Introduction to CFT, Lecture 5 (23.03.11)

Today: moduli of conformal (2,0) structures on 2D surfaces

Correction for previous lecture: the correct way to associate a conf. str. to a cx. str: pick any volume form $\sigma \in \Gamma(\Sigma, \Lambda^2 T^*\Sigma)$, consistent with orientation of Σ and set

$$g_x(\vec{u},\vec{v}) := \sigma_x(\vec{u}, J_x\vec{v})$$

Exercise: check that
1) conf. class $\gamma_J := g_J/\text{Weyl transf.}$ does not depend on σ
2) g_J is symmetric and non-degenerate
3) association $J \mapsto \gamma_J$ inverts the map $J \mapsto \gamma_J$ given last time

We saw that

$$\left\{\begin{matrix}\text{conf. str. } \gamma \\ \text{on } \Sigma\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{cx. str. } J \\ \text{on } \Sigma\end{matrix}\right\}$$

but also equivalences are the same:

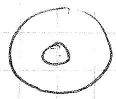
$$\left(\begin{matrix}\text{diffeo } (\Sigma,\gamma) \xrightarrow{\varphi} (\Sigma',\gamma') \\ \text{is a conf. equivalence}\end{matrix}\right) \iff \left(\begin{matrix}(\Sigma, J_\gamma) \xrightarrow{\varphi} (\Sigma', J_{\gamma'}) \\ \text{is a biholom. map}\end{matrix}\right)$$

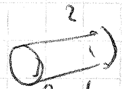
Uniformization theorem (Klein-Koebe-Poincaré)

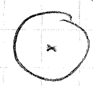
A Riemann surface $(\Sigma_{g,n,m},\gamma)$ is conf. equivalent to one of the following:

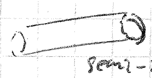
① $\mathbb{C}P^1$

② $\mathbb{C}$, $\mathbb{C}\setminus\{0\}$, annulus, disk without origin i.e. $D\setminus\{0\}$, torus $\mathbb{C}/\mathbb{Z}^2$









$\mathbb{C}\setminus\{0\}$ ≈ infinite cylinder $\sim \mathbb{C}/\mathbb{Z}$; annulus ≈ finite cylinder ~ strip $\{0 \le \mathrm{Im}\, z \le 1\}/\mathbb{Z}$; $D\setminus\{0\}$ ≈ semi-infinite cylinder $\sim \Pi_+/\mathbb{Z}$

③ Π_+/Γ

for some $\Gamma \subset PSL_2(\mathbb{R})$ — a discrete subgroup of the gr. of Möbius trans. of Π_+ ("Fuchsian group")

(all with standard cx / conf structure)

Rem Surface ① admits (unique) metric (representative of std. conf. str.) of curvature +1 — the Fubini-Study metric $\frac{4\,dz\,d\bar{z}}{(1+z\bar{z})^2}$ (normalization?)

surfaces ② admit flat metric, s.t. — a geodesic cannot run into puncture in finite time — boundaries are geodesics

surfaces ③ admit hyper metric of curvature −1 ("hyperbolic metric") coming from $g = \frac{4}{y^2}\left((dx)^2+(dy)^2\right)$ on Π_+; this metric is complete

• Terminology: ① elliptic surface(s)
② parabolic
③ hyperbolic ⟵ $\chi = 2-2g-n-m < 0$ (or a disk) ⟵ disk

<Rem: hyp. surfaces also admit non-complete flat metrics with ... ; hemi-sphere disk admits elliptic metric>

- Hyperbolic upper half-plane Π_+ with $g = \frac{1}{y^2}((dx)^2 + (dy)^2)$

has other (isometric) models:
- Poincaré disk $D = \{ z \in \mathbb{C} \mid |z| < 1 \}$, $g \overset{?}{=} \frac{4\, dz d\bar{z}}{(1 - z\bar{z})^2}$
- hyperboloid $\{ (x, y, t) \in \mathbb{R}^{2,1} \mid x^2 + y^2 - t^2 = -1, \ t > 0 \}$ with metric induced from that one on $\mathbb{R}^{2,1}$

Exercise: check that the induced metric is hyperbolic

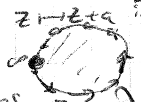
- for Π_+

$\left\{ \begin{matrix} \text{isometry} \\ \text{group} \\ \text{(for hyperb. metric)} \end{matrix} \right\} = \left\{ \begin{matrix} \text{group of} \\ \text{conformal} \\ \text{autom.} \end{matrix} \right\}$ (imposing D as compactification) $= PSL_2(\mathbb{R})$ ← Möbius transf. of Π_+

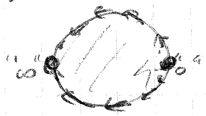
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longmapsto \frac{az+b}{cz+d}$, $ad - bc = 1$

