Introduction to CFT

Lecture 8 (20.04.11)

Today:
- Noether theorem in class. mechanics & classical field theory
- Stress-energy tensor

Class. (Lagrangian) mechanics (reminder): X - configuration space, $L \in C^\infty(TX)$ - Lagrangian

$$S : \mathrm{Maps}([t_0,t_1], X) \longrightarrow \mathbb{R} \qquad \text{- \underline{action}}$$
$$[x(t)]_{t_0}^{t_1} \longmapsto \int_{t_0}^{t_1} dt\, L(x(t), \dot{x}(t))$$

variation of action: $$\delta S = \int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\right)\delta x^i + \left.\frac{\partial L}{\partial \dot{x}^i}\,\delta x^i\right|_{t_0}^{t_1}$$
$$\shortparallel$$
$$\left(\frac{d}{d\alpha}\Big|_{\alpha=0} S[x(t)+\alpha\,\delta x(t)]\right)$$

Classical trajectories: $[x(t)]_{t_0}^{t_1}$: extremals of S (with fixed boundary values $\delta x^i|_{t_0,t_1} = 0$) $\Leftrightarrow$ solutions of Euler-Lagrange equations $\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = 0$

Ex: a particle of mass m on $\mathbb{R}^n$ is a potential $U \in C^\infty(\mathbb{R}^n)$

$X = \mathbb{R}^n$, $L = \frac{m|\vec{v}|^2}{2} - U(x)$

E-L eq.: $m\ddot{x}^i + \partial_i U(x) = 0$ - Newton's eq. of motion for a particle in a force field $F^i(x) = -\partial_i U(x)$

Exercise: free particle on a Riemannian manifold (X,g):

$$L = \frac{m}{2} g_{ij}(x) v^i v^j$$

check that E-L eq. is: $\ddot{x}^i + \Gamma^i_{jk}(x)\dot{x}^j\dot{x}^k = 0$ - eq. of geodesic motion

$\uparrow$ Christoffel symbol for metric g

Symmetries

- (continuous) target symmetry: group action $F : \mathbb{R} \circlearrowright X$ $\qquad F_\alpha \circ F_\beta = F_{\alpha+\beta}$
  $\alpha \longmapsto (F_\alpha : X \xrightarrow{\text{diffeo}} X)$

$$\rightsquigarrow (F_\alpha)_* : \mathrm{Maps}([t_0,t_1], X) \longrightarrow \mathrm{Maps}([t_0,t_1], X)$$
$$[x(t)]_{t_0}^{t_1} \longmapsto [F_\alpha(x(t))]_{t_0}^{t_1}$$

F is called a <u>symmetry</u> if S is invariant under $(F_\alpha)_*$

infinitesimally: $f = \left.\frac{d}{d\alpha}\right|_{\alpha=0} F_\alpha \in \mathrm{Vect}(X)$, "$x^i(t) \longmapsto x^i(t) + f^i(x(t))$"
(infinitesimal variation)

$\overset{''}{f(x)\partial_i}$ is a symmetry if $\delta_f S = 0$

$$\overset{''}{\left.\frac{d}{d\alpha}\right|_{\alpha=0}} S[F_\alpha(x(t))]_{t_0}^{t_1}$$

<u>Computation</u>

$$\delta_f S = \underbrace{\int_{t_0}^{t_1} dt \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\right) f^i(x(t))}_{=0 \text{ modulo } \mathcal{E}\text{-L}} + \left|\frac{\partial L}{\partial \dot{x}^i} f^i\right|_{t_0}^{t_1}$$

<u>Therefore:</u>

$f \in \mathrm{Vect}(X)$ is a symmetry $\Rightarrow$ $I_f \equiv \frac{\partial L}{\partial v^i} f^i(x) \in C^\infty(TX)$ is an <u>integral of motion</u>, i.e,
$\frac{d}{dt} I_f(x(t), \dot{x}(t)) = 0$ if $[x(t)]$ is a classical trajectory

