\mathbb{Z}} \pi_n(t)\, e^{in\sigma}\cdot\frac{1}{2\pi}$

$\{\varphi_n, \pi_m\} = \delta_{n,-m}$ Fields are real $\Rightarrow$ $\varphi_{-n} = \overline{\varphi_n}$, $\pi_{-n} = \overline{\pi_n}$

$$H = \sum_{n\in\mathbb{Z}} \frac{1}{2}\frac{1}{2\pi\varkappa}\pi_n\pi_{-n} + \frac{1}{2}2\pi\varkappa n^2\varphi_n\varphi_{-n} = H_{\substack{\text{free particle} \\ \text{on } \mathbb{R} \text{ of mass} \\ \frac{1}{2\pi\varkappa}}} + \sum_{n\neq 0} H_{\substack{\text{Harm. osc.} \\ \text{with } \omega_n = |n|}}$$

Ham. eq. $\begin{cases}\dot\varphi_n = \frac{1}{2\pi\varkappa}\pi_n \\ \dot\pi_n = -2\pi\varkappa n^2\varphi_n\end{cases}$

$$\Phi = T^*\mathbb{R} \oplus \bigoplus_{n\neq 0} T^*\mathbb{R}$$

Canonical quantization

Choose $\varkappa = \frac{1}{4\pi}$. Promote φ_n, π_n to operators $\hat{\varphi}_n, \hat{\pi}_n$ satisfying $[\hat{\varphi}_n, \hat{\pi}_m] = i\delta_{n,-m}$

Introduce creation/annihilation operators $\hat{a}_n, \hat{\bar{a}}_n$ for $n \neq 0$:

$$\hat{\varphi}_n = \frac{i}{n}(-\hat{a}_{-n} + \hat{\bar{a}}_n), \quad \hat{\pi}_n = \frac{\hat{a}_{-n} + \hat{\bar{a}}_n}{2}$$

$$\begin{cases} [\hat{a}_n, \hat{a}_m] = n\delta_{n,-m} \\ [\hat{\bar{a}}_n, \hat{\bar{a}}_m] = n\delta_{n,-m} \\ [\hat{a}_n, \hat{\bar{a}}_m] = 0 \end{cases} \quad (*)$$

$$\hat{H} = \sum_{n\neq 0} \frac{\hat{a}_{-n}\hat{a}_n + \hat{\bar{a}}_{-n}\hat{\bar{a}}_n}{2} + (\hat{\pi}_0)^2$$

May define $\hat{a}_0 := \hat{\pi}_0 =: \hat{\bar{a}}_0$ then comm. relations $(*)$ are unchanged and

$$\hat{H} = \frac{1}{2}\sum_{n\in\mathbb{Z}} (\hat{a}_{-n}\hat{a}_n + \hat{\bar{a}}_{-n}\hat{\bar{a}}_n), \quad \text{total momentum operator: } \hat{P} = \frac{1}{2}\sum_{n\in\mathbb{Z}} (\hat{a}_{-n}\hat{a}_n - \hat{\bar{a}}_{-n}\hat{\bar{a}}_n)$$

Rem Lie algebra $\mathrm{Span}_{\mathbb{C}}(\{a_n\}_{n\in\mathbb{Z}} \cup \{K\})$ with comm. relations $\begin{cases}[a_n, a_m] = n\delta_{n,-m}K \\ [a_n, K] = 0\end{cases}$ is called "Heisenberg Lie algebra" $\simeq$ central extension of the (abelian) Lie algebra of formal Laurent series $\{f(z) = \sum_{n\in\mathbb{Z}} \underbrace{f_n z^{-n}}_{\text{"}a_n\text{"}}\}$

$[f, g]' = K \cdot \mathrm{res}_{z=0}\, f\, dg$

$$[\hat{H}, \hat{a}_n] = -n\hat{a}_n \qquad [\hat{P}, \hat{a}_n] = -n\hat{a}_n$$

$$[\hat{H}, \hat{\bar{a}}_n] = -n\hat{\bar{a}}_n \qquad [\hat{P}, \hat{\bar{a}}_n] = +n\hat{\bar{a}}_n$$

for $n>0$:	annihilation operator	creation operator
right-mover	$\hat{a}_n$	$\hat{a}_{-n}$
left-mover	$\hat{\bar{a}}_n$	$\hat{\bar{a}}_{-n}$

