Introduction to CFT

Lecture 7 (06.04.11)

Today:
- coordinates on $\mathcal{M}_{g,n}$ - some remarks
- field theory & symmetries

$\mathcal{M}_{g,n}$ and (moduli of) flat $PSL_2(\mathbb{R})$-connections

(in hyp. case $\chi = 2-2g-n<0$)

$$\left\{\begin{matrix}\text{conf. str.}\\ \text{on } \Sigma_{g,n}\end{matrix}\right\} \xrightarrow[\text{uniformization thm}]{} \left\{\begin{matrix}\text{subgroups } \Gamma\subset PSL_2(\mathbb{R})\\ \text{s.t. } \Gamma\simeq \pi_1(\Sigma_{g,n})\end{matrix}\right\}/PSL_2(\mathbb{R})$$

more specifically: $T_{g,n} \xrightarrow{p} \mathrm{Hom}(\pi_1(\Sigma_{g,n}), PSL_2(\mathbb{R}))/PSL_2(\mathbb{R})$

$$\overset{\shortparallel}{\left\{\begin{matrix}\text{moduli space of}\\ \text{flat } PSL_2(\mathbb{R})\text{-bundles on } \Sigma_{g,n}\end{matrix}\right\}}$$

Aside on moduli of flat connections:

$$\left\{\begin{matrix}\text{flat connections}\\ \text{in a principal}\\ G\text{-bundle } P \text{ on } M\end{matrix}\right\}\Big/\left\{\begin{matrix}\text{Gauge}\\ \text{transformations}\\ \text{trivial over the}\\ \text{base point } b\end{matrix}\right\} \xrightarrow[\text{of the flat conn. around loops based at } b]{\text{holonomy}} \mathrm{Hom}(\pi_1(M,b), G)$$

also: $\left\{\begin{matrix}\text{flat conn.}\\ \text{in } P\end{matrix}\right\}\Big/\left\{\begin{matrix}\text{all gauge}\\ \text{transf.}\end{matrix}\right\} \longrightarrow \mathrm{Hom}(\pi_1(M), G)/G$ ⟵ injective but in general not surjective

and $\left\{\begin{matrix}\text{flat bundles}\\ \text{over } M\end{matrix}\right\}\Big/\sim \xrightarrow{\sim} \mathrm{Hom}(\pi_1(M), G)/G$

In fact, p is injective and $\boxed{T_{g,n} \overset{p}{\xrightarrow{\sim}} \mathrm{Hom}^{dfp}(\pi_1(\Sigma_{g,n}), PSL_2(\mathbb{R}))/PSL_2(\mathbb{R})}$

where: d - "discrete" (so that $\mathbb{1}\in PSL_2(\mathbb{R})$ is not an accum. point of the image of π_1)
f - "faithful" (injective)
p - "mapping peripherals to parabolics" $|\mathrm{tr}\,\rho(c_i)|=2$

Moduli space: $\mathcal{M}_{g,n} = MCG_{g,n}\backslash \mathrm{Hom}^{dfp}(\pi_1(\Sigma_{g,n}), PSL_2(\mathbb{R}))/PSL_2(\mathbb{R})$

(Towards) coordinates on $T_{g,n}$:

$$T_{g,n} \simeq PSL_2(\mathbb{R})\Big\backslash\left\{\begin{matrix}\rho(\alpha_1),\dots,\rho(\alpha_g), \\ \rho(\beta_1),\dots,\rho(\beta_g)\end{matrix} \in PSL_2^{hyp}(\mathbb{R}) \atop \rho(c_1),\dots,\rho(c_n)\in PSL_2^{parab}(\mathbb{R}) \;\middle|\; \begin{matrix}\text{all } \rho(\alpha_1)\dots\rho(c_n) \text{ distinct}\\ +\text{relation}\\ \rho(\alpha_1)\rho(\beta_1)\rho(\alpha_1)^{-1}\rho(\beta_1)^{-1}\dots\\ \dots\rho(\alpha_g)\rho(\beta_g)\rho(\alpha_g)^{-1}\rho(\beta_g)^{-1}\\ \rho(c_1)\dots\rho(c_n)=\mathbb{1}\end{matrix}\right\}$$

