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Free boson (= free massless scalar field)

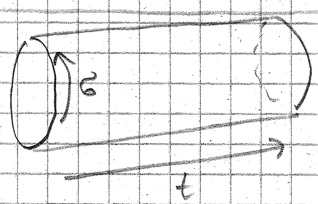
Underlying classical Lagrangian field theory

- Action: $S_{\Sigma,g}[\varphi] = \frac{\varkappa}{2}\int_\Sigma \sqrt{\det g(x)}\, d^D x \cdot g^{\mu\nu}(x)\cdot\partial_\mu\varphi(x)\cdot\partial_\nu\varphi(x)$

  normalization
- Fields: $\varphi\in C^\infty(\Sigma)$
- Euler-Lagrange equation: $\underbrace{\Delta}_{\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}\cdot g^{\mu\nu}\partial_\nu}\varphi = 0$

  Recall: generally $T^{\mu\nu}(x) := -2\,\frac{\delta S_g}{\sqrt{g(x)}\,\delta g_{\mu\nu}(x)}$
- Stress-energy tensor: $T_{\mu\nu} = \varkappa\left(\partial_\mu\varphi\cdot\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\,g^{\alpha\beta}\cdot\partial_\alpha\varphi\cdot\partial_\beta\varphi\right)$
- In $D=2$, theory is conformal, i.e. $S_{\Sigma,g} = S_{\Sigma,\Omega\cdot g} \Leftrightarrow T^\mu{}_\mu = 0$

Free boson on a Minkowski cylinder



$\sigma\in\mathbb{R}/2\pi\mathbb{Z}$, $t\in\mathbb{R}$

$$g_{\mu\nu} = \begin{array}{cc} t & \sigma \end{array} \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$S = \frac{\varkappa}{2}\int_\Sigma dt\, d\sigma\left(\dot\varphi^2 - (\partial_\sigma\varphi)^2\right)$$

– class. mechanics with configuration space $X = C^\infty(S^1) \ni \varphi(\sigma)$

and Lagrangian $L = \frac{\varkappa}{2}\oint d\sigma\left(\dot\varphi^2 - (\partial_\sigma\varphi)^2\right)$

Hamiltonian formalism:

phase space $\Phi = T^*X$

coordinates on Φ: $\varphi(\sigma), \pi(\sigma)$; $\{\varphi(\sigma), \pi(\sigma')\} = \delta_{per}(\sigma-\sigma')$

Legendre transform: $TX \longrightarrow T^*X$ $\quad \pi(\sigma) = \frac{\delta L}{\delta\dot\varphi(\sigma)} = \varkappa\dot\varphi(\sigma)$

Hamiltonian: $H = \oint d\sigma\left(\frac{\pi(\sigma)^2}{2\varkappa} + \frac{\varkappa}{2}(\partial_\sigma\varphi)^2\right)$ $\quad$ Ham. eq. $\begin{cases}\dot\varphi = \frac{\pi}{\varkappa}\\ \dot\pi = \varkappa\,\partial_\sigma^2\varphi\end{cases}$

- Rem. In terms of $T_{\mu\nu}$: $T_{00} = T_{11} = \frac{\varkappa}{2}\left(\dot\varphi^2 + (\partial_\sigma\varphi)^2\right)$ $\quad H = \oint d\sigma\, T_{00}$ – total energy

  $T_{01} = T_{10} = \varkappa\,\dot\varphi\,\partial_\sigma\varphi$ $\quad P = \oint d\sigma\, T_{01}$ – total (source-) momentum

Fourier modes:

$\varphi(\sigma,t) = \sum_{n\in\mathbb{Z}}\varphi_n(t)\,e^{in\sigma}$, $\quad \pi(\sigma,t) = \sum_{n\in\mathbb{Z}}\pi_n(t)\,e^{in\sigma}\cdot\frac{1}{2\pi}$