Space of states: $\mathcal{H} = \mathcal{H}_{\text{free particle}} \otimes \bigotimes_{n\neq 0} \mathcal{H}_{\text{Harm. osc.}, \omega_n = |n|} =$

$$= \mathrm{Span}_{\mathbb{C}}\left\{ \underbrace{\prod_{n,\bar{n}>0} (\hat{a}_{-n})^{k_n} (\hat{\bar{a}}_{-\bar{n}})^{\bar{k}_{\bar{n}}} |\pi_0\rangle}_{=: |\pi_0; \{k_n\}, \{\bar{k}_{\bar{n}}\}\rangle} \;\middle|\; \begin{array}{l} k_n, \bar{k}_{\bar{n}} \geq 0 \\ \text{only finitely many} \\ k_n, \bar{k}_{\bar{n}} \text{ are non-zero} \end{array} \right\} = \text{"}\bigoplus_{\pi_0\in\mathbb{R}}\text{"}\, V^{\mathrm{Heis}\otimes\overline{\mathrm{Heis}}}_{\pi_0, \bar{\pi}_0}$$

zero-mode momentum; occupation numbers for modes

$$:\hat{H}: |\pi_0; \{k_n\}, \{\bar{k}_{\bar{n}}\}\rangle = \left(\pi_0^2 + \sum_{n>0} k_n \cdot n + \sum_{\bar{n}>0} \bar{k}_{\bar{n}} \cdot \bar{n}\right) |\pi_0; \{k_n\}, \{\bar{k}_{\bar{n}}\}\rangle$$

$$:\hat{P}: |\pi_0; \{k_n\}, \{\bar{k}_{\bar{n}}\}\rangle = \left(\sum_{n>0} k_n \cdot n - \sum_{\bar{n}>0} \bar{k}_{\bar{n}} \cdot \bar{n}\right) |\pi_0; \{k_n\}, \{\bar{k}_{\bar{n}}\}\rangle$$

Normal ordering: $\mathrm{FreeAssocAlg}(\{\hat{a}_n, \hat{\bar{a}}_n\}_{n\in\mathbb{Z}}) \circlearrowleft$

$\mathcal{O} \longmapsto :\mathcal{O}:$ putting $a_{>0}, \bar{a}_{>0}$ to the right; $a_{<0}, \bar{a}_{<0}$ to the left

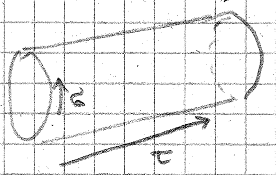
Time-dependence of fields (Schrödinger $\to$ Heisenberg picture)

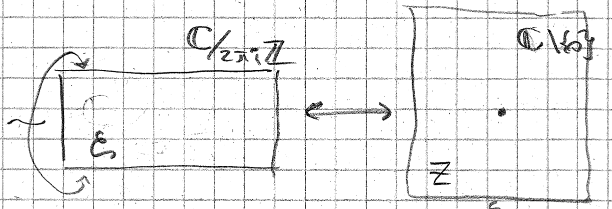
$$\hat{\mathcal{O}}^{\text{Schrödinger}} \longmapsto \hat{\mathcal{O}}^{\text{Heis}}(t) = U^{-1}(t - t_{\text{ref}})\, \hat{\mathcal{O}}^{\text{Schröd}} \circ U(t - t_{\text{ref}})$$

$$\hat\varphi(\sigma) \longmapsto \hat\varphi(\sigma,t) = e^{i\hat H t}\,\hat\varphi(\sigma)\,e^{-i\hat H t} =$$

$$= \hat\varphi_0 + 2t\hat\pi_0 + \sum_{n\neq 0}\frac{i}{n}\left(-\hat a_{-n}e^{in(\sigma+t)} + \hat{\bar a}_{-n}e^{in(\sigma-t)}\right)$$

