<u>Partition function for real fermion on torus</u>

$$Z(\tau) = (q\bar q)^{-\frac{1}{48}} \operatorname{tr}_{\mathcal{H}^{even}} q^{L_0}\bar q^{\bar L_0} =$$

($-\frac{1}{48} = -\frac{c}{24}$)

$$= (q\bar q)^{-\frac{1}{48}}\frac{1}{2}\Big(\operatorname{tr}_{\mathcal{H}_A} + \operatorname{Str}_{\mathcal{H}_A} + \operatorname{tr}_{\mathcal{H}_P} + \underbrace{\operatorname{Str}_{\mathcal{H}_P}}_{=0}\Big) q^{L_0}\bar q^{\bar L_0} =$$

two pseudo-vacua in P sector

"vacuum energy" in P sector (L_0 eigenvalue for highest vector)

$$= (q\bar q)^{-\frac{1}{48}}\frac{1}{2}\Big(\prod_{n\geq 1}(1+q^{n-\frac12})(1+\bar q^{n-\frac12}) + \prod_{n\geq 1}(1-q^{n-\frac12})(1-\bar q^{n-\frac12}) + 2(q\bar q)^{\frac{1}{16}}\prod_{n\geq 1}(1+q^n)(1+\bar q^n)\Big)$$

<u>Aside</u> • Jacobi triple product identity:

$$\prod_{n=1}^{\infty}(1-q^n)(1+tq^{n-\frac12})(1+t^{-1}q^{n-\frac12}) = \sum_{k=-\infty}^{\infty} t^k q^{k^2/2} =: \vartheta_3(w;\tau), \quad |q|<1$$

$t = e^{2\pi i w}$, $q = e^{2\pi i\tau}$

$\vartheta_4(w;\tau) := \vartheta_3(w+\frac12;\tau)$

$\vartheta_2(w;\tau) := \vartheta_3(w+\frac{\tau}{2};\tau)\cdot q^{\frac18}t^{\frac12}$

$\vartheta_1(w;\tau) := -i\,\vartheta_3(w+\frac12+\frac{\tau}{2};\tau)\cdot q^{\frac18}t^{\frac12}$

$\vartheta_i(\tau) \equiv \vartheta_i(0;\tau)$, $i=1,\dots,4$ – Jacobi ϑ-functions

$\vartheta(w+a+b\tau;\tau) = q^{-\frac{b^2}{2}}t^{-b}\vartheta(w;\tau)$

$\vartheta(w;\tau+1) = \vartheta_4(w;\tau)$

$\vartheta(\frac{w}{\tau};-\frac{1}{\tau}) = \vartheta(w;\tau)\,e^{\frac{\pi i w^2}{\tau}}(-i\tau)^{\frac12}$

$\vdots$

Cases:

<u>$t=1$</u>: $\prod_{n\geq 1}(1-q^n)(1+q^{n-\frac12})^2 = \sum_{k\in\mathbb{Z}} q^{k^2/2} = \vartheta_3(\tau) \equiv \vartheta_3(0;\tau)$

<u>$t=-1$</u>: $\prod_{n\geq 1}(1-q^n)(1-q^{n-\frac12})^2 = \sum_{k\in\mathbb{Z}}(-1)^k q^{k^2/2} = \vartheta_4(\tau)$

$t = q^{1/2}$: $2\prod_{n\geq 1}(1-q^n)(1+q^n)^2 = \sum_{k\in\mathbb{Z}} q^{\frac{k(k+1)}{2}} = \vartheta_2(\tau)\cdot q^{-\frac18}$

So: $$Z^{\text{real fermion}}_{\text{torus}}(\tau) = \left|\frac{\vartheta_3(\tau)}{\eta(\tau)}\right| + \left|\frac{\vartheta_4(\tau)}{\eta(\tau)}\right| + \left|\frac{\vartheta_2(\tau)}{\eta(\tau)}\right|$$

$$= \frac{\eta(\tau)^2}{\eta(\frac{\tau}{2})\eta(2\tau)}\cdot\frac{\eta(\bar\tau)^2}{\eta(\frac{\bar\tau}{2})\eta(2\bar\tau)} + \frac{\eta(\frac{\tau}{2})}{\eta(\tau)}\cdot\frac{\eta(\frac{\bar\tau}{2})}{\eta(\bar\tau)} + 2\,\frac{\eta(2\tau)}{\eta(\tau)}\cdot\frac{\eta(2\bar\tau)}{\eta(\bar\tau)}$$

← purely in terms of Dedekind eta-fun.

