CFT Lecture 10, 09.05.11

Today: Stress-energy tensor

Last time: in Lagr. class. F.T. for a field transformation

$\Phi^a(x) \longmapsto \Phi^a(x) + f^a(\Phi(x))$ we have

$$\left(\begin{array}{c} \delta_f S_N = 0 \\ \text{for any } N \subset \Sigma \end{array} \right) \Longrightarrow \left(\begin{array}{l} \text{the Noether current associated } \underline{J}_f := \underline{P}_a f^a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} f^a \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad J^\mu(x)\partial_\mu \\ \text{is "conserved", i.e. } \operatorname{div} \underline{J}_f \underset{\text{mod } \mathcal{E}\text{-}L}{\sim} 0 \end{array} \right)$$

$$\boxed{\begin{array}{l} \text{Notation: } \underline{J} = J^\mu(x)\partial_\mu \\ \underline{P}_a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \partial_\mu \\ \operatorname{div} \underline{J} = \nabla_\mu J^\mu = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g}\, J^\mu) \\ \qquad = d^*(\underline{J})_\flat \end{array}}$$

Example: complex scalar field ("charged")

$\mathcal{F}_\Sigma = \mathrm{Maps}(\Sigma, \mathbb{C})$

can put any $U(|\Phi|^2)$ here

$$S = \int_\Sigma \sqrt{g}\, dx \left(\frac{1}{2} \partial_\mu \Phi \cdot \partial^\mu \bar{\Phi} + \frac{m^2}{2} |\Phi|^2 \right) \in \mathbb{R}$$

$$\mathcal{E}\text{-}L \text{ eq.: } \begin{cases} (\Delta - m^2)\Phi = 0 \\ (\Delta - m^2)\bar{\Phi} = 0 \end{cases}$$

symmetry - phase rotation $F_\alpha: \begin{cases} \Phi(x) \longmapsto e^{i\alpha}\Phi(x) \\ \bar{\Phi}(x) \longmapsto e^{-i\alpha}\bar{\Phi}(x) \end{cases}$

infinitesimally: $\delta\Phi = i\Phi$, $\delta\bar{\Phi} = -i\bar{\Phi}$

Noether current: $J^\mu = \frac{1}{2i}(\bar{\Phi}\partial^\mu \Phi - \Phi \partial^\mu \bar{\Phi}) = \mathrm{Im}(\bar{\Phi}\partial^\mu \Phi)$

or: $\underline{J} = (\mathrm{Im}\, \bar{\Phi} d\Phi)^\#$

conservation: $\operatorname{div} \underline{J} = \frac{1}{2i}(\bar{\Phi}\Delta\Phi - \Phi\Delta\bar{\Phi}) \underset{\mathcal{E}\text{-}L}{\sim} 0$

Rem: Noether current $\underline{J}$ can be considered modulo equivalence

$\underline{J} \sim \underline{J} + \operatorname{div} \underline{\underline{B}}$ for any field-dependent bi-vector $\underline{\underline{B}}$ on Σ

- then $\operatorname{div}\operatorname{div} \underline{\underline{B}} \equiv 0$ implies $(\operatorname{div} \underline{J} \sim 0) \Rightarrow (\operatorname{div}(\underline{J} + \operatorname{div}\underline{\underline{B}}) \sim 0)$

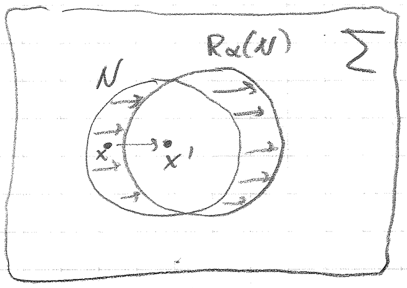
in coords: $J^\mu \sim J^\mu + \nabla_\nu B^{\nu\mu}$ for any skew-symmetric $B^{\mu\nu} = -B^{\nu\mu}$

Mixed (source-target) symmetry

$$\left.\begin{aligned} x \mapsto x' &= R_\alpha(x) \\ \Phi(x) \mapsto \Phi'(x') &= F_\alpha(\Phi(x)) \end{aligned}\right\} \quad \text{or} \quad \Phi(x) \longmapsto F_\alpha(\Phi(R_\alpha^{-1}(x)))$$

infinitesimally: $x^\mu \longmapsto x^\mu + r^\mu(x)$

$$\Phi^a(x) \longmapsto \Phi^a(x) + f^a(\Phi(x)) - r^\mu \partial_\mu \Phi(x)$$



for any $N \subset \Sigma$ we have:

$$\underset{\substack{\parallel \\ \left.\frac{d}{d\alpha}\right|_{\alpha=0} S_{R_\alpha(N)}[F_\alpha(\Phi(R_\alpha^{-1}(x)))]}}{\delta_{f,r} S_N} = \delta^{\text{integration domain}} S_N + \delta^{\text{fields}} S_N$$

flux of $\mathcal{L}\underline{r}$ through ∂N

$\underset{\mathcal{E}\text{-}L}{\sim}$ flux of $\underline{P}_a(f^a - r^\mu \partial_\mu \Phi^a)$ through ∂N

