Introduction to CFT, Lecture 6

30.03.11

Today: bits of Teichmüller theory

The idea is to do the quotient $\mathcal{M}_{g,n} = \left\{ \begin{matrix} \text{conf. str.} \\ \text{on } \Sigma_{g,n} \end{matrix} \right\} \Big/ \mathrm{Diff}_+(\Sigma_{g,n})$

in two steps:

① $T_{g,n} := \left\{ \begin{matrix} \text{conf. str.} \\ \text{on } \Sigma_{g,n} \end{matrix} \right\} \Big/ \mathrm{Diff}_0(\Sigma_{g,n})$ ② $\mathcal{M}_{g,n} = T_{g,n} \big/ MCG_{g,n}$

Teichmüller space

Rem. We have ex. seq. $0 \to \mathrm{Diff}_0(\Sigma) \hookrightarrow \mathrm{Diff}_+(\Sigma) \twoheadrightarrow MCG(\Sigma) \to 0$

In hyperbolic case (i.e. $\chi = 2-2g-n < 0$), $T_{g,n}$ is a smooth mfd diffeo. to $\mathbb{R}^{6g-6+2n}$. It has natural cx. str., sympl. str., several natural choices of metric, in particular Weil-Petersson Kähler metric, several natural coord. systems, cell decomp., structure of "cluster variety", MCG action.

Rem. If boundary components are present, one can still define $T_{g,n,m}$, $\sim \mathbb{R}^{6g-6+2n+3m}$ but it has no natural cx. str. on it.

- MCG action on $T_{g,n}$ has a discrete set of fixed points $\rightsquigarrow \mathcal{M}_{g,n} = T_{g,n} \big/ MCG_{g,n}$ is an orbifold

- $T_{g,n} = \left\{ \begin{matrix} \text{conf. str.} \\ \text{on } \Sigma_{g,n} \end{matrix} \right\} \Big/ \mathrm{Diff}_0(\Sigma_{g,n}) = \left\{ \begin{matrix} \text{equiv. classes of conf. str. on } \Sigma_{g,n} \\ \text{endowed with a diffeo} \\ \Sigma^{\circ}_{g,n} \longrightarrow \Sigma_{g,n} \text{ up to homotopy} \end{matrix} \right\}$

"marking"

fixed reference surface

General structure

$$\text{conf. equivalence}\left\{\begin{array}{c}\begin{Bmatrix}\text{Weyl}\\ \text{transf.}\end{Bmatrix} \curvearrowright \begin{Bmatrix}\text{Riem. metrics}\\ \text{on } \Sigma\end{Bmatrix}\\ \downarrow \\ \mathrm{Diff}_+(\Sigma) \curvearrowright \begin{Bmatrix}\text{Conf. str.}\\ \text{on } \Sigma\end{Bmatrix}\end{array}\right.$$

$$\downarrow$$

$$\mathcal{M}_\Sigma$$

$$\begin{Bmatrix}\text{Weyl}\\ \text{transf}\end{Bmatrix} \curvearrowright \begin{Bmatrix}\text{Riem. met}\\ \text{on } \Sigma\end{Bmatrix}$$

$$\downarrow$$

$$\mathrm{Conf}(\Sigma) \hookrightarrow \mathrm{Diff}_0(\Sigma) \curvearrowright \begin{Bmatrix}\text{Conf. str.}\\ \text{on } \Sigma\end{Bmatrix} = \begin{Bmatrix}\text{cx. str.}\\ \text{on } \Sigma\end{Bmatrix} \left(= \begin{Bmatrix}\text{hyperb.}\\ \text{met. on } \Sigma\end{Bmatrix}\right)$$

$$\downarrow$$

$$\pi_0(\mathrm{Conf}(\Sigma)) \nearrow MCG(\Sigma) \curvearrowright \text{Teichmüller space } T_\Sigma$$

$$\downarrow$$

$$\text{moduli space } \mathcal{M}_\Sigma$$

More detailed picture (for $n \geq 2$ punctures):

$$\begin{Bmatrix}\text{Weyl}\\ \text{transf}\end{Bmatrix} \curvearrowright \begin{Bmatrix}\text{Riem. met}\\ \text{on } \Sigma\end{Bmatrix}$$

$$\downarrow$$

$$0 \to \mathrm{Conf}_0(\Sigma,\gamma) \hookrightarrow \mathrm{Diff}_0(\Sigma) \curvearrowright \begin{Bmatrix}\text{Conf. str. } \gamma\\ \text{on } \Sigma\end{Bmatrix}$$

$$\downarrow$$

$$0 \to \begin{pmatrix}\text{isotopy}\\ \text{classes}\\ \text{of conf.}\\ \text{maps not}\\ \text{permuting}\\ \text{punctures}\end{pmatrix} \hookrightarrow pMCG(\Sigma) \curvearrowright T_\Sigma$$