It is modular invariant: $Z(\tau+1) = Z(\tau)$

$Z(-\tau^{-1}) = Z(\tau)$

In path integral formalism:

$$Z(\tau) = \sum_{\substack{\text{4 spin structures}\\ \text{on torus}\\ \varepsilon_1=\pm1,\ \varepsilon_2=\pm1}} \int \mathcal{D}\psi\mathcal{D}\bar\psi\ e^{-S[\psi,\bar\psi]} \underset{\text{symbolically}}{=} {}^{A}\square_{A} + {}^{P}\square_{A} + {}^{A}\square_{P} + {}^{P}\square_{P}$$

$\psi(\xi+2\pi i) = \varepsilon_1\psi(\xi)$

$\bar\psi(\xi+2\pi i) = \varepsilon_1\bar\psi(\xi)$

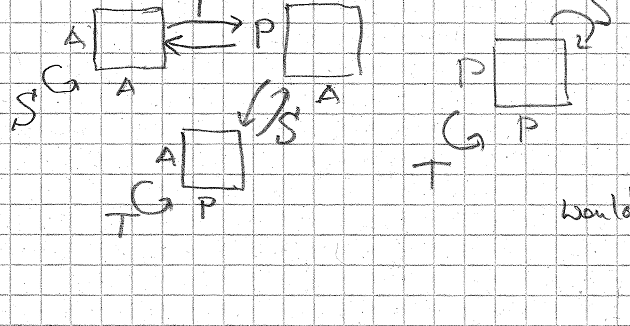
$\psi(\xi+2\pi i\tau) = \varepsilon_2\psi(\xi)$

$\bar\psi(\xi+2\pi i\tau) = \varepsilon_2\bar\psi(\xi)$

Correspondence ~~to~~ operator language:

${}^{A}\square_{A} = \frac12\operatorname{tr}_{\mathcal{H}_A}(\dots)$, ${}^{P}\square_{A} = \frac12\operatorname{Str}_{\mathcal{H}_A}(\dots)$, ${}^{A}\square_{P} = \frac12\operatorname{tr}_{\mathcal{H}_P}(\dots)$, ${}^{P}\square_{P} = \frac12\operatorname{Str}_{\mathcal{H}_P}(\dots) = 0$

<u>Rem</u> modular group $PSL_2(\mathbb{Z})$ acts on spin-structures on torus:



where $T: \tau\mapsto\tau+1$, $S: \tau\mapsto-\tau^{-1}$ are generators of $PSL_2(\mathbb{Z})$

So, a general modular-invariant expression for $Z(\tau)$ would be $C_1\left({}^{A}\square_{A} + {}^{P}\square_{A} + {}^{A}\square_{P}\right) + C_2\left({}^{P}\square_{P}\right)$

Bosonization

Vague idea: $\underbrace{\text{two } c=\tfrac12 \text{ real fermions}}$ $\sim$ one $c=1$ free boson

in the sense of "product of CFTs"

Dirac fermion

$$\psi_\pm = \frac{\psi_1 \pm i\psi_2}{\sqrt2}$$

Action: $S \propto \int \Psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \Psi$ where $\Psi = \begin{pmatrix}\psi_+ \\ \bar\psi_+\end{pmatrix}$, $\Psi^\dagger = (\psi_-, \bar\psi_-)$

or $S_{Dirac}[\psi_\pm, \bar\psi_\pm] = S_{real}[\psi_1, \bar\psi_1] + S_{real}[\psi_2, \bar\psi_2]$

• Impose that $\psi_{1,2}$ have synchronized periodicity condition A/P, so

$$\mathcal{H}_{Dirac} = \mathcal{H}_A^{real} \otimes \mathcal{H}_A^{real} \oplus \mathcal{H}_P^{real} \otimes \mathcal{H}_P^{real}$$

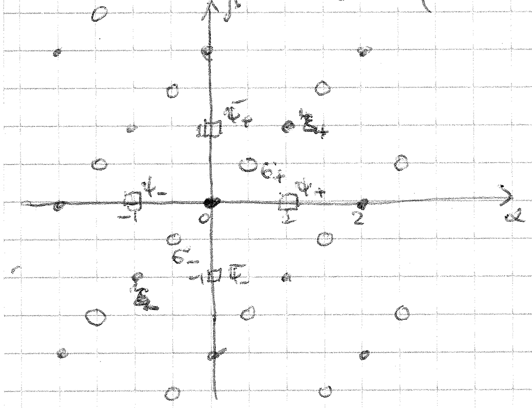
$$Z^{Dirac}_{torus}(\tau) = \frac12\Big(tr_{\mathcal{H}_A\otimes\mathcal{H}_A} + Str_{\mathcal{H}_A\otimes\mathcal{H}_A} + tr_{\mathcal{H}_P\otimes\mathcal{H}_P} + \underbrace{Str_{\mathcal{H}_P\otimes\mathcal{H}_P}}_{0}\Big)\, q^{-\frac{c}{24}+L_0}\, \bar q^{-\frac{\bar c}{24}+\bar L_0} =$$