- Noether thm (for target symmetry)

<u>Rem:</u> One defines $\alpha := \frac{\partial L}{\partial v^i} dx^i \in \Omega^1(TX)$ - "Liouville 1-form"

then $\boxed{I_f = \iota_{\tilde{f}} \alpha}$ where $\tilde{f} \in \mathrm{Vect}(TX)$ is the lift of f:

$$\tilde{f} = f^i(x) \frac{\partial}{\partial x^i} + \partial_j f^i(x) \cdot v^j \frac{\partial}{\partial v^i}$$

<u>Ex</u> ① free particle in $\mathbb{R}^D$, (target) translations

$X = \mathbb{R}^D$, $L = \frac{m|\vec{v}|^2}{2}$ $\qquad F_{\alpha,\vec{u}}: X \longrightarrow X$ for any fixed $\vec{u} \in \mathbb{R}^D$
$\qquad\qquad \vec{x} \longmapsto \vec{x} + \alpha\vec{u}$

$I_{f_{\vec{u}}} = m(\dot{\vec{x}}, \vec{u})$ is an integral of motion for any $\vec{u}$

$\Rightarrow$ $m\dot{\vec{x}}$ is a (vector-valued) integral of motion - <u>momentum</u>

(1') n particles in $\mathbb{R}^D$ with pairwise interaction, simultaneous translation of all particles

$$X = \underbrace{\mathbb{R}^D \times \cdots \times \mathbb{R}^D}_{n} \qquad L = \sum_{i=1}^{n} \frac{m|\vec{v}_i|^2}{2} - \sum_{1 \le i < j \le n} U(|\vec{x}_i - \vec{x}_j|)$$
$\vec{x}_1, \ldots, \vec{x}_n$

$$F_{\alpha,\vec{u}}: X \longrightarrow X, \quad (\vec{x}_1, \ldots, \vec{x}_n) \longmapsto (\vec{x}_1 + \vec{u}\alpha, \ldots, \vec{x}_n + \vec{u}\alpha), \quad \vec{u} \in \mathbb{R}^D \text{ fixed}$$

$\Rightarrow I_{f_{\vec{u}}} = m(\dot{\vec{x}}_1 + \cdots + \dot{\vec{x}}_n, \vec{u}) = (\vec{P}, \vec{u})$

$\vec{P} = m\dot{\vec{x}}_1 + \cdots + m\dot{\vec{x}}_n$ - total momentum

② Particle on $\mathbb{R}^2$ in a radially-symmetric potential, rotations

$X = \mathbb{R}^2 \qquad L = \frac{m|\vec{v}|^2}{2} - U(|\vec{x}|)$

$$F_\alpha : \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \longmapsto \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

$$f = -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}$$

$I_f = m(x^1 v^2 - \cancel{x}^2 v^1)$ - angular momentum

(2') Same in $\mathbb{R}^3$, $SO(3)$-rotations

$SO(3) \circlearrowleft \mathbb{R}^3$, infinitesimally $\quad so(3) \longrightarrow Vect(\mathbb{R}^3)$

$$\vec{u} \longmapsto \underbrace{(\vec{u} \times \vec{x})^i \frac{\partial}{\partial x^i}}_{f_{\vec{u}}}$$

$I_{f_{\vec{u}}} = (\vec{J}, \vec{u})$

$\vec{J} = m\,\vec{x} \times \vec{v}$ - angular momentum

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Source symmetry

family of diffeo $R_\alpha : \mathbb{R} \to \mathbb{R}$, $\qquad$ ← $\mathbb{R}$-action on source

$$\underset{[t_0,t_1]}{\cup} \qquad \underset{[R_\alpha(t_0),\, R_\alpha(t_1)]}{\cup}$$

$$(R_\alpha^{-1})^* : Maps([t_0,t_1], X) \longrightarrow Maps([R_\alpha(t_0), R_\alpha(t_1)], X)$$

$$[x(t)]_{t_0}^{t_1} \longmapsto [x'(t') = x(t)]_{t_0' = R_\alpha(t_0)}^{t_1' = R_\alpha(t_1)}$$

- right action of $\overset{\alpha}{\overset{\in}{\mathbb{R}}}$ on trajectories

R is a symmetry if S is $(R_\alpha^{-1})^*$-invariant.

