Introduction to Conformal Field Theory

Lecture 1 [23.02.11]

Plan for today: 1. Introduction → 1.1. Geometric "pre-definition" of QFT
→ 1.2. CFT as a set of correlators
→ 1.3. Motivation (applications of CFT)
→ 1.4. Plan of the course

Geometric "pre-definition" of QFT
(after G. Segal)

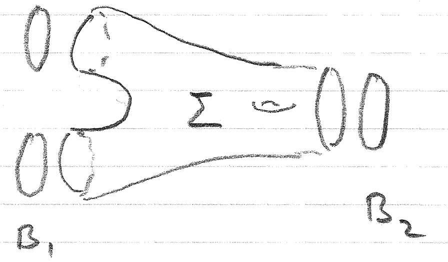
Fix $D \geq 1$. A general QFT is an association:

- $(D-1)$-dimensional closed mfds $\longrightarrow$ vector spaces over $\mathbb{C}$

  $B \longmapsto \mathcal{H}_B$ "space of states"

- D-cobordisms $\longrightarrow$ linear operators

  $B_1 \xrightarrow{\Sigma} B_2 \longmapsto Z_\Sigma : \mathcal{H}_{B_1} \longrightarrow \mathcal{H}_{B_2}$ "evolution operator" or "partition function"

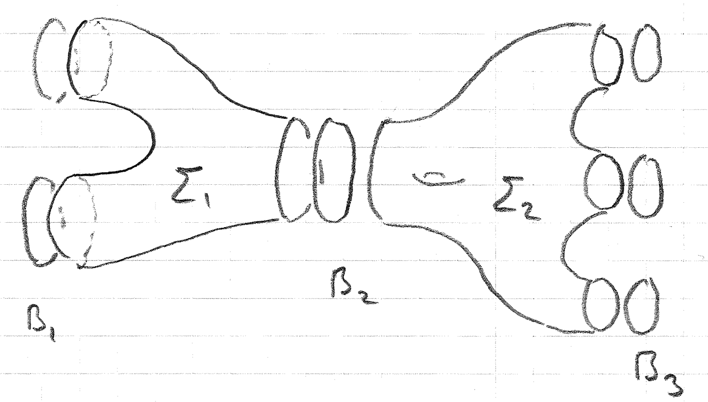


- Sewing axiom "$\cup \to \circ$": sewing of cobordisms is mapped to composition of evol. operators

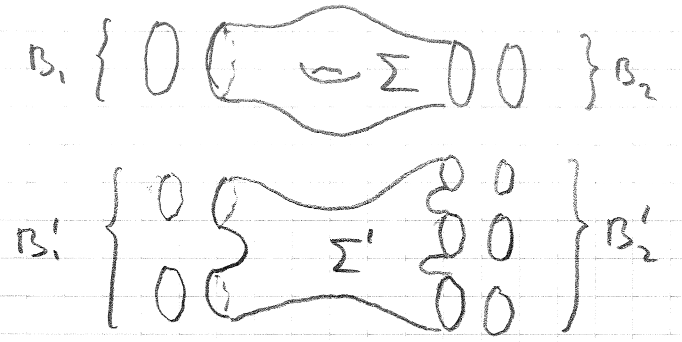
$$B_1 \xrightarrow{\Sigma_1} B_2,\; B_2 \xrightarrow{\Sigma_2} B_3 \;\overset{\text{sewing}}{\rightsquigarrow}\; B_1 \xrightarrow{\Sigma_1 \cup \Sigma_2} B_3$$

$$\Downarrow$$

$$\mathcal{H}_{B_1} \xrightarrow{Z_{\Sigma_1}} \mathcal{H}_{B_2},\; \mathcal{H}_{B_2} \xrightarrow{Z_{\Sigma_2}} \mathcal{H}_{B_3} \;\overset{\text{composition}}{\rightsquigarrow}\; \mathcal{H}_{B_1} \xrightarrow{Z_{\Sigma_2} \circ Z_{\Sigma_1}} \mathcal{H}_{B_3}$$



- Multiplicativity "$\sqcup \to \otimes$": disjoint unions are mapped to tensor products
    - $\mathcal{H}_\emptyset = \mathbb{C}$ (we consider $\emptyset$ as a $(D-1)$-dim. closed mfd)
    - $\mathcal{H}_{B_1 \sqcup B_2} = \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2}$
    