$$= \frac12\Big((q\bar q)^{-\frac1{24}} \prod_{n\geq1}(1+q^{n-\frac12})^2(1+\bar q^{n-\frac12})^2 + (q\bar q)^{-\frac1{24}}\prod_{n\geq1}(1-q^{n-\frac12})^2(1-\bar q^{n-\frac12})^2 + (q\bar q)^{\frac1{12}}\cdot 4\prod_{n\geq1}(1+q^n)^2(1+\bar q^n)^2\Big) =$$

$$\underset{\substack{\text{Jacobi}\\\text{triple product}\\\text{identity}}}{=} \frac{1}{\eta(\tau)\eta(\bar\tau)}\cdot\frac12\sum_{(k,l)\in\mathbb{Z}^2}\Big(q^{k^2/2}\bar q^{l^2/2} + (-1)^{k+l} q^{k^2/2}\bar q^{l^2/2} + q^{\frac12(k+\frac12)^2}\bar q^{\frac12(l+\frac12)^2} + \underbrace{(-1)^{k+l} q^{\frac12(k+\frac12)^2}\bar q^{\frac12(l+\frac12)^2}}_{\substack{\text{sums up to zero,}\\\text{as it changes sign}\\\text{under } k\to -k-1}}\Big)$$

$$= \frac{1}{\eta(\tau)\eta(\bar\tau)}\sum_{(e,m)\in\mathbb{Z}^2} q^{\frac12(\frac e2+m)^2}\bar q^{\frac12(\frac e2-m)^2} \boxed{= Z^{\text{Compactified boson},\,R=2}_{torus}(\tau)}$$

More precisely: $(\mathcal{H}^{Dirac})^{even} \simeq \mathcal{H}^{comp.\ boson,\, R=2}$ as a $Vir\oplus\overline{Vir}$ - module



• - lattice $\lambda_1 \subset \mathbb{R}^2$ (e even, m any)

o - lattice $\lambda_2 \subset \mathbb{R}^2$ (e odd, m any)

$$\left(\mathcal{H}^{real}_A \otimes \mathcal{H}^{real}_A\right)^{even} \simeq \bigoplus_{(\alpha,\beta)\in\lambda_1} V^{\overline{Heis\otimes Heis}}_{(\alpha,\beta)}$$

$$\left(\mathcal{H}^{real}_P \otimes \mathcal{H}^{real}_P\right)^{even} \simeq \bigoplus_{(\alpha,\beta)\in\lambda_2} V^{\overline{Heis\otimes Heis}}_{(\alpha,\beta)}$$

Explicit correspondence of fields

$\hat\psi^\pm(z) = \,:e^{\pm i\hat\chi(z)}:$ - $(\pm1,0)$ chiral vertex operator [but there are no such fields in $R=2$ comp. boson theory!

↑ chiral part of boson

but even composites of $\psi^\pm$ do correspond to some fields]

or one can write $\psi_1 = \sqrt2 :\cos\chi(z):$

$\psi_2 = \sqrt2 :\sin\chi(z):$

E.g. ~~$\varepsilon_\pm(z,\bar z) = \varepsilon^1_4(z,\bar z)\,\varepsilon^2(z,\bar z)$~~

Dirac $U(1)$ current

$j = :\psi^+(z)\psi^-(z): = -i\psi^1\psi^2 \;\underset{\text{in terms of boson}}{\Big|}\; i\partial\varphi(z)$ · Heisenberg current

Stress-energy

$T(z) = \frac{1}{2}:j(z)j(z): = \frac{1}{2}:\psi^+\psi^-\psi^+\psi^-: = -\frac{1}{2}:\psi^1\partial\psi^1 + \psi^2\partial\psi^2: = -\frac{1}{2}:\partial\varphi\partial\varphi:$

