CFT Lecture 11

11.05.03

Today: - some remarks on classical CFTs
- quantum free scalar field

Reminder: Hilbert Stress-Energy tensor:

- $T^{\mu\nu}(x) := -\frac{2}{\sqrt{g}} \frac{\delta S'_{\Sigma,g}[\Phi]}{\delta g_{\mu\nu}(x)}$

- properties: ① symmetric $T^{\mu\nu} = T^{\nu\mu}$, ② conserved $\nabla_\mu T^{\mu\nu} \underset{E-L}{\sim} 0$ (due to covariance),

③ transforms source-symmetries $\Gamma \in Vect(\Sigma)$ s.t. $\overset{(g\text{ fixed})}{\delta_\Gamma} S_N = 0$ into conserved currents by $j^\mu_\Gamma = T^\mu{}_\nu \Gamma^\nu$,

④ coincides with $T^{\mu\nu}_{canonical}$ for σ-models, for flat Σ

- for scalar field: $T^{\mu\nu} = \partial^\mu\Phi \cdot \partial^\nu\Phi - (g^{-1})^{\mu\nu}\left(\frac{1}{2}\partial_\lambda\Phi\cdot\partial^\lambda\Phi + \frac{m^2}{2}\Phi^2\right)$
- for E-M field: $T^{\mu\nu} = g_{\alpha\beta} F^{\mu\alpha}F^{\nu\beta} - \frac{1}{4}(g^{-1})^{\mu\nu} F_{\alpha\beta}F^{\alpha\beta}$

Class. CFT

def Call a Lagr. F.T. conformal if

$$S_{\Sigma,g}[\Phi] = S_{\Sigma,\Omega\cdot g}[\Phi] \quad \forall \underset{\Omega>0}{\Omega \in C^\infty(\Sigma)}$$

($\underset{covar.}{\Leftrightarrow}$ $conf(\Sigma,g)$ is source-symmetry of action S')

(Weyl-invariance of S') $\Leftrightarrow$ $\delta_{g\mapsto(1+\omega)g} S' = 0$

$$\overset{\shortparallel}{-\frac{1}{2}\int\sqrt{g}\,dx\; T^{\mu\nu}(x)\, g_{\mu\nu}(x)\cdot\omega(x)} \quad \forall \omega(x)$$

$$\Updownarrow$$

$$T^\mu{}_\mu(x) = T^{\mu\nu}(x)\, g_{\mu\nu}(x) = 0$$

So: <u>a F.T. is conformal iff the stress-energy tensor is traceless</u>

Rem: a F.T. is topological (i.e. $S'_{\Sigma,g}[\Phi]$ does not depend on g) iff $T^{\mu\nu} \equiv 0$

Ex: 3D Chern-Simons: $\dim\Sigma = 3$, $\mathcal{F}_\Sigma = \mathfrak{g} \otimes \Omega^0(\Sigma)$ (simple Lie algebra)

$$S_\Sigma[A] = tr\int\left(\frac{1}{2}A\wedge dA + \frac{1}{3}A\wedge A\wedge A\right)$$

- to a conf. v.f. $\underline{r} \in \mathrm{conf}(\Sigma, g)$ the associated Noether current is $J^\mu_{\underline{r}} = T^\mu{}_\nu r^\nu$ (conservation of J follows automatically from conservation, symm., tracelessness of $T^{\mu\nu}$ and from $\nabla_\mu r_\nu + \nabla_\nu r_\mu = 2\omega\, g_{\mu\nu}$ - conf. property)

- $T^{\mu\nu}$ depends on background metric and transforms under Weyl transf. $g \mapsto \Omega \cdot g$ as $T^{\mu\nu} \mapsto \Omega^{-\frac{n}{2}-1} T^{\mu\nu}$

<u>Examples</u> ① scalar field $T^{\mu\nu} = \partial^\mu\phi \cdot \partial^\nu\phi - (g^{-1})^{\mu\nu}\left(\frac{1}{2}\partial_\lambda\phi \cdot \partial^\lambda\phi + \frac{m^2}{2}\phi^2\right)$

trace: $T^\mu{}_\mu = \frac{2-n}{2}\partial_\mu\phi \cdot \partial^\mu\phi - n \cdot \frac{m^2}{2}\phi^2$

so: $T^\mu{}_\mu \equiv 0 \iff n=2$ and $m=0$

i.e. only massless 2D scalar field is conformal (cf. dilatation symmetry for massless scalar in any dimension)

Explicit check of Weyl-invariance of S:

$$S_{\Sigma,g}[\Phi] = \int_\Sigma dx\, \underbrace{(\det g)^{1/2}}_{\sim\Omega^{\frac{n}{2}}} \cdot \left(\underbrace{(g^{-1})^{\mu\nu}}_{\sim\Omega^{-1}}\frac{1}{2}\partial_\mu\phi \cdot \partial_\nu\phi + \underbrace{\frac{m^2}{2}\phi^2}_{\sim\Omega^0}\right)$$