- elements of $PSL_2(\mathbb{R})$ [$\mu = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$] are classified into:
  - elliptic with $|\mathrm{tr}\, \mu| < 2$, e.g. rotations of $D \simeq \Pi_+$; they have 1 fixed pt. inside Π_+

  - parabolic with $|\mathrm{tr}\, \mu| = 2$, e.g. real translations $z \mapsto z + a$ in Π_+; they have unique fixed pt. on the bdry of Π_+



  - hyperbolic with $|\mathrm{tr}\, \mu| > 2$, e.g. dilatations $z \mapsto \lambda z$; they have 2 fixed pts on bdry of $\Pi_+ \simeq D$



?• Π_+/Γ is smooth iff Γ contains no elliptic elements ($\Leftrightarrow$ if Γ is discrete in $PSL_2(\mathbb{R})$ and contains no finite-order elements) (then Γ acts on Π_+ freely)

(otherwise, Π_+/Γ is an orbifold)

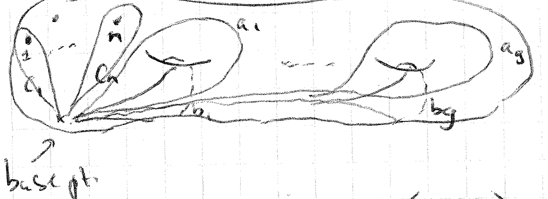
(• for $\Gamma' = \alpha \Gamma \alpha^{-1}$, $\alpha \in PSL_2(\mathbb{R})$, surfaces Π_+/Γ and Π_+/Γ' are isometric)

- for hyp. surfaces Σ, Π_+ is the universal cover $\quad \begin{matrix} \Pi_+ \circlearrowleft \Gamma \\ \downarrow \\ \Sigma \end{matrix}$

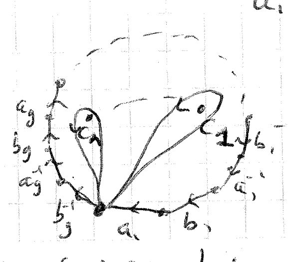
so, $\boxed{\Gamma \simeq \pi_1(\Sigma)}$ – group isomorphism

- fund. group for 2D surfaces via generators & relations:

$$\pi_1(\Sigma_{g,n}) = \langle a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_n \rangle \Big/ a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} c_1 \cdots c_n = 1$$



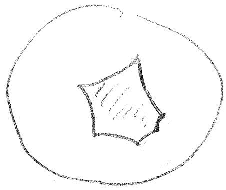
or:



Examples: – $\pi_1(\Sigma_{0,n}) = \langle c_1 \dots c_n \rangle / c_1 \cdots c_n = 1$ ≅ free gp. on $(n-1)$ generators

– $\pi_1(\Sigma_{1,0}) = \mathbb{Z}^2$

- One can always choose a fund. domain $\mathfrak{A}$ for $\Gamma \curvearrowright \mathbb{H}_2$ as a polygon bounded by geodesics in $\mathbb{H}_2$



the $\{\alpha \mathfrak{A} \mid \alpha \in \Gamma\}$ ~~prodn~~ is a tesselation of $\mathbb{H}_2$ by hyp. polygons

Mapping Class Group

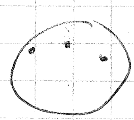
def: The mapping class group of a mfd. M is
$$MCG(M) := \pi_0(\mathrm{Diff}_+(M))$$

i.e. we have $0 \to \mathrm{Diff}_0(M) \hookrightarrow \mathrm{Diff}_+(M) \twoheadrightarrow MCG(M) \to 0$

For 2D surfaces $MCG(\Sigma)$ is also called the Teichmüller modular group

Ex.
- $MCG(\Sigma_{1,0}) = \{$group of lattice preserving automorphisms of $\mathbb{Z}^2\} \simeq SL_2(\mathbb{Z})$ (?)

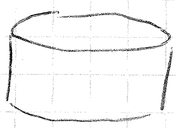
- $MCG(\Sigma_{0,3}) = S_3$ - sym. group permuting punctures



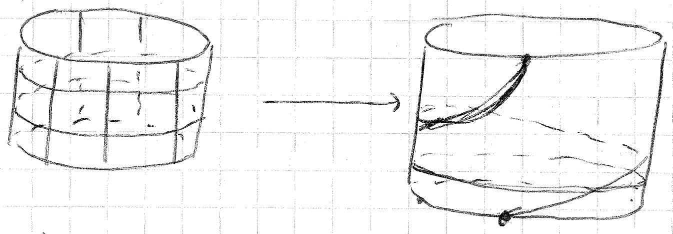
- $MCG(\Sigma_{0,n}) = \mathcal{B}_n(S^2)$ - "spherical braid group on n strands"
  $= \pi_1(\underbrace{S^2 \times \dots \times S^2}_{n} \setminus \text{all diagonals})$