Twist fields   energy field $\varepsilon(z,\bar z) = j\bar j = \psi^1\bar\psi^1\psi^2\bar\psi^2 = -\partial\varphi\bar\partial\varphi$

$$\sigma^{\pm} := \frac{\sigma^1(z,\bar z)\sigma^2(z,\bar z) \pm i\mu^1(z,\bar z)\mu^2(z,\bar z)}{\sqrt{2}} = :e^{\pm\frac{i}{2}\hat\varphi(z,\bar z)}: = V_{(\pm\frac{1}{2},\pm\frac{1}{2})}(z,\bar z) \text{ - vertex operator}$$

or

$$\sigma^1\sigma^2 = \sqrt{2}:\cos\frac{\varphi}{2}:$$
$$\mu^1\mu^2 = \sqrt{2}:\sin\frac{\varphi}{2}:$$

Correlators of real fermion theory via bosonization

Correspondences $\underset{\substack{\uparrow\\ \text{some field in real fermion theory}}}{\Phi^1(z,\bar z)\Phi^2(z,\bar z)} = \underset{\substack{\uparrow\\ \text{some field in boson theory}}}{X(z,\bar z)}$

lead to relations for correlators of form

$$\left(\langle\Phi_{(1)}\cdots\Phi_{(n)}\rangle_{\text{real fermion}}\right)^2 = \langle X_{(1)}\cdots X_{(n)}\rangle_{\text{boson}}$$

Examples   1) $\Phi = \psi$ - fermion field itself

$$\left(\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\right)^2 = \langle\psi(z_1)\cdots\psi(z_{2n})\rangle^2_{\text{fermion}} = \langle\partial\varphi(z_1)\cdots\partial\varphi(z_{2n})\rangle_{\text{boson}} = \frac{1}{2^n n!}\sum_{\pi\in S_{2n}}(z_{\pi(1)}-z_{\pi(2)})^{-2}\cdots(z_{\pi(2n-1)}-z_{\pi(2n)})^{-2}$$

e.g. for $n=2$: $\langle\psi_1\psi_2\psi_3\psi_4\rangle^2 = \langle\partial\varphi_1\partial\varphi_2\partial\varphi_3\partial\varphi_4\rangle = \frac{1}{(z_1-z_2)^2(z_3-z_4)^2} + \frac{1}{(z_1-z_3)^2(z_2-z_4)^2} + \frac{1}{(z_1-z_4)^2(z_2-z_3)^2}$

$$\overset{\shortparallel}{\left(\frac{1}{z_1-z_2}\cdot\frac{1}{z_3-z_4} - \frac{1}{z_1-z_3}\cdot\frac{1}{z_2-z_4} + \frac{1}{z_1-z_4}\cdot\frac{1}{z_2-z_3}\right)^2}$$

2) $\Phi = \sigma$ - twist field [or spin-field in terms of Ising]

$$\langle\sigma(z_1,\bar z_1)\cdots\sigma(z_n,\bar z_n)\rangle^2_{\text{fermion}} = 2^{n/2}\left\langle\cos\frac{\varphi(z_1,\bar z_1)}{2}\cdots\cos\frac{\varphi(z_n,\bar z_n)}{2}\right\rangle_{\text{boson}} =$$

$$= 2^{-n/2}\left\langle\left(V_{\frac{1}{2}}(z_1,\bar z_1)+V_{-\frac{1}{2}}(z_1,\bar z_1)\right)\cdots\left(V_{\frac{1}{2}}(z_n,\bar z_n)+V_{-\frac{1}{2}}(z_n,\bar z_n)\right)\right\rangle_{\text{boson}} = \quad \text{(we know correlators of vertex operators)}$$

$$= 2^{-n/2}\sum_{\substack{k_1,\ldots,k_n\in\{+1,-1\}\\ \text{s.t. } k_1+\ldots+k_n=0}}\prod_{1\le i<j\le n}|z_i - z_j|^{k_ik_j/2}$$

e.g. $n=2$: $\langle\sigma(z_1,\bar z_1)\,\sigma(z_2,\bar z_2)\rangle = |z_1-z_2|^{-1/4}$ (- spin-spin corr. in critical Ising)

<u>Ex</u>: recover the result $\langle\psi(z)\psi(w)\rangle_P = \frac{1}{2}\frac{(z/w)^{1/2}+(w/z)^{1/2}}{z-w}$ from bosonization.