<u>Euclidean cylinder</u>





$$\xi \longmapsto z = e^{\xi}$$

$$S_{Eucl} = \frac{\varkappa}{2}\int d\tau\, d\sigma\left((\partial_\tau\varphi)^2 + (\partial_\sigma\varphi)^2\right)$$

Complex coordinates: $\xi = \tau + i\sigma$, $\bar\xi = \tau - i\sigma$

$$S_{Eucl} = 2\varkappa\int_{\mathbb{C}/2\pi i\mathbb{Z}}\frac{i}{2}\,d\xi\wedge d\bar\xi\ \partial_\xi\varphi\cdot\partial_{\bar\xi}\varphi$$

<u>Stress-energy tensor</u>: $T_{\xi\xi} = \varkappa(\partial_\xi\varphi)^2$, $T_{\bar\xi\bar\xi} = \varkappa(\partial_{\bar\xi}\varphi)^2$, $T_{\xi\bar\xi} = T_{\bar\xi\xi} = 0$

Minkowski metric $\longrightarrow$ Euclidean metric

formal substitution $\quad t \longmapsto -i\tau \quad$ ("Wick rotation")

evolution $\quad e^{-i\hat H t} \longmapsto e^{-\hat H\tau}$

Space of states $\mathcal{H}$ and Hamiltonian $\hat H \in \mathrm{End}(\mathcal{H})$ do not change!

(Heisenberg) field: $\hat\varphi(\xi,\bar\xi) = \hat\varphi_0 - i\hat\pi_0(\xi+\bar\xi) + i\sum_{n\neq 0}\dfrac{\hat a_n e^{-n\xi} + \hat{\bar a}_n e^{-n\bar\xi}}{n} =$

$$= \boxed{\hat\varphi_0 - i\hat\pi_0\log(z\bar z) + \sum_{n\neq 0}\frac{i}{n}\left(\hat a_n z^{-n} + \hat{\bar a}_n\bar z^{-n}\right)}$$

<u>Propagator</u> for $|z|>|w|$, $\hat\varphi(z,\bar z)\hat\varphi(w,\bar w) - :\hat\varphi(z,\bar z)\hat\varphi(w,\bar w): =$

$$= \sum_{n>0}\frac{1}{n^2}\left(\underbrace{[\hat a_n,\hat a_{-n}]}_{n} z^{-n}w^n + \underbrace{[\hat{\bar a}_n,\hat{\bar a}_{-n}]}_{n}\bar z^{-n}\bar w^n\right) - i\underbrace{[\hat\pi_0,\hat\varphi_0]}_{-i}\log(z\bar z) =$$

$$= \sum_{n>0}\frac{1}{n}\left(\left(\frac{w}{z}\right)^n + \left(\frac{\bar w}{\bar z}\right)^n\right) - \log(z\bar z) = -\log\left(1-\frac{w}{z}\right) - \log\left(1-\frac{\bar w}{\bar z}\right) - \log(z\bar z) = -2\log|z-w|$$

<u>Rem</u> We supplement the normal ordering prescription by putting $\hat\pi_0$ to the right of $\hat\varphi_0$

Thus $\quad R\left(\hat\varphi(z,\bar z)\,\hat\varphi(w,\bar w)\right) = :\hat\varphi(z,\bar z)\hat\varphi(w,\bar w): - 2\log|z-w|$

radial ordering

<u>2-point correlator</u> $\quad \langle\varphi(z,\bar z)\varphi(w,\bar w)\rangle = \langle vac|R\,\hat\varphi(z,\bar z)\hat\varphi(w,\bar w)|vac\rangle =$

$$= -2\log|z-w| + C =: g(z-w,\bar z-\bar w) \quad \text{— Propagator}$$

infinite constant $\langle vac|(\hat\varphi_0)^2|vac\rangle$

n-point correlators calculated by Wick's theorem, e.g.

$$\langle\varphi_1\varphi_2\varphi_3\varphi_4\rangle = g_{12}g_{34} + g_{13}g_{24} + g_{14}g_{23}$$

Radially-ordered products reduced to normal-ordered, e.g. $\quad R(\varphi_1\varphi_2\varphi_3) = :\varphi_1\varphi_2\varphi_3: + g_{12}\varphi_3 + g_{13}\varphi_2 + g_{23}\varphi_1$