Infinitesimally: $\quad r = \frac{d}{d\alpha}\Big|_{\alpha=0} R_\alpha \in Vect(\mathbb{R})$, $\quad$ inf. transformation: $t \longmapsto t' = t + r(t)$

$r = r(t)\frac{\partial}{\partial t}$

$-r^* : x'(t) = x(t - r(t)) = x(t) - r(t)\,\dot{x}(t) \quad (*)$

$$\delta_r S = \underbrace{-\int_{t_0}^{t_1} dt\; r\,\dot{x}^i\left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\right) - \left|r\left(\dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - \boxed{L}\right)\right|_{t_0}^{t_1}}_{} \quad \sim$$

$\delta_r S = \frac{d}{d\alpha}\Big|_{\alpha=0} S[x(R_\alpha^{-1}(t))]_{R_\alpha(t_0)}^{R_\alpha(t_1)}$

usual variation of fields $(*)$

contribution of variation of t_0, t_1

$\sim$ modulo E-L $\quad -\left|r(t)\, H(x,\dot{x})\right|_{t_0}^{t_1}$ $\quad$ where $\boxed{H := \dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L} \in C^\infty(TX)$

Since constant r (indep. of t) is always a symmetry, i.e. $\delta_r S = 0$ $\Rightarrow$ H is an integral of motion if some non-constant $r \in \mathrm{Vect}(\mathbb{R})$ is a symmetry, then $H \equiv 0$ and in fact the theory is reparameterization-invariant, i.e. any $r \in \mathrm{Vect}(\mathbb{R})$ is a symmetry

Exercise: check that $\boxed{\delta_r S = -\int_{t_0}^{t_1} dt\, \dot{r}(t)\, H(x(t),\dot{x}(t))}$ ← "off-shell" identity, i.e. not modulo E-L

Ex ① $X = \mathbb{R}^D$, $L = \frac{m|\vec{v}|^2}{2} - U(\vec{x})$, $H = \frac{m|\vec{v}|^2}{2} + U(\vec{x})$

- invariant only under constant time-translations, but not under general reparameterizations

② Relativistic particle

$X = \mathbb{R}^{3,1}$ $\quad L = m\sqrt{-(\vec{v},\vec{v})}$, $S = \int_{t_0}^{t_1} dt\, m\sqrt{-\eta_{ij}\dot{x}^i\dot{x}^j}$, $\boxed{H = 0}$

this is a reparam.-invariant theory!

extremals of S = {parameterized geodesics with fixed boundary conditions} / reparametrizations =

= {non-parameterized geodesics}

E-L: $\frac{d}{dt}\left(\frac{\dot{x}^i}{\sqrt{-\eta_{kl}\dot{x}^k\dot{x}^l}}\right) = 0$ - problem with uniqueness of solutions

$$\Uparrow \qquad \not\Downarrow$$

$\ddot{x}^i = 0$

Rem: one can also consider mixed source-target symmetries

$x'(t') = F_\alpha(x(t))$ $\quad$, i.e. $(F_\alpha)_*(R_\alpha^{-1})^* : \mathrm{Maps}([t_0,t_1],X) \to \mathrm{Maps}([R_\alpha(t_0),R_\alpha(t_1)],X)$

$(= F_\alpha(x(R_\alpha^{-1}(t'))))$

infinitesimally: $x'^i(t) = x^i(t) - r(t)\dot{x}^i(t) + f^i(x(t))$

$t_{0,1} \longmapsto t_{0,1} + r(t_{0,1})$

$$\delta_{r,f} S \underset{\mathrm{mod}\,E\text{-}L}{\sim} \Big| \underbrace{\frac{\partial L}{\partial \dot{x}^i} f^i - H\cdot r}_{I_{r,f}\ \in C^\infty(TX\times\mathbb{R})} \Big|_{t_0}^{t_1}$$