Idea: 1) $\langle\psi(z)\psi(w)\rangle_P = \lim_{\substack{u\to\infty\\ v\to 0}} u^{1/8}\langle\sigma(u)\psi(z)\psi(w)\sigma(0)\rangle_{\text{chiral fermion}}$ ($\uparrow$ $2h_\sigma$)

2) $\langle\sigma(u)\bar\sigma(\bar u)\psi(z)\bar\psi(\bar z)\psi(w)\bar\psi(\bar w)\sigma(v)\bar\sigma(\bar v)\rangle^2_{\text{real fermion}} = 2\left\langle\cos\frac{\varphi(u)}{2}\cdot(\partial\varphi\bar\partial\varphi)_z(\partial\varphi\bar\partial\varphi)_w\cos\frac{\varphi(v)}{2}\right\rangle_{\text{boson}}$

$\shortparallel\; |\langle\sigma(u)\psi(z)\psi(w)\sigma(0)\rangle|^4$

"bc system" (or "reparametrization ghosts")

$$S = \frac{1}{2\pi}\int d^2x\, b_{\mu\nu}\partial^\mu c^\nu$$

$b_{\mu\nu}$ - odd traceless symmetric (tensor)
c^ν - odd (vector field)

In complex coordinates: $S \propto \int dz\,d\bar z\,(\underbrace{b}_{b_{zz}}\bar\partial \underbrace{c}_{c^z} + \bar b\,\partial\bar c)$

Propagator: $\langle b(z)\,c(w)\rangle = \frac{1}{z-w}$, basic OPE: $b(z)\,c(w) \sim \frac{1}{z-w}$ + reg.

Stress-energy: $\hat T(z) = \,:2\partial\hat c(z)\,\hat b(z) + \hat c(z)\,\partial\hat b(z):$, $\bar T = \dots$

OPE Tb, Tc, TT $\xrightarrow{\text{using Wick's thm}}$
- b, c are primary with conf. weight $(h,\bar h) = (2,0)$ and $(-1,0)$ respectively
- central charge is $(c,\bar c) = (-26,-26)$

— Exercise!

Exercise: describe the space of states

- What does this have to do with (Polyakov) bosonic string theory? (Rough idea)

We want
$$\underbrace{Z_{string}(\Sigma,\mathbb{R}^D)}_{\text{as a topological surface}} = \int_{Met(\Sigma)} \mathcal{D}g \int_{Maps(\Sigma,\mathbb{R}^D)} \mathcal{D}X^i\; e^{-S_{Polyakov}(g,X)} \propto$$

$S_{Polyakov} = \int_\Sigma \sqrt{g}\,d^2x\,(g^{-1})^{\mu\nu}(x)\,\partial_\mu X^i\,\partial_\nu X^i$ = action for D free bosons on (Σ, g)

$$\propto \int_{(Met(\Sigma)\times Maps(\Sigma,\mathbb{R}^D))/Diff(\Sigma)} e^{-S_{Polyakov}} \overset{\text{write } g = dz\,d\bar z\cdot e^{2\varphi}}{=} \int_{\left(\left(\substack{\text{conformal}\\\text{structures}\\\text{on }\Sigma}\right)\times\left(\substack{\text{Weyl}\\\text{factors}\\\Omega = e^{2\varphi}}\right)\times Maps\right)/Diff(\Sigma)} e^{-S_{Polyakov}} =$$

quotient over $Diff(\Sigma)$

φ: "Liouville field"

$$\overset{\text{Faddeev-Popov gauge-fixing}}{=} \int_{\mathcal{M}_\Sigma} \int_{\substack{\text{Liouv. field}\\ \varphi\in C^\infty(\Sigma)}} \underbrace{\int \mathcal{D}b\,\mathcal{D}c\; e^{-S_{bc}}}_{\text{Gaussian }\int\text{ computing FP Jacobian}} \int_{Maps(\Sigma,\mathbb{R}^D)} e^{-S_{Polyakov}} =$$

$\mathcal{M}_\Sigma$: moduli of conf. str.

(arising from the choice of section of $\{\text{conf. str.}\}/Diff$)

$$= \int_{\mathcal{M}_\Sigma} d\gamma \int \mathcal{D}\varphi\; Z_{CFT}\left(\substack{D\text{ bosons}\\ +\,bc\text{ system}},\gamma\right)\cdot e^{\frac{D-26}{24}S_{Liouville}(\varphi)}$$

$D-26$: central charge of CFT consisting of D bosons and one bc system

if $D = D_{crit} = 26$, then integrand is independent of φ.

A modification of bc system: $T = \,:\partial c\cdot b:$, then $h_b = +1$, $h_c = 0$, $c = -2$ "simple ghosts"

More generally: $T = \,:(1+j)\,\partial c\cdot b + j\,c\cdot\partial b:$, then $h_b = 1+j$, $h_c = -j$, $c = -12j^2+12j-2$
$j \in \mathbb{R}$ - parameter