Dimension count

$$\begin{pmatrix}\text{expected}\\ \text{dimension}\\ \text{of } \mathcal{M}_{g,n}\end{pmatrix} = \underbrace{2g\cdot \underbrace{\dim\left(PSL_2^{\text{hyp}}(\mathbb{R})\right)}_{3}}_{\text{images of } \alpha,\beta\text{-cycles}} + \underbrace{n\cdot \underbrace{\left(\dim PSL_2^{\text{parab}}(\mathbb{R})\right)}_{2}}_{\text{images of peripherals } C_1 \dots C_n} - \underbrace{\underbrace{\dim PSL_2(\mathbb{R})}_{3}}_{\substack{\text{relation in } \pi_1(\Sigma)\\ \text{represented in}\\ PSL_2(\mathbb{R})}} - \underbrace{\underbrace{\dim PSL_2(\mathbb{R})}_{3}}_{\substack{\text{quotient}\\ \text{by}\\ PSL_2(\mathbb{R})}}$$

$$= \boxed{6g-6+2n}$$

Strebel's theory - a glimpse

Strebel's thm: Given a Riemann surface $\Sigma_{g,n}$ (with a cx. str.) with $n \geq 1$ and weights $a_1, \dots, a_n > 0$, there exists a unique meromorphic quadratic differential (locally $f = f(z)\,dz^2$) on $\Sigma_{g,n}$ such that

- f has 2nd order poles in punctures, $f = \left(-\frac{a_k^2}{z^2} + \frac{\text{const}}{z} + \text{reg. part}\right)(dz)^2$ and is holomorphic outside punctures
- non-singular horizontal trajectories are closed

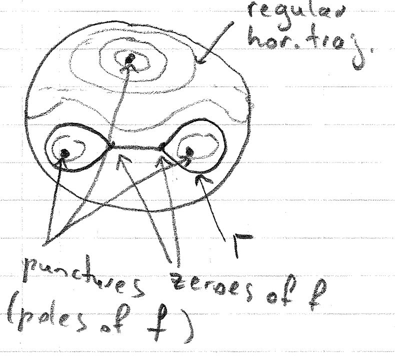
[$\gamma \subset \Sigma_{g,n}$ is "horizontal" if $f|_\gamma > 0$
"singular" trajectory is the one passing through a zero of f]

Such quadratic differential f is called **horocyclic**.

- $|f(z)|\,dz\,d\bar{z}$ is an almost-everywhere flat Riem. metric on $\Sigma_{g,n}$, singular at zeroes of f
- Singular horiz. trajectories form a special ribbon graph $\Gamma \subset \Sigma_{g,n}$ with $\{\text{vertices}\} = \{\text{zeroes of } f\}$, $\{\text{valence of vertex at } z_i\} = 2 + \begin{pmatrix}\text{order of}\\ \text{zero of } f\\ \text{at } z_i\end{pmatrix}$,
$\{\text{faces of } \Gamma\} \longleftrightarrow \{\text{poles of } f\} = \{\text{punctures}\}$

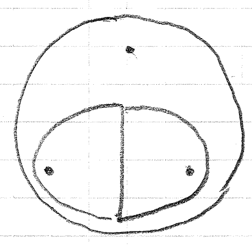
Ex: $\Sigma_{0,3} = \mathbb{C}P^1 \setminus \{0,1,\infty\}$

horocyclic quad. differential explicitly: $f = \left(-\frac{a_0^2}{z^2(1-z)} - \frac{a_1^2}{z(z-1)^2} - \frac{a_\infty^2}{z(z-1)}\right)dz^2$



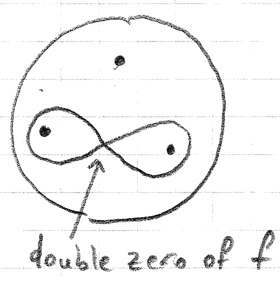
if triangle inequality is strictly violated for a_0, a_1, a_∞
e.g. $a_\infty > a_0 + a_1$

or



if strict triangle inequality is satisfied for a_0, a_1, a_∞

or



if triangle equality is satisfied, e.g. $a_\infty = a_0 + a_1$

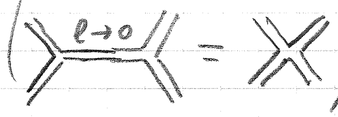
- lengths of edges of $\Gamma /$ ribbon graph automorphisms of Γ give coordinates on $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$ (perimeters of peripherals are a_k)

residual MCG action

- Combinatorial model for $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$

$$\mathcal{M}_{g,n} \times \mathbb{R}_+^n \simeq \left(\bigsqcup_{\substack{\text{connected ribbon graphs}\\ \Gamma \text{ of genus } g \\ \text{with } n \text{ faces}}} \mathbb{R}_+^{E_\Gamma} \big/ \mathrm{Aut}(\Gamma) \right) \Big/\!\!\Big/ \left(\text{gluing maps} \right)$$

orbifold homeomorphism



- Top-dimension cells correspond to 3-valent ribbon graphs

Dimension count:
$$\begin{cases} V - E + F = 2 - 2g & (\text{Euler char. of } \Sigma_g) \\ F = n & \\ 2E = 3V & (\#\text{ half-edges in 3-valent case}) \end{cases}$$

$$\Rightarrow \frac{2}{3}E - E + n = 2 - 2g \Rightarrow \boxed{\begin{array}{c} E = 6g - 6 + 3n \\ \parallel \\ \dim(\mathcal{M}_{g,n} \times \mathbb{R}_+^n) \end{array}}$$