$\Rightarrow$ Ω-dependence cancels for $m=0$, $n=2$

② E-M field $T^{\mu\nu} = g_{\alpha\beta}F^{\mu\alpha}F^{\nu\beta} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}(g^{-1})^{\mu\nu}$

trace: $T^\mu{}_\mu = \frac{4-n}{4}F_{\alpha\beta}F^{\alpha\beta}$

so $T^\mu{}_\mu \equiv 0 \iff n=4$

explicitly:
$$S_{\Sigma,g}[A] = \frac{1}{4}\int_\Sigma dx\, \underbrace{(\det g)^{1/2}}_{\sim\Omega^{n/2}} \underbrace{(g^{-1})^{\mu\alpha}}_{\sim\Omega^{-1}}\underbrace{(g^{-1})^{\nu\beta}}_{\sim\Omega^{-1}} F_{\mu\nu}F_{\alpha\beta}$$

$\Rightarrow$ Ω-dep. cancels for $n=4$

2D class. CFT on $\mathbb{C} \simeq \mathbb{R}^2$

Sym. + tracelessness: $T_{\mu\nu} = \begin{pmatrix} T_{11} & T_{12} \\ T_{12} & -T_{11} \end{pmatrix}$, conservation: $\partial^\mu T_{\mu\nu} = 0 \iff \begin{cases} \partial_1 T_{11} + \partial_2 T_{12} \sim 0 \\ \partial_1 T_{12} - \partial_2 T_{11} \sim 0 \end{cases}$

in complex coords $z = x+iy$, $\bar{z} = x-iy$ we have:

$$T_{\mu\nu}dx^\mu dx^\nu = T_{11}\underbrace{\left((dx)^2 - (dy)^2\right)}_{\frac{1}{2}((dz)^2+(d\bar z)^2)} + T_{12}\underbrace{2\,dx\,dy}_{\frac{1}{2i}((dz)^2-(d\bar z)^2)} = \underbrace{\frac{T_{11} - iT_{12}}{2}}_{T_{zz}}(dz)^2 + \underbrace{\frac{T_{11}+iT_{12}}{2}}_{T_{\bar z\bar z}}(d\bar z)^2$$

- no mixed $dz\,d\bar z$ term!

<u>Rem</u> in dimension 2, $T_{\mu\nu}$ is Weyl-invariant

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Conservation: $\partial_{\bar z} T_{zz} \underset{\mathcal{E}\text{-}L}{\sim} 0$

$\partial_z T_{\bar z \bar z} \underset{\mathcal{E}\text{-}L}{\sim} 0$

$$\left\langle \text{check: } \underbrace{\frac{\partial_1 + i\partial_2}{2}}_{\partial_{\bar z}} \underbrace{\frac{T_{11} - iT_{12}}{2}}_{T_{zz}} = \frac{1}{4}(\partial_1 T_{11} + \partial_2 T_{12}) + \frac{i}{4}(\partial_2 T_{11} - \partial_1 T_{12}) \underset{\mathcal{E}\text{-}L}{\sim} 0 \right\rangle$$

Standard notation: $T := T_{zz}, \quad \bar T := T_{\bar z \bar z}$

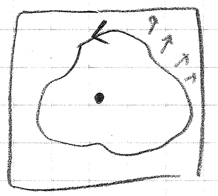
So: $T_{\mu\nu}\, dx^\mu dx^\nu = T (dz)^2 + \bar T (d\bar z)^2$,

$\bar\partial T \underset{\mathcal{E}\text{-}L}{\sim} 0, \quad \partial \bar T \underset{\mathcal{E}\text{-}L}{\sim} 0$

For a conf. v.f. $\underline{\varepsilon} = \varepsilon \partial + \bar\varepsilon \bar\partial$, the assoc. Noether current is

$\underline{J} = T\varepsilon\bar\partial + \bar T \bar\varepsilon \partial$ (ε-holomorphic)

Noether charges assoc. to conf. symmetry on $\mathbb{C}\setminus\{0\}$:



$C_{\underline{\varepsilon}} = (\text{Flux of } T\varepsilon\bar\partial + \bar T\bar\varepsilon\partial \text{ through } \gamma) =$

γ - a loop in $\mathbb{C}\setminus\{0\}$ with winding number $+1$

$$= \oint_\gamma \iota_{\underline{J}} \underbrace{\mathrm{vol}}_{\frac{i}{2} dz\wedge d\bar z} = \frac{1}{2i}\oint_\gamma T\varepsilon\, dz - \frac{1}{2i}\oint_\gamma \bar T \bar\varepsilon\, d\bar z = \mathrm{Im} \oint_\gamma T\varepsilon\, dz$$

on a sol. of $\mathcal{E}$-L, C_ε does not depend on choice of γ due to Cauchy thm.