$\{\varphi_n, \pi_m\} = \delta_{n,-m}$ $\quad$ Fields are real $\Rightarrow \varphi_{-n} = \overline{\varphi_n}$, $\pi_{-n} = \overline{\pi_n}$

$$H = \sum_{n\in\mathbb{Z}}\frac{1}{2}\frac{1}{2\pi\varkappa}\pi_n\pi_{-n} + \frac{1}{2}2\pi\varkappa n^2\varphi_n\varphi_{-n} = \underbrace{H_0}_{\substack{\text{free particle}\\ \text{on } \mathbb{R} \text{ of mass}\\ \frac{1}{2\pi\varkappa}}} + \sum_{n\neq 0}\underbrace{H_n}_{\substack{\text{Harm. osc.}\\ \text{with } \omega_n = |n|}}$$

Ham. eq. $\begin{cases}\dot\varphi_n = \frac{1}{2\pi\varkappa}\pi_n\\ \dot\pi_n = -2\pi\varkappa n^2\varphi_n\end{cases}$ $\quad \Phi = T^*\mathbb{R}\oplus\bigoplus_{n\neq 0}T^*\mathbb{R}$

Rem: Oscillators split in variables $(\mathrm{Re}\,\varphi_n, \mathrm{Re}\,\pi_n)$, $(\mathrm{Im}\,\varphi_n, \mathrm{Im}\,\pi_n)$ for $n>0$

Canonical quantization

Choose $\varkappa = \frac{1}{4\pi}$. Promote φ_n, π_n to operators $\hat\varphi_n, \hat\pi_n$ satisfying $[\hat\varphi_n, \hat\pi_m] = i\delta_{n,-m}$

Introduce creation/annihilation operators $\hat a_n, \hat{\bar a}_n$ for $n \neq 0$:

$$\hat\varphi_n = \frac{i}{n}\left(-\hat a_{-n} + \hat{\bar a}_n\right), \quad \hat\pi_n = \frac{\hat a_{-n} + \hat{\bar a}_n}{2} \qquad \begin{cases} [\hat a_n, \hat a_m] = n\,\delta_{n,-m} \\ [\hat{\bar a}_n, \hat{\bar a}_m] = n\,\delta_{n,-m} \\ [\hat a_n, \hat{\bar a}_m] = 0 \end{cases} \quad (*)$$

$$\hat H = \sum_{n\neq 0} \frac{\hat a_{-n}\hat a_n + \hat{\bar a}_{-n}\hat{\bar a}_n}{2} + (\hat\pi_0)^2$$

Rem adjoints: $(\hat a_n)^\dagger = \hat a_{-n}$, $(\hat{\bar a}_n)^\dagger = \hat{\bar a}_{-n}$

May define $\hat a_0 := \hat\pi_0 =: \hat{\bar a}_0$ then comm. relations $(*)$ are unchanged and

$$\hat H = \frac12 \sum_{n\in\mathbb Z}\left(\hat a_{-n}\hat a_n + \hat{\bar a}_{-n}\hat{\bar a}_n\right), \quad \text{total momentum operator: } \hat P = \frac12\sum_{n\in\mathbb Z}\left(\hat a_{-n}\hat a_n - \hat{\bar a}_{-n}\hat{\bar a}_n\right)$$

Rem Lie algebra $\mathrm{Span}_{\mathbb C}\left(\{a_n\}_{n\in\mathbb Z} \cup \{K\}\right)$ with comm. relations $\begin{cases}[a_n, a_m] = n\delta_{n,-m}K \\ [a_n, K] = 0\end{cases}$ is called "Heisenberg Lie algebra" $\simeq$ central extension of the (abelian) Lie algebra of formal Laurent series $\{f(z) = \sum_{n\in\mathbb Z} f_n \underbrace{z^{-n}}_{\text{"}a_n\text{"}}\}$

$$[f, g]' = K\cdot \mathrm{res}_{z=0}\, f\,dg$$

$$[\hat H, \hat a_n] = -n\,\hat a_n \qquad [\hat P, \hat a_n] = -n\,\hat a_n$$
$$[\hat H, \hat{\bar a}_n] = -n\,\hat{\bar a}_n \qquad [\hat P, \hat{\bar a}_n] = +n\,\hat{\bar a}_n$$

for $n>0$:	annihilation operator	creation operator
right-mover	$\hat a_n$	$\hat a_{-n}$
left-mover	$\hat{\bar a}_n$	$\hat{\bar a}_{-n}$