Thus

$$\left(\begin{array}{l} \delta_{f,r} S_N = 0 \\ \forall\, N \subset \Sigma \\ \quad\ \text{submersed} \end{array}\right) \Rightarrow \left(\begin{array}{l} \text{the associated Noether current} \\ \boxed{\underline{J}_{f,r} = \mathcal{L}\underline{r} + \underline{P}_a (f^a - r^\mu \partial_\mu \Phi^a)} \\ \text{is conserved:} \quad \operatorname{div} \underline{J}_{f,r} \underset{\mathcal{E}\text{-}L}{\sim} 0 \end{array}\right)$$

Explicitly: $$J^\mu_{f,r}(x) = r^\nu \left(\mathcal{L} \delta^\mu_\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \partial_\nu \Phi^a \right) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} f^a$$

Examples

- $\Sigma = \mathbb{R}^{p,q}$ (flat), source-translations $x^\mu \mapsto x^\mu + a^\mu$ are symmetries (since isometries are automatically symmetries due to covariance)

Noether current:

$$J^\mu_{x \mapsto x + \underline{a}}(x) = -T^\mu{}_\nu(x)\, a^\nu$$

where $$\boxed{T^\mu{}_\nu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \partial_\nu \Phi^a - \mathcal{L}\, \delta^\mu_\nu}$$ - "canonical" stress-energy tensor

or: $$T^\mu{}_\nu\, \partial_\mu \otimes dx^\nu = \underline{P}_a \otimes d\Phi^a - \mathcal{L} \cdot \mathrm{id}$$
$$\rotatebox{90}{$\in$}$$
$$\Gamma(\Sigma, \mathrm{End}(T\Sigma)) \otimes \mathrm{Fun}(\mathcal{F}_\Sigma)$$

conservation: $\partial_\mu T^\mu{}_\nu \underset{\mathcal{E}\text{-}L}{\sim} 0$

$$P_\mu(x^0) = \int_{\left(\substack{x^0\text{-fixed} \\ \text{slice}}\right) \subset \Sigma} T^0{}_\mu(x)$$ - conserved Energy-Momentum (co-)vector $\longleftarrow$ i.e. $\frac{d}{dx^0} P_\mu(x^0) \underset{\mathcal{E}\text{-}L}{\sim} 0$

interpretation: $T^0{}_0$ - energy density, $T^0{}_i$ - momentum density, $T^i{}_0$ - energy flux in direction ∂_i, $T^i{}_j$ - momentum flux

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Ex: • scalar field, $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\cdot\partial^\mu\phi + \frac{m^2}{2}\phi^2$

$$T^\mu{}_\nu = \partial^\mu\phi\cdot\partial_\nu\phi - \delta^\mu{}_\nu\left(\tfrac{1}{2}\partial_\lambda\phi\cdot\partial^\lambda\phi + \tfrac{m^2}{2}\phi^2\right)$$

• E-M field, $\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$

$$T^\mu{}_\nu = F^{\mu\lambda}\partial_\nu A_\lambda - \tfrac{1}{4}\delta^\mu_\nu F_{\lambda\rho}F^{\lambda\rho}$$

non-gauge-invariant, $T^{\mu\nu} \neq T^{\nu\mu}$
- there is a better version, Hamilton's stress-energy tensor (later)

(Exercise: compute $T^\mu{}_\nu$ for a complex scalar field, in particular for ϕ = plane-wave $e^{i(k,x)}$ with $k^2 = -m^2$)

• $\Sigma = \mathbb{R}^{p,q}$, $SO(p,q)$-rotations are automatically symmetries (due to covariance)

$\omega \in so(p,q)$: $\quad x^\mu \longmapsto x^\mu + \omega^\mu{}_\nu x^\nu$, $\quad \omega_{\mu\nu} = -\omega_{\nu\mu}$

$$\phi^a \longmapsto \phi^a + \underbrace{f^a_\omega(\phi)}_{\text{inf. action of } so(p,q) \text{ on } X}$$

then $j^\mu_\omega = -T^{\mu\nu}(x)\,\omega_{\nu\rho}x^\rho + \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}f^a_\omega$ is conserved

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• $\Sigma = \mathbb{R}^{p,q}$, $p+q=n$, massless scalar field $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\cdot\partial^\mu\phi$

dilatation: $x^\mu \longmapsto e^\alpha x^\mu$

$\phi \longmapsto e^{\frac{2-n}{2}\alpha}\phi$

$$j^\mu_{dil.} = \tfrac{1}{2}\partial_\nu\phi\cdot\partial^\nu\phi\cdot x^\mu - \partial^\mu\phi\cdot\partial_\nu\phi\cdot x^\nu + \tfrac{2-n}{2}\partial^\mu\phi\cdot\phi$$