Field theory & symmetries

Classical Lagrangian mechanics

X - smooth (f.d.) mfd without boundary - "Configuration space" (coords x^i on X are called "degrees of freedom")

trajectories $[t_0, t_1] \longrightarrow X$, $t \longmapsto x^i(t)$ Notation: $[x^i(t)]_{t_0}^{t_1}$

tangent lift of a trajectory $[t_0, t_1] \longrightarrow TX$, $t \longmapsto (x^i(t), v^i(t) = \dot{x}^i(t))$ where $\dot{x}^i(t) = \frac{d}{dt} x^i(t)$

$x^i(t)$: base coord., $v^i(t)$: fiber coord.

Lagrangian: $L \in C^\infty(TX)$, we usually assume a non-degeneracy condition:
$$\det\left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right) \neq 0 \quad (\text{otherwise it is a gauge theory})$$

Classical trajectories are given by the action principle:

action: $\mathrm{Maps}([t_0,t_1],X) \longrightarrow \mathbb{R}$

$$[x(t)] \longmapsto S[x(t)] = \int_{t_0}^{t_1} dt\, \underbrace{L(x^i(t), \dot{x}^i(t))}_{\overset{\text{"}}{L(x^i(t),v^i(t))}\big|_{v^i=\dot{x}^i(t)}}$$

(Rem: Notion of tangent lift of a trajectory and "dt" in action refer to a choice of parametrization (or Riem. metric) on $[t_0,t_1]$ - geom. data in class. mech.)

Class. traj. are given by variational principle: $\left.\begin{array}{l}\delta S = 0 \\ x(t_0), x(t_1) \text{ fixed}\end{array}\right\} \longrightarrow$ Euler-Lagrange equations of motion

$$\delta S = \int_{t_0}^{t_1} dt\, \frac{\partial L}{\partial x^i}\Big|_{x^i(t),\dot{x}^i(t)} \delta x^i(t) + \frac{\partial L}{\partial v^i}\Big|_{x^i(t),\dot{x}^i(t)} \delta \dot{x}^i(t) = \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial x^i}\Big|_{x^i(t),\dot{x}^i(t)} - \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\Big|_{x^i(t),\dot{x}^i(t)}\right)\right)\delta x^i(t) + \left(\frac{\partial L}{\partial v^i}\Big|_{x^i(t),\dot{x}^i(t)} \delta x^i(t)\right)\Bigg|_{t_0}^{t_1}$$

$$\| $$

$$\left(S[x(t)+\delta x(t)] - S[x(t)]\right)_{\text{infinitesimally}}$$

One usually just writes:

$$\delta S = \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}\right)\delta x^i + \left. \frac{\partial L}{\partial \dot{x}^i}\delta x^i \right|_{t_0}^{t_1}$$

Euler-Lagrange equations of motion:

$$\boxed{\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = 0}$$

$$\left(\frac{\delta S}{\delta x^i(t)} \;\text{"}\right)$$

Ex: a particle on $\mathbb{R}^n$ in a potential $U \in C^\infty(\mathbb{R}^n)$

$$X = \mathbb{R}^n, \quad L = \frac{m|\vec{v}|^2}{2} - U(x)$$

(m: mass)

E-L eq.: $m\ddot{x}^i + \partial_i U(x) = 0$ - Newton's eq. of motion in a force field $F(x) = -\partial_i U(x)$

Symmetries

- target symmetry (continuous): group action $\mathbb{R} \circlearrowright X$, $\quad F_\alpha \circ F_\beta = F_{\alpha+\beta}$

$$\alpha \longmapsto (F_\alpha : X \xrightarrow{\text{diffeo}} X)$$

$$\rightsquigarrow (F_\alpha)_* : \mathrm{Maps}([t_0,t_1],X) \longrightarrow \mathrm{Maps}([t_0,t_1],X)$$

$$[x(t)] \longmapsto [F_\alpha(x(t))]$$

is called a "symmetry" if S is invariant under $(F_\alpha)_*$

Rem: a less restrictive definition of symmetry is to require that F_α takes solutions of E-L to solutions of E-L