"pure MCG" i.e. not permuting punctures

$$\downarrow$$

$$0 \to \begin{pmatrix}\text{permutations}\\ \text{of punctures}\\ \text{induced by}\\ \text{conf. autom. of } \Sigma\end{pmatrix} \hookrightarrow S_n \curvearrowright \widetilde{\mathcal{M}}_\Sigma$$

moduli space with ordered punctures

$$\downarrow$$

$$\mathcal{M}_\Sigma$$

moduli space with indistinguishable (unordered) punctures

Rem: We implicitly use the exact sequence

$$0 \longrightarrow pMCG_{g,n} \hookrightarrow MCG_{g,n} \twoheadrightarrow S_n \longrightarrow 0$$

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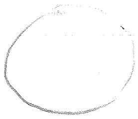
Examples

Ex. 1 sphere $S^2 = \Sigma_{0,0}$



$\begin{Bmatrix}\text{Weyl}\\ \text{transf}\end{Bmatrix} \circlearrowright \begin{Bmatrix}\text{Riem. met}\\ \text{on } S^2\end{Bmatrix}$

$$0 \to \underset{\overset{\wr\wr}{PSL_2(\mathbb{C})}}{Conf(S^2,\gamma)} \hookrightarrow Diff_+(S^2) \circlearrowright \begin{Bmatrix}\text{conf. str. } \gamma\\ \text{on } S^2\end{Bmatrix}$$

$$\downarrow$$

$$\underset{\overset{\shortparallel}{0}}{MCG(S^2)} \circlearrowright T_{0,0} = pt \leftarrow \text{"Standard } \mathbb{CP}^1\text{"}$$

$$\downarrow$$

$$\mathcal{M}_{0,0} = pt$$

Ex. 2 torus $\mathbb{R}^2/\mathbb{Z}^2 = \Sigma_{1,0}$

$$0 \longrightarrow Diff_0 \circlearrowright \begin{Bmatrix}\text{conf. str.}\\ \text{on torus}\end{Bmatrix}$$

(since there are no 2-periodic holom. v.f. on $\mathbb{C}$)

$$\downarrow$$

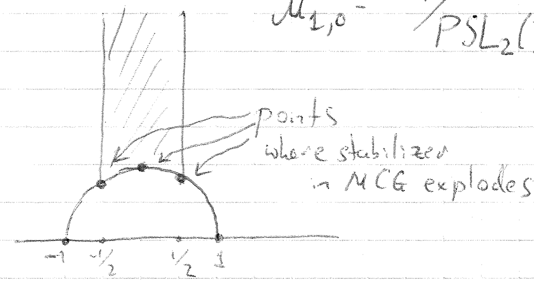
$$0 \to \mathbb{Z}_2 \hookrightarrow \overbrace{SL_2(\mathbb{Z})}^{MCG} \circlearrowright T_{1,0} = \mathbb{H}_+ \leftarrow \begin{pmatrix}\text{point } \tau \in \mathbb{H}_+ \\ \text{is associated to} \\ \text{a standard cx. torus} \\ \mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}\end{pmatrix}$$

$(z \mapsto -z)$ (autom. of $\mathbb{Z}^2$ lattice)

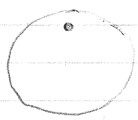
(Möbius action)

$$\downarrow$$

$$\mathcal{M}_{1,0} = \mathbb{H}_+ / PSL_2(\mathbb{Z})$$



Ex. 3 plane $\mathbb{R}^2 = \Sigma_{0,1}$

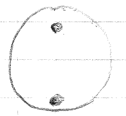


$$0 \longrightarrow \underset{PSL_2(\mathbb{C})}{\{z \mapsto az+b\}} \hookrightarrow Diff_+ \circlearrowright \begin{Bmatrix}\text{conf. str.}\\ \text{on } \Sigma_{0,1}\end{Bmatrix}$$