MCG is generated by Dehn twists

Dehn twist:

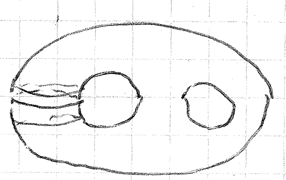


$S^1 \times [0,\varepsilon] \longrightarrow S^1 \times [0,\varepsilon]$ - diffeo rel. to bdry
$(\varphi, \tau) \longmapsto (\varphi + 2\pi \frac{\tau}{\varepsilon}, \tau)$

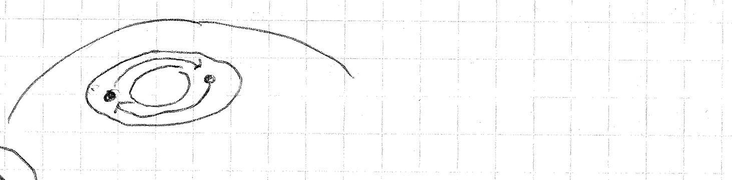


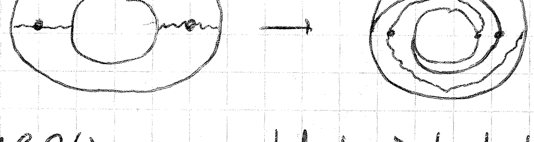
Dehn twist of Σ along a cycle $\alpha \subset \Sigma$:
$\alpha \simeq S^1$

→ cut out a tubular nbhd of α in Σ $\simeq S^1 \times (0,\varepsilon)$ and do the twist



for punctures: half-twists

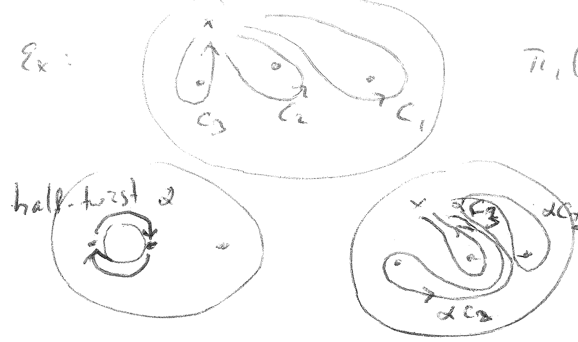




$MCG(\Sigma)$ is generated by Dehn twists & half-twists along a finite set of cycles (modulo relations) (more precisely, on conjugacy classes of elts of π_1 (a diffeo may have moved the basepoint around a non-trivial loop))

- $MCG(\Sigma)$ acts on $\pi_1(\Sigma)$ (since diffeo. act on loops)

Ex:



$\pi_1(\Sigma_{0,3}) = \langle c_1, c_2, c_3 \rangle / c_1 c_2 c_3 = 1$

α: $c_1 \longmapsto c_1$
$c_2 \longmapsto c_1^{-1} c_2^{-1}$
$c_3 \longmapsto c_2$

Thm (Dehn-Nielsen-Baer)

$MCG^{\pm}(\Sigma_g) \simeq \underbrace{Out}_{\text{outer autom.} = Aut/Inn}(\pi_1(\Sigma_g))$ where $MCG^{\pm} := \pi_0(Diff(\Sigma))$,

$MCG(\Sigma_g)$ is a subgroup of index 2 in $MCG^{\pm}(\Sigma_g)$

Ref: www.math.utah.edu/~margalit/primer ← detailed textbook on MCG

In Teichmüller theory one does the quotient $\mathcal{M}_{g,n} = \{\text{conf. str. on } \Sigma_{g,n}\} / Diff_+(\Sigma_{g,n})$ in 2 steps:

$T_{g,n} := \{\text{conf. str. on } \Sigma_{g,n}\} / Diff_0(\Sigma_{g,n})$, and then $\mathcal{M}_{g,n} = T_{g,n} / MCG_{g,n}$

↑ Teichmüller space

(In hyperbolic case $\chi = 2-2g-n < 0$) $T_{g,n}$ is a smooth (non-compact) mfd diffeo. to $\mathbb{R}^{6g-6+2n}$

It has natural complex structure, sympl. str., several natural choices of metric, in part. Weil-Petersson Kähler metric, several natural coord. systems, cell decompositions, structure of "cluster variety", MCG action

Rem: when boundary is present, $T_{g,n,m} \underset{\text{diffeo}}{\simeq} \mathbb{R}^{6g-6+2n+3m}$, but there is no natural $\mathbb{C}$ str.!

- MCG action on $T_{g,n}$ has a discrete set of fixed pts

$\mathcal{M}_{g,n} = T_{g,n} / MCG_{g,n}$ is an orbifold