Stress-energy tensor

- canonical - defined only for flat space-times, not necessarily symmetric
- Bellinfante tensor $T^{\mu\nu}_B = T^{\mu\nu}_{can} + \partial_\lambda C^{\lambda\mu\nu}$ for some $C^{\lambda\mu\nu} = -C^{\mu\lambda\nu}$ s.t. $T^{\mu\nu}_B = -T^{\nu\mu}_B$ (it is not always possible to define such $T^{\mu\nu}_B$)

- Hilbert stress-energy tensor

$$T^{\mu\nu}(x) := -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}(x)} \quad \text{i.e. } \delta_g S = -\tfrac{1}{2}\int_\Sigma \sqrt{g}\,dx\; T^{\mu\nu}(x)\,\delta g_{\mu\nu}(x)$$

- manifestly symmetric, $T^{\mu\nu} = T^{\nu\mu}$
- conserved, $\nabla_\mu T^{\mu\nu} \underset{E\text{-}L}{\sim} 0$ due to covariance: $\dfrac{d}{d\alpha}\Big|_{\alpha=0} S_{R_\alpha(N),(R_\alpha^{-1})^* g}\left((R_\alpha^{-1})^*\phi\right) = 0$

$$= -\tfrac{1}{2}\int_N \sqrt{g}\,T^{\mu\nu}(\nabla_\mu r_\nu + \nabla_\nu r_\mu) + \underbrace{\delta_r \overset{(g\text{ fixed})}{S_N}}_{\underset{E\text{-}L}{\sim}\text{ surface term}}$$

- $\delta_r^{(g\,\text{fixed})} S_N = \int_N \sqrt{g}\,dx \cdot T^{\mu\nu} \nabla_\mu r_\nu$

$$\overset{\shortparallel}{\frac{d}{d\alpha}\Big|_{\alpha=0} S_{R_\alpha(N),\, g|_N}\left((R_\alpha^{-1})^* \phi\right)}$$

$\Rightarrow$ if r is a source-symmetry i.e. $\delta_r^{(g\,\text{fixed})} S_N = 0$ then $j^\mu = T^\mu{}_\nu r^\nu$ is conserved

Rem: thus $T^\mu{}_\nu \partial_\mu \otimes dx^\nu \in \Gamma(\Sigma, \mathrm{End}(T\Sigma))$ can be viewed as a tensor, transforming source-symmetries $\underline{r}$ into associated conserved currents $\underline{j}$
$\qquad\qquad\qquad\qquad\overset{\shortparallel}{T \circ \underline{r}}$

- for a scalar field, or more generally for a sigma-model with $\mathcal{F}_\Sigma = \mathrm{Maps}(\Sigma, X)$ (with $\mathrm{Diff}(\Sigma)$ not acting on X)

one has $T^{\mu\nu}_{Hilbert} = T^{\mu\nu}_{canonical}$ for flat Σ.

Exercise: show that for a scalar field
$$T^{\mu\nu}_{Hilb} = \partial^\mu\phi \cdot \partial^\nu\phi - (g^{-1})^{\mu\nu}\left(\tfrac{1}{2}\partial_\lambda\phi \cdot \partial^\lambda\phi + \tfrac{m^2}{2}\phi^2\right)$$
and for E-M field:
$$T^{\mu\nu}_{Hilb} = F^{\mu\lambda} F^\nu{}_\lambda - \tfrac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$

For conformally-invariant theories

$S_{\Sigma,g}$ is Weyl-invariant, i.e. $S_{\Sigma,g}[\Phi] = S_{\Sigma,\Omega\cdot g}[\Phi] \quad \forall\, \Omega(x) > 0$

$\Rightarrow \quad 0 = \delta_{g \mapsto (1+\omega)g} S = -\frac{1}{2}\int \sqrt{g}\,dx\, T^{\mu\nu}(x)\, g_{\mu\nu}(x)\, \omega(x) \quad \forall\, \omega(x)$

$\Leftrightarrow \quad T^\mu{}_\mu = T^{\mu\nu}(x)\, g_{\mu\nu}(x) \equiv 0$

i.e. (theory is conformally-invariant) $\Leftrightarrow$ (Hilbert stress-energy tensor is traceless)

- Weyl-invariance $\underset{\text{due to Covariance}}{\Leftrightarrow}$ $\delta_r^{(g\,\text{fixed})} S_N = 0$ for any $r \in \mathrm{conf}(N, g)$

$\Rightarrow$ for any conf. v.f. $r^\mu \partial_\mu$, $\quad j^\mu_{\underline{r}} = T^\mu{}_\nu r^\nu$ is a conserved current

- $T^{\mu\nu}$ depends on metric and Weyl transf. act on $T^{\mu\nu}$ as $\qquad g_{\mu\nu} \longmapsto \Omega \cdot g_{\mu\nu}$

$\qquad\qquad T^{\mu\nu} \longmapsto \Omega^{-\frac{D}{2}-1}\, T^{\mu\nu}$

⟨ therefore, for 2-dimensional CFT, $T_{\mu\nu}$ is Weyl-invariant ⟩