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massless scalar field on $\mathbb{C}$

$$S = \int dx\,dy\ \frac{1}{2}(g^{-1})^{\mu\nu}\partial_\mu\phi\cdot\partial_\nu\phi = \int \frac{i}{2} dz\wedge d\bar z\ \partial\phi\cdot\bar\partial\phi$$

$\mathcal{E}$-L eq: $\Delta\phi = 0 \iff \partial\bar\partial\phi = 0$ i.e. ϕ is harmonic

Rem: equation $0 = \Delta\phi = \left(\frac{1}{\sqrt{g}}\partial_\mu \underbrace{\sqrt{g}\,(g^{-1})^{\mu\nu}}_{\sim \Omega^{\frac{n}{2}-1}} \partial_\nu\right)\phi$ is Weyl-invariant in dimension 2.

Stress-energy tensor:

$$T = \partial\phi\cdot\partial\phi, \quad \bar T = \bar\partial\phi\cdot\bar\partial\phi$$

Quantum free scalar field

(quantum) Harmonic oscillator - reminder

Class. mech. system (Hamiltonian formalism):

phase space $\Phi = \mathbb{R}^2$, $\quad H = \frac{p^2}{2} + \alpha^2 \frac{x^2}{2}$ $\quad$ eq. of motion: $\dot{x} = p$, $\dot{p} = -\alpha^2 x$

α "frequency"

$\omega = dp \wedge dx$

$\{p, x\} = 1$

solutions: $x(t) = x_0 \cos \alpha(t-t_0)$, $p(t) = -\alpha x_0 \sin \alpha(t-t_0)$

canonical quantization:

$$\left.\begin{array}{l} x \longmapsto \hat{x} \\ p \longmapsto \hat{p} \end{array}\right\} \in \mathrm{End}(\mathcal{H}) \quad , \quad \hat{H} = \frac{\hat{p}^2}{2} + \alpha^2 \frac{\hat{x}^2}{2} \quad \text{(no ordering issue)}$$

$$[\hat{p}, \hat{x}] = -i\hbar$$

Schroedinger representation:

$$\mathcal{H} = L_2(\mathbb{R}) \quad , \quad \hat{x} : f(x) \longmapsto x \cdot f(x) \quad , \quad \hat{p} : f(x) \longmapsto -i\hbar \frac{\partial}{\partial x} f(x)$$

Spectral problem $\hat{H} f = E f$ can be solved explicitly

with $E_n = \hbar\alpha \cdot (n + \frac{1}{2})$, $f_n = C_n \cdot e^{-\frac{\alpha x^2}{2\hbar}} H_n\left(\sqrt{\frac{\alpha}{\hbar}}\, x\right)$, $n \geq 0$

C_n: normalization constant s.t. $(f_n, f_n) = 1$

H_n: Hermite polynomials $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$

Creation / annihilation operators

$$\left.\begin{array}{l} \hat{a} = \sqrt{\frac{\alpha}{2\hbar}} \left(\hat{x} + \frac{i}{\alpha} \hat{p}\right) \\ \hat{a}^+ = \sqrt{\frac{\alpha}{2\hbar}} \left(\hat{x} - \frac{i}{\alpha} \hat{p}\right) \end{array}\right\} \text{ or } \left.\begin{array}{l} \hat{x} = \sqrt{\frac{\hbar}{2\alpha}} \left(\hat{a}^+ + \hat{a}\right) \\ \hat{p} = i\sqrt{\frac{\hbar\alpha}{2}} \left(\hat{a}^+ - \hat{a}\right) \end{array}\right\} \quad , \quad \hat{H} = \hbar\omega\left(\hat{a}^+ \hat{a} + \frac{1}{2}\right)$$

then $[\hat{a}, \hat{a}^+] = 1$, $\quad [\hat{H}, \hat{a}] = -\hbar\alpha \hat{a}$, $\quad [\hat{H}, \hat{a}^+] = \hbar\alpha \hat{a}^+$

eigenstates of $\hat{H}$:

"vacuum" $|0\rangle \in \mathcal{H}$ s.t. $\hat{a}|0\rangle = 0 \Rightarrow \hat{H}|0\rangle = \frac{1}{2}\hbar\omega|0\rangle$

excited states $|n\rangle \in \mathcal{H}$, $|n\rangle = (\hat{a}^+)^n |0\rangle \Rightarrow \hat{H}|n\rangle = (n + \frac{1}{2})\hbar\omega|n\rangle$

$$\mathcal{H} = \mathrm{Span}_{\mathbb{C}}(|0\rangle, |1\rangle, |2\rangle, \dots)$$