Space of states: $\mathcal H = \mathcal H_{\text{free particle}} \otimes \bigotimes_{n\neq 0} \mathcal H_{\text{harm. osc.}, \omega_n = |n|}$

$$= \mathrm{Span}_{\mathbb C}\left\{ \underbrace{\prod_{n,\bar n>0} (\hat a_{-n})^{k_n} (\hat{\bar a}_{-\bar n})^{\bar k_{\bar n}} |\pi_0\rangle}_{=:\, |\pi_0;\, \{k_n\}, \{\bar k_{\bar n}\}\rangle} \;\middle|\; \begin{array}{l} k_n, \bar k_{\bar n} \geq 0 \\ \text{only finitely many} \\ k_n, \bar k_{\bar n} \text{ are non-zero}\end{array} \right\} = \text{"}\bigoplus_{\pi_0\in\mathbb R}\text{"}\, V^{\mathrm{Heis}\otimes\overline{\mathrm{Heis}}}_{\pi_0, \pi_0}$$

zero-mode momentum; occupation numbers for modes

$$:\hat H:\, |\pi_0; \{k_n\}, \{\bar k_{\bar n}\}\rangle = \left(\pi_0^2 + \sum_{n>0} k_n\cdot n + \sum_{\bar n>0} \bar k_{\bar n}\cdot \bar n\right) |\pi_0; \{k_n\}, \{\bar k_{\bar n}\}\rangle$$

$$:\hat P:\, |\pi_0; \{k_n\}, \{\bar k_{\bar n}\}\rangle = \left(\sum_{n>0} k_n\cdot n - \sum_{\bar n>0} \bar k_{\bar n}\cdot \bar n\right) |\pi_0; \{k_n\}, \{\bar k_{\bar n}\}\rangle$$

Normal ordering: $\mathrm{FreeAssocAlg}\left(\{\hat a_n, \hat{\bar a}_n\}_{n\in\mathbb Z}\right) \circlearrowleft$

$\mathcal O \longmapsto\; :\mathcal O:$ putting $a_{>0}, \bar a_{>0}$ to the right, $a_{<0}, \bar a_{<0}$ to the left

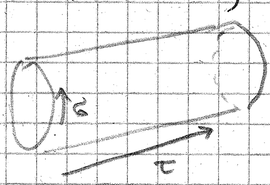
Time-dependence of fields (Schrödinger → Heisenberg picture)

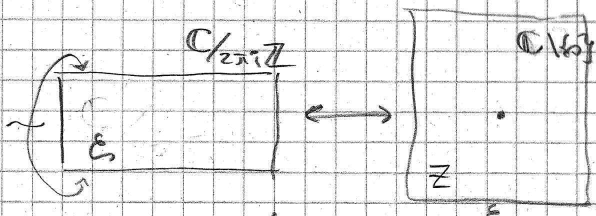
$$\hat{\mathcal O}^{\text{Schrödinger}} \longmapsto \hat{\mathcal O}^{\text{Heis}}(t) = U^{-1}(t - t_{\text{ref}})\, \hat{\mathcal O}^{\text{Schröd}}\, U(t - t_{\text{ref}})$$

$$\hat\varphi(\sigma) \longmapsto \hat\varphi(\sigma,t) = e^{i\hat H t}\,\hat\varphi(\sigma)\,e^{-i\hat H t} =$$

$$= \hat\varphi_0 + 2t\hat\pi_0 + \sum_{n\neq 0}\frac{i}{n}\left(-\hat a_{-n}\, e^{in(\sigma+t)} + \hat{\bar a}_{-n}\, e^{in(\sigma-t)}\right)$$