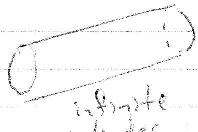
$$\downarrow$$

$$T_{0,1} = \mathcal{M}_{0,1} = \text{"standard } \mathbb{C}\text{"}$$

Ex. 4 $\Sigma_{0,2} = \mathbb{R}^2/\mathbb{Z}$



or



$$0 \longrightarrow \underset{\text{dilatations + rotations of } \mathbb{C}\setminus\{0\}}{\{z \mapsto az\}} \hookrightarrow Diff_0 \circlearrowright \begin{Bmatrix}\text{conf. str.}\\ \text{on } \mathbb{R}^2/\mathbb{Z}\end{Bmatrix}$$

$$\downarrow$$

$$\underset{\overset{\shortparallel}{0}}{pMCG_{0,2}} \circlearrowright T_{0,2} = pt \leftarrow \begin{pmatrix}\text{standard}\\ \mathbb{C}/\mathbb{Z}\end{pmatrix}$$

$$\downarrow$$

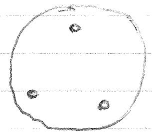
$$0 \to \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2 \circlearrowright \widetilde{\mathcal{M}}_{0,2} = pt$$

$\begin{pmatrix} z \mapsto 1/z \\ \text{permutes punctures} \\ 0 \leftrightarrow \infty \end{pmatrix}$

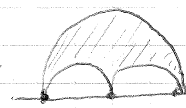
$$\downarrow$$

$$\mathcal{M}_{0,2} = pt$$

Ex. 5 $\Sigma_{0,3}$



$\sim$



$\mathbb{H}_+/\Gamma_2$, $\Gamma_2 \cong \pi_1(\Sigma_{0,3}) \cong F_2$

$$0 \longrightarrow Diff_0 \circlearrowright \begin{Bmatrix}\text{conf.}\\ \text{str. on } \Sigma_{0,3}\end{Bmatrix}$$

(there are no holom. v.f. on $\mathbb{CP}^1\setminus\{0,1,\infty\}$ vanishing at $\{0,1,\infty\}$)

$$\downarrow$$

$$\underset{0}{\underbrace{pMCG_{0,3}}} \circlearrowright T_{0,3} = pt \leftarrow \begin{pmatrix}\text{standard}\\ \mathbb{CP}^1\setminus\{0,1,\infty\}\end{pmatrix}$$

$$\downarrow$$

$$0 \longrightarrow S_3 \hookrightarrow S_3 \circlearrowright \widetilde{\mathcal{M}}_{0,3} = pt$$

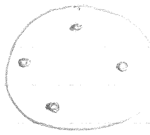
conf. maps permuting $\{0,1,\infty\}$:

- $(0\,1\,\infty)$ $z \mapsto z$
- $(\infty\,1\,0)$ $z \mapsto 1/z$
- $(1\,0\,\infty)$ $z \mapsto 1-z$
- $(1\,\infty\,0)$ $z \mapsto \frac{1}{1-z}$
- $(\infty\,0\,1)$ $z \mapsto \frac{z-1}{z}$
- $(0\,\infty\,1)$ $z \mapsto \frac{z}{z-1}$

$$\downarrow$$

$$\mathcal{M}_{0,3} = pt$$

Ex. 6 $\Sigma_{0,4}$



$$0 \to \mathrm{Diff}_0 \circlearrowright \left\{ \begin{matrix} \text{conf. str.} \\ \text{on } \Sigma_{0,4} \end{matrix} \right\}$$

$$\downarrow$$

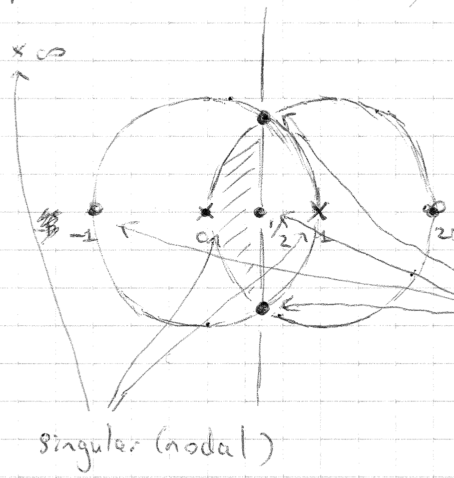
$$0 \to \underset{\substack{\| \\ pB_4(S^2) \sim F_2}}{pMCG_{0,4}} \circlearrowright T_{0,4} = \mathbb{C}$$

$$\Big\downarrow \text{ univ. cover}$$

$$0 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \hookrightarrow S_4 \circlearrowright \widetilde{\mathcal{M}}_{0,4} = \mathbb{C} \setminus \{0,1,\infty\} \leftarrow$$ (point $z \neq 0,1,\infty$ corresponds to "standard" $\mathbb{CP}^1 \setminus \{0,1,\infty,z\}$)