Correlators of field φ are ill-defined (due to zero-mode $\hat{\varphi}_0$), but for fields $\partial\varphi$, $\bar\partial\varphi$ they are well-defined.

$$i\partial\hat{\varphi}(z) = \sum_{n\in\mathbb{Z}} \hat{a}_n z^{-n-1}, \quad i\bar\partial\hat{\varphi}(z) = \sum_{n\in\mathbb{Z}} \hat{\bar{a}}_n \bar{z}^{-n-1}$$

$$\langle \partial\varphi(z)\cdot\partial\varphi(w)\rangle = -\frac{1}{(z-w)^2}, \quad \langle \bar\partial\varphi(\bar z)\cdot\bar\partial\varphi(\bar w)\rangle = -\frac{1}{(\bar z-\bar w)^2}, \quad \langle \partial\varphi(z)\cdot\bar\partial\varphi(\bar w)\rangle = 0$$

Exercise: check that $\langle vac|\partial\hat\varphi(z)\circ\partial\hat\varphi(w)|vac\rangle$ diverges if $|z|<|w|$

(quantum) Stress-energy tensor: $\hat{T}(z) = -\frac{1}{2} :\partial\hat\varphi(z)\partial\hat\varphi(z):$

$$\hat{\bar{T}}(\bar z) = -\frac{1}{2} :\bar\partial\hat\varphi(\bar z)\bar\partial\hat\varphi(\bar z):$$

Rem
- normal ordering removes the infinite constant from definition of $\hat T$, $\hat{\bar T}$
- we use a new normalization, $T_{new} = -2\pi T_{old}$

Operator product expansions

- $$R\,\partial\hat\varphi(z)\partial\hat\varphi(w) = -\frac{1}{(z-w)^2}\underbrace{\mathbb{1}}_{id\in End(\mathcal{H})} + \underbrace{:\partial\hat\varphi(z)\partial\hat\varphi(w):}_{\text{regular as } z\to w}$$
$$= \sum_{n\geq 0}\frac{1}{n!}(z-w)^n :\partial^{n+1}\varphi(w)\cdot\partial\varphi(w):$$

Likewise $$R\,\bar\partial\hat\varphi(\bar z)\bar\partial\hat\varphi(\bar w) = -\frac{1}{(\bar z-\bar w)^2}\mathbb{1} + \text{reg.}$$

$$R\,\partial\hat\varphi(z)\bar\partial\hat\varphi(\bar w) = \text{reg.}$$

- $$R\underbrace{\hat T(z)}_{-\frac{1}{2}:\partial\hat\varphi(z)\cdot\partial\hat\varphi(z):}\partial\hat\varphi(w) = \frac{\partial\hat\varphi(z)}{(z-w)^2} + \underbrace{:-\frac{1}{2}\partial\hat\varphi(z)\cdot\partial\hat\varphi(z)\cdot\partial\hat\varphi(w):}_{\text{reg.}} = \frac{\partial\hat\varphi(w)}{(z-w)^2} + \frac{\partial^2\hat\varphi(w)}{z-w} + \text{reg.}$$

$$R\,\hat{\bar T}(\bar z)\bar\partial\hat\varphi(\bar w) = \frac{\bar\partial\hat\varphi(\bar w)}{(\bar z-\bar w)^2} + \frac{\bar\partial^2\hat\varphi(\bar w)}{\bar z-\bar w} + \text{reg.}$$

$$R\,\hat T\cdot\bar\partial\hat\varphi, \quad R\,\hat{\bar T}\cdot\partial\hat\varphi = \text{reg.}$$

- $$R\,\hat T(z)\hat T(w) = \frac{1/2}{(z-w)^4}\mathbb{1} + \frac{2}{(z-w)^2}\hat T(w) + \frac{1}{z-w}\partial\hat T(w) + \text{reg.}$$

$R\,\hat{\bar T}(\bar z)\hat{\bar T}(\bar w)$ = c.conjugate of ——"——

$$R\,\hat T(z)\hat{\bar T}(w) = \text{reg.}$$