Euclidean cylinder





$$\xi \longrightarrow z = e^{\xi}$$

$$S_{Eucl} = \frac{\varkappa}{2}\int d\tau\, d\sigma\left((\partial_\tau\varphi)^2 + (\partial_\sigma\varphi)^2\right)$$

Complex coordinates: $\xi = \tau + i\sigma$, $\bar\xi = \tau - i\sigma$

$$S_{Eucl} = 2\varkappa\int_{\mathbb{C}/2\pi i\mathbb{Z}} \frac{i}{2}\, d\xi\wedge d\bar\xi\;\partial_\xi\varphi\cdot\partial_{\bar\xi}\varphi$$

Stress-energy tensor: $T_{\xi\xi} = \varkappa(\partial_\xi\varphi)^2$, $T_{\bar\xi\bar\xi} = \varkappa(\partial_{\bar\xi}\varphi)^2$, $T_{\xi\bar\xi} = T_{\bar\xi\xi} = 0$

Minkowski metric $\longrightarrow$ Euclidean metric

formal substitution: $t \longrightarrow -i\tau$ ("Wick rotation")

evolution: $e^{-i\hat H t} \longmapsto e^{-\hat H\tau}$

Space of states $\mathcal{H}$ and Hamiltonian $\hat H \in \mathrm{End}(\mathcal{H})$ do not change!

(Heisenberg) field:
$$\hat\varphi(\xi,\bar\xi) = \hat\varphi_0 - i\hat\pi_0(\xi+\bar\xi) + i\sum_{n\neq 0}\frac{\hat a_n e^{-n\xi} + \hat{\bar a}_n e^{-n\bar\xi}}{n} =$$

$$= \boxed{\hat\varphi_0 - i\hat\pi_0 \log(z\bar z) + \sum_{n\neq 0}\frac{i}{n}\left(\hat a_n z^{-n} + \hat{\bar a}_n \bar z^{-n}\right)}$$

Propagator For $|z|>|w|$, $\hat\varphi(z,\bar z)\hat\varphi(w,\bar w) - :\hat\varphi(z,\bar z)\hat\varphi(w,\bar w): =$

$$= \sum_{n>0}\frac{1}{n^2}\left(\underbrace{[\hat a_n, \hat a_{-n}]}_{n} z^{-n} w^{n} + \underbrace{[\hat{\bar a}_{n}, \hat{\bar a}_{-n}]}_{n} \bar z^{-n}\bar w^{n}\right) - i\underbrace{[\hat\pi_0,\hat\varphi_0]}_{-i}\log(z\bar z) =$$

$$= \sum_{n>0}\frac{1}{n}\left(\left(\frac{w}{z}\right)^n + \left(\frac{\bar w}{\bar z}\right)^n\right) - \log(z\bar z) = -\log\left(1-\frac{w}{z}\right) - \log\left(1-\frac{\bar w}{\bar z}\right) - \log(z\bar z) = -2\log|z-w|$$

Rem We supplement the normal ordering prescription by putting $\hat\pi_0$ to the right of $\hat\varphi_0$.

Thus
$$R\left(\hat\varphi(z,\bar z)\,\hat\varphi(w,\bar w)\right) = :\hat\varphi(z,\bar z)\,\hat\varphi(w,\bar w): - 2\log|z-w|$$
radial ordering

2-point correlator $\langle\varphi(z,\bar z)\varphi(w,\bar w)\rangle = \langle vac|R\,\hat\varphi(z,\bar z)\hat\varphi(w,\bar w)|vac\rangle =$

$$= -2\log|z-w| + \underbrace{C}_{\text{infinite constant } \langle vac|(\hat\varphi_0)^2|vac\rangle} =: g(z-w,\bar z-\bar w) \quad \text{— Propagator}$$

n-point correlators calculated by Wick's theorem, e.g.

$$\langle\varphi_1\varphi_2\varphi_3\varphi_4\rangle = g_{12}g_{34} + g_{13}g_{24} + g_{14}g_{23}$$