If (r,f) is an infinitesimal symmetry, then $I_{r,f}$ is an integral of motion

Ex: free particle on $\mathbb{R}^D$, source/target dilatation

$X = \mathbb{R}^D$, $L = \frac{m|\vec{v}|^2}{2}$, $\quad R_\alpha : t \longmapsto e^\alpha \cdot t$ $\quad$ ← Exercise: find $I_{r,f}$

$F_\alpha : \vec{X} \longmapsto e^{\alpha/2} \cdot \vec{X}$

Rem: One may introduce the Poincaré-Cartan 1-form

$$\beta := \underbrace{\alpha}_{\frac{\partial L}{\partial v^i} dx^i} + H\,dt \in \Omega^1(TX \times \mathbb{R})$$

and the symmetry vector field $-r + \tilde{f} = -r(t)\frac{\partial}{\partial t} + f^i(x)\frac{\partial}{\partial x^i} + \partial_j f^i(x)\, v^j \frac{\partial}{\partial v^i} \in \mathrm{Vect}(TX \times \mathbb{R})$

then one can write the Noether charge for mixed symmetry as

$$\boxed{I_{r,f} = \iota_{-r+\tilde{f}}\,\beta}$$

Hamiltonian mechanics

(Φ, ω) - symplectic mfd - the "phase space", $H \in C^\infty(\Phi)$ - Hamiltonian

$H \rightsquigarrow \underset{\text{the Ham. vector field}}{\check{H} := \{H, \cdot\} \in \mathrm{Vect}(\Phi)}$, $\iota_{\check{H}}\,\omega = dH$

time-evolution = flow of $\check{H}$, i.e. $\dot{Y}^a = \{H, Y^a\}$ for $\{Y^a\}$ -coords on Φ

- "Hamilton's equations of motion"

phase-space symmetry: $\underset{\{\psi,\cdot\}}{\underset{\parallel}{\check{\psi}}} \in \mathrm{Vect}(\Phi)$ such that $[\check{H}, \check{\psi}] \underset{\{H,\psi\}^\vee}{\underset{\parallel}{}} = 0$ $(\Leftrightarrow \{H,\psi\} = C$ -const on $\Phi)$

- if $\{H,\psi\} = 0$ then $\check{H}\psi = 0 \Rightarrow \frac{d}{dt}\psi(Y(t)) = 0 \Rightarrow \psi$ is an integral of motion
- if $\{H,\psi\} = C$ then $\psi - C \cdot t$ is an integral of motion

Lagrangien → Hamiltonian formalism

given X, $L \in C^\infty(TX)$, one sets $\Phi = T^*X$, $\omega = d\alpha$ where $\alpha = p_i\, dx^i$

$$\underbrace{T^*}_{p_i}\underbrace{X}_{x^i} \text{ -coords}$$

assuming the non-degeneracy condition

$$\det\left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right) \neq 0 ,$$

one has fiberwise diffeo

$$\begin{tikzcd}
\lambda : TX \arrow[r, "\sim"] & T^*X = \Phi \\
(x^i, v^i) \arrow[r, mapsto] \arrow[dr] & (x^i, p_i = \frac{\partial L}{\partial v^i}) \arrow[d] \\
& X
\end{tikzcd}$$

one sets $H = (\lambda^{-1})^*\left(v^i \frac{\partial L}{\partial v^i} - L\right)$

- Legendre transform of L

Rem: global description:

$$\begin{tikzcd}
T^*(TX) \arrow[r, "\text{natural}", "\text{iso}"'] & T^*(T^*X) \\
\mathrm{graph}(dL) \arrow[u, hook] \arrow[r, "\sim"] \arrow[d] & \mathrm{graph}(dH) \arrow[u, hook] \arrow[d] \\
TX \arrow[r, "\lambda"] & T^*X
\end{tikzcd}$$