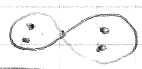
Exercise! ⇝ (Klein 4-group: perm. $(0\,1\,\infty\,z), (1\,0\,z\,\infty), (\infty\,z\,0\,1), (z\,\infty\,1\,0)$ are representable by certain Möbius maps $\mathbb{CP}^1 \setminus \{0,1,\infty,z\} \circlearrowleft$)

$$\mathcal{M}_{0,4} = \left(\mathbb{CP}^1 \setminus \{0,1,\infty\}\right) \Big/ {\scriptstyle z \sim \frac{1}{z} \sim 1-z \sim \frac{z-1}{z} \sim \frac{1}{1-z} \sim \frac{z}{z-1}}$$



at these points stabilizer of $S^4 \circlearrowright \widetilde{\mathcal{M}}_{0,4}$ explodes

singular (nodal) curves



Aside on cross-ratios

- Given 4 points z_1, z_2, z_3, z_4 on (standard) $\mathbb{CP}^1$, one defines

$$[z_1, z_2 : z_3, z_4] := \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \quad \text{– the cross-ratio.}$$

(*) • $[\ ]$ is invar. under conf. (Möbius) autom. of $\mathbb{CP}^1$, i.e. defines a cx. coordinate on $(\mathbb{CP}^1)^{\times 4} / PSL_2(\mathbb{C})$

- $PSL_2(\mathbb{C})$ acts on $\mathbb{CP}^1$ 3-transitively, so one can map

$$\{z_1, z_2, z_3, z_4\} \xrightarrow[\substack{\text{some} \\ PSL_2(\mathbb{C})\text{-transf.}}]{} \{0, 1, \infty, \lambda\} \quad \text{for some } \lambda$$

By (*) we have: $\frac{\lambda - 1}{\lambda} = [z_1, z_2 : z_3, z_4]$

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- S_4 (permutations of $z_1 \dots z_4$) acts on cross-ratio $[\cdot]$

$$\lambda \sim \frac{1}{\lambda} \sim 1-\lambda \sim \frac{1}{1-\lambda} \sim \frac{\lambda-1}{\lambda} \sim \frac{\lambda}{\lambda-1}$$

but there is a stabilizer: the Klein 4-group:

$$[z_1, z_2 : z_3, z_4] = [z_2, z_1 : z_4, z_3] = [z_3, z_4 : z_1, z_2] = [z_4, z_3 ; z_2, z_1]$$

i.e. we have
$$0 \to \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2}_{\substack{\text{symmetries} \\ \text{of cross-ratio}}} \hookrightarrow \underbrace{S_4}_{\substack{\text{perm. of} \\ \text{4 points}}} \longrightarrow \underbrace{S_3}_{\substack{\text{transf.} \\ \text{of cross-ratio}}} \longrightarrow 0$$

- cross-ratio defines a coordinate on $\tilde{\mathcal{M}}_{0,4}$:

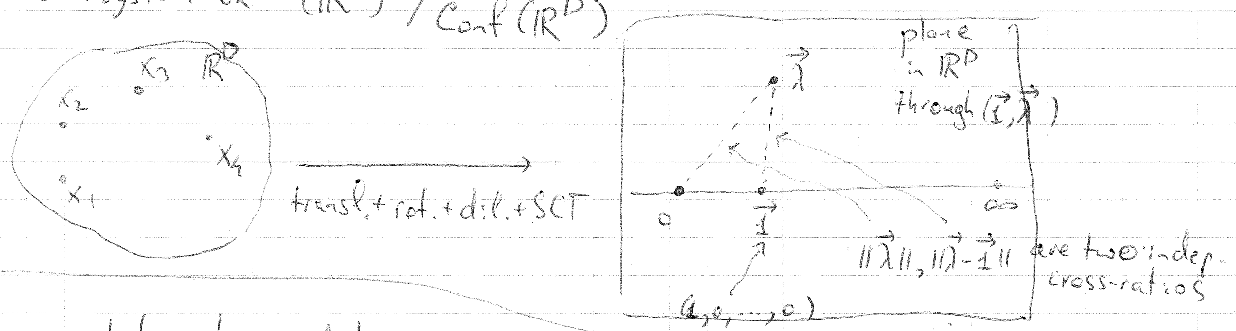
$$\underset{\substack{\text{endowed with} \\ \text{conf. str.}}}{S^2 \setminus 4 \text{ points}} \xrightarrow[\text{of } S^2]{\text{uniformization}} \left(\mathbb{CP}^1 \setminus \{z_1, \dots, z_4\}\right) / PSL_2(\mathbb{C}) \quad \leftarrow \text{a point in } \tilde{\mathcal{M}}_{0,4}$$

and $\mathcal{M}_{0,4} = \tilde{\mathcal{M}}_{0,4} / S_3$ - transf. of cross-ratio