Radially-ordered products reduced to normal-ordered, e.g. $R(\varphi_1\varphi_2\varphi_3) = :\varphi_1\varphi_2\varphi_3: + g_{12}\varphi_3 + g_{13}\varphi_2 + g_{23}\varphi_1$

Correlators of field φ are ill-defined (due to zero-mode $\hat{\varphi}_0$), but for fields $\partial\varphi$, $\bar{\partial}\varphi$ they are well-defined.

$$i\partial\hat{\varphi}(z) = \sum_{n\in\mathbb{Z}} \hat{a}_n z^{-n-1} \quad , \quad i\bar{\partial}\hat{\varphi}(\bar{z}) = \sum_{n\in\mathbb{Z}} \hat{\bar{a}}_n \bar{z}^{-n-1}$$

$$\langle \partial\varphi(z) \cdot \partial\varphi(w)\rangle = -\frac{1}{(z-w)^2} \quad , \quad \langle \bar{\partial}\varphi(\bar{z}) \cdot \bar{\partial}\varphi(\bar{w})\rangle = -\frac{1}{(\bar{z}-\bar{w})^2} \quad , \quad \langle \partial\varphi(z)\cdot\bar{\partial}\varphi(\bar{w})\rangle = 0$$

Exercise: check that $\langle vac| \partial\hat{\varphi}(z) \circ \partial\hat{\varphi}(w) |vac\rangle$ diverges if $|z|<|w|$

(quantum) Stress-energy tensor: $\hat{T}(z) = -\frac{1}{2} :\partial\hat{\varphi}(z)\,\partial\hat{\varphi}(z):$

$$\hat{\bar{T}}(\bar{z}) = -\frac{1}{2} :\bar{\partial}\hat{\varphi}(\bar{z})\,\bar{\partial}\hat{\varphi}(\bar{z}):$$

Rem
- normal ordering removes the infinite constant from definition of $\hat{T}$, $\hat{\bar{T}}$
- we use a new normalization, $T_{new} = -2\pi\, T_{old}$

Operator product expansions

- $$R\,\partial\hat{\varphi}(z)\,\partial\hat{\varphi}(w) = -\frac{1}{(z-w)^2}\underbrace{\mathbb{1}}_{id\in End(\mathcal{H})} + \underbrace{:\partial\hat{\varphi}(z)\,\partial\hat{\varphi}(w):}_{\text{regular as } z\to w}$$

Check that!

$$= \sum_{n\geq 0} \frac{1}{n!}(z-w)^n :\partial^{n+1}\varphi(w)\cdot\partial\varphi(w):$$

Likewise $$R\,\bar{\partial}\hat{\varphi}(\bar{z})\,\bar{\partial}\hat{\varphi}(\bar{w}) = -\frac{1}{(\bar{z}-\bar{w})^2}\mathbb{1} + \text{reg.}$$

$$R\,\partial\hat{\varphi}(z)\,\bar{\partial}\hat{\varphi}(\bar{w}) = \text{reg.}$$

- $$R\,\underbrace{\hat{T}(z)}_{-\frac{1}{2}:\partial\hat{\varphi}(z)\cdot\partial\hat{\varphi}(z):}\,\partial\hat{\varphi}(w) = \frac{\partial\hat{\varphi}(z)}{(z-w)^2} + \underbrace{:-\frac{1}{2}\partial\hat{\varphi}(z)\cdot\partial\hat{\varphi}(z)\cdot\partial\hat{\varphi}(w):}_{\text{reg.}} = \frac{\partial\hat{\varphi}(w)}{(z-w)^2} + \frac{\partial^2\hat{\varphi}(w)}{z-w} + \text{reg.}$$

$$R\,\hat{\bar{T}}(\bar{z})\,\bar{\partial}\hat{\varphi}(\bar{w}) = \frac{\bar{\partial}\hat{\varphi}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}^2\hat{\varphi}(\bar{w})}{\bar{z}-\bar{w}} + \text{reg.}$$

$$R\,\hat{T}\cdot\bar{\partial}\hat{\varphi}\ ,\ R\,\hat{\bar{T}}\cdot\partial\hat{\varphi} = \text{reg.}$$

- $$R\,\hat{T}(z)\,\hat{T}(w) = \frac{1/2}{(z-w)^4}\mathbb{1} + \frac{2}{(z-w)^2}\hat{T}(w) + \frac{1}{z-w}\partial\hat{T}(w) + \text{reg.}$$

$R\,\hat{\bar{T}}(\bar{z})\,\hat{\bar{T}}(\bar{w})$ = c.conjugate of ——"——

$$R\,\hat{T}(z)\,\hat{\bar{T}}(\bar{w}) = \text{reg.}$$

Virasoro action on $\mathcal{H}$

Generally (not just for free boson):

$$\rho\Big(\varepsilon(z)\frac{\partial}{\partial z}+\bar\varepsilon(\bar z)\frac{\partial}{\partial \bar z}\Big) = -\frac{1}{2\pi i}\oint dz\,\varepsilon(z)\hat T(z) + \frac{1}{2\pi i}\oint d\bar z\,\bar\varepsilon(\bar z)\hat{\bar T}(\bar z) \quad \in \mathrm{End}(\mathcal H)$$

In particular
$$\hat L_n := \rho\Big(-z^{n+1}\frac{\partial}{\partial z}\Big) = \frac{1}{2\pi i}\oint dz\cdot z^{n+1}\hat T(z) \quad \in \mathrm{End}(\mathcal H) \qquad (*)$$

likewise
$$\hat{\bar L}_n = -\frac{1}{2\pi i}\oint d\bar z\cdot \bar z^{n+1}\hat{\bar T}(\bar z)$$

Exercise: ① show that OPE
$$R\,\hat T(z)\hat T(w) = \frac{c/2}{(z-w)^4} + \frac{2\hat T(w)}{(z-w)^2} + \frac{\partial \hat T(w)}{z-w} + \text{reg.}$$
implies the Virasoro comm. relations:
$$[\hat L_n,\hat L_m] = (n-m)\hat L_{n+m} + c\,\delta_{n,-m}\frac{n^3-n}{12}\cdot\hat{\mathbb 1}$$

② Show that
$$R\,\hat T(z)\hat\Phi(w,\bar w) = \frac{h\hat\Phi(w,\bar w)}{(z-w)^2} + \frac{\partial\hat\Phi(w,\bar w)}{z-w} + \text{reg.}$$
+ similar for $\bar T\Phi$ imply
$$\delta_{\varepsilon,\bar\varepsilon}\hat\Phi := [\rho(\varepsilon\partial+\bar\varepsilon\bar\partial),\hat\Phi] =$$
$$= -\varepsilon(z)\partial\hat\Phi(z,\bar z) - h\,\partial\varepsilon(z)\cdot\hat\Phi(z,\bar z) - \bar\varepsilon(\bar z)\bar\partial\hat\Phi(z,\bar z) - \bar h\,\bar\partial\bar\varepsilon(\bar z)\cdot\hat\Phi(z,\bar z)$$
– infinitesimal $(h,\bar h)$-tensor transformation rule

Inverse formula for $(*)$ is the Fourier expansion for $\hat T(z)$:
$$\hat T(z) = \sum_{n\in\mathbb Z}\hat L_n z^{-n-2}, \qquad \hat{\bar T}(\bar z) = \sum_{n\in\mathbb Z}\hat{\bar L}_n \bar z^{-n-2}$$

Back to free boson

$$\hat T(z) = -\frac12 :\partial\hat\varphi(z)\cdot\partial\hat\varphi(z): \quad \Rightarrow \quad \hat L_n = \frac12\sum_{m\in\mathbb Z} :\hat a_m\hat a_{n-m}: \;(**), \quad \text{likewise } \hat{\bar L}_n = \frac12\sum_{m\in\mathbb Z}:\hat{\bar a}_m\hat{\bar a}_{n-m}:$$

Exercise

① Check Virasoro relations for $(**)$ from Heisenberg relations for $\hat a_n$

② Check that $\hat L_0\underbrace{|\pi_0\rangle}_{\text{pseudo-vacuum with } \pi_0 \text{ being the eigenvalue of } \hat\pi_0} = \hat{\bar L}_0|\pi_0\rangle = \frac12(\pi_0)^2|\pi_0\rangle$

③ • $\lim_{z\to 0}\hat T(z)|vac\rangle = ?$ (state associated to the field $T(z)$)

• Does $\lim_{z\to0}\hat T(z)|\pi_0\rangle$ exist for $\pi_0\neq 0$? What about $\lim_{z\to 0}\hat T(z)\hat T(0)|vac\rangle$?

④* How $\mathcal H$ splits into irreducible highest weight modules for $Vir\oplus\overline{Vir}$?

Vertex operators

Define $\boxed{\hat{V}_\alpha(z,\bar z) = :e^{i\alpha\hat\varphi(z,\bar z)}:} = e^{-\alpha\sum_{n<0}\frac{\hat a_n z^{-n}+\hat{\bar a}_n \bar z^{-n}}{n}} \circ e^{-\alpha\sum_{n>0}\frac{\hat a_n z^{-n}+\hat{\bar a}_n \bar z^{-n}}{n}} \circ e^{i\alpha\hat\varphi_0} \circ e^{\alpha\hat\pi_0 \log(z\bar z)}$

$\alpha \in \mathbb{R}$ - parameter ("charge")

Exercise: prove the following properties of vertex operators:

① V_α is a primary field with $(h,\bar h) = \left(\frac{\alpha^2}{2}, \frac{\alpha^2}{2}\right)$ (by computing $T\cdot V_\alpha$ OPE)

② $\lim_{z,\bar z\to 0} \hat V_\alpha(z,\bar z)|Vac\rangle = |\alpha\rangle$

pseudo-vacuum with $\hat\pi_0|\alpha\rangle = \alpha|\alpha\rangle$

③ • $\langle V_\alpha(z,\bar z)\, V_\beta(w,\bar w)\rangle = \begin{cases} \frac{1}{|z-w|^{2\alpha^2}} & \text{if } \beta=-\alpha \\ 0 & \text{otherwise} \end{cases}$

Use the following corollary of BCH f-la:
$e^A e^B = e^B e^A e^{[A,B]}$
if $[[A,B],A]=0$ and $[[A,B],B]=0$

• $\langle V_\alpha(z,\bar z)\, V_\beta(w,\bar w)\, V_\gamma(u,\bar u)\rangle = \begin{cases} |z-w|^{2\alpha\beta}\cdot|z-u|^{2\alpha\gamma}\cdot|w-u|^{2\beta\gamma} & \text{if } \alpha+\beta+\gamma=0 \\ 0 & \text{otherwise} \end{cases}$

"neutrality" condition

• $\langle \prod_{k=1}^{n} V_{\alpha_k}(z_k,\bar z_k)\rangle = \begin{cases} e^{\sum_{1\le j<k\le n} 2\alpha_j\alpha_k \log|z_j-z_k|} & \text{if } \sum\alpha_k=0 \\ 0 & \text{otherwise} \end{cases}$

• asymptotics $\sim |z_k|^{-2\alpha_k^2}$ as $z_k\to\infty$

④ Compute $V_\alpha\cdot V_\beta$ OPE (most singular term)

⑤ $V_\alpha(z,\bar z)|\beta\rangle = |z|^{2\alpha\beta}\, e^{\alpha\sum_{n>0}\frac{1}{n}(\hat a_{-n}z^n + \hat{\bar a}_{-n}\bar z^n)}|\alpha+\beta\rangle$

⑥ OPE $i\partial\varphi(z)\cdot V_\alpha(w,\bar w) \sim \frac{\alpha}{z-w}V_\alpha(w,\bar w) + \text{reg.}$