- For $D > 2$, $\mathrm{Conf}(\mathbb{R}^D)$ also acts on $\mathbb{R}^D$ 3-transitively

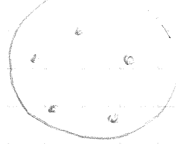
for 4 points, $[x_1, x_2 ; x_3, x_4] = \dfrac{\|x_1 - x_3\| \cdot \|x_2 - x_4\|}{\|x_1 - x_4\| \cdot \|x_2 - x_3\|}$ is Conf-invariant

Moreover, one can pick 2 independent cross-ratios (e.g. $[x_1, x_2 : x_3, x_4]$ and $[x_2, x_1 : x_3, x_4]$) and they give a coord. system on $(\mathbb{R}^D)^{\times 4} / \mathrm{Conf}(\mathbb{R}^D)$



However, no relation to moduli spaces as there is no uniformization thm for $D > 2$

Ex. 7 $\Sigma_{0,5}$



$$0 \to \mathrm{Diff}_0 \circlearrowright \left\{ \begin{matrix} \text{conf. str.} \\ \text{on } \Sigma_{0,5} \end{matrix} \right\}$$

$$\downarrow$$

$$0 \to \underset{\substack{\wr\wr \\ F_2 \times F_3}}{pMCG_{0,5}} \circlearrowright T_{0,5} \sim \mathbb{C}^2$$

$$\downarrow \text{ univ. cover}$$

$$0 \to S_5 \circlearrowright \widetilde{\mathcal{M}}_{0,5} = \left(\mathbb{CP}^1 \setminus \{0,1,\infty\}\right)^{\times 2} \setminus \text{diagonal} \leftarrow \left(\begin{matrix} \text{point } (z,w) \\ \text{corresponds to} \\ \mathbb{CP}^1 \setminus \{0,1,\infty,z,w\} \end{matrix} \right)$$

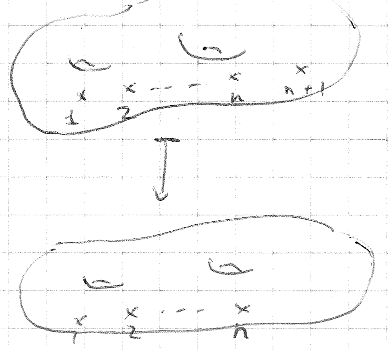
$\uparrow$ (2 cross-ratios $[z_1, z_2 : z_3, z_4]$, $[z_1, z_2 : z_3, z_5]$)

$$\downarrow$$

$$\mathcal{M}_{0,5}$$

Rem "Forgetting the last puncture" map

$$\widetilde{\mathcal{M}}_{g,n+1} \leftarrow \Sigma_{g,n}$$
$$\downarrow$$
$$\widetilde{\mathcal{M}}_{g,n}$$



Conf. structures as flat $PSL_2(\mathbb{R})$-connections
(in hyp. case $\chi = 6g-6+2n < 0$)

$$\left\{ \begin{matrix} \text{conf. str.} \\ \text{on } \Sigma_{g,n} \end{matrix} \right\} \xrightarrow[\text{uniformization thm}]{} \left\{ \begin{matrix} \text{subgroups } \Gamma \subset PSL_2(\mathbb{R}) \\ \text{s.t. } \Gamma \simeq \pi_1(\Sigma_{g,n}) \end{matrix} \right\} \Big/ PSL_2(\mathbb{R})$$

more specifically, we have $\quad T_{g,n} \xrightarrow{\ p\ } \mathrm{Hom}\left(\pi_1(\Sigma_{g,n}), PSL_2(\mathbb{R})\right) \big/ PSL_2(\mathbb{R})$

$$\parallel$$

$$\left\{ \begin{matrix} \text{moduli space of} \\ \text{flat } PSL_2(\mathbb{R})\text{-bundles} \\ \text{on } \Sigma_{g,n} \end{matrix} \right\}$$

In fact, p is injective and the image is:

$$\mathrm{im}(p) = \mathrm{Hom}^{dfp}\left(\pi_1(\Sigma_{g,n}), PSL_2(\mathbb{R})\right) / PSL_2(\mathbb{R})$$

where: d - "discrete" (so that $1 \in PSL_2(\mathbb{R})$ is not an accum. point of image of π_1)
f - "faithful" (injective)
p - "mapping peripherals to parabolics" $|\mathrm{tr}\, \rho(C_i)| = 2$

Moduli space $\quad \mathcal{M}_{g,n} = MCG_{g,n} \backslash \mathrm{Hom}^{dfp}\left(\pi_1(\Sigma_{g,n}), PSL_2(\mathbb{R})\right) / PSL_2(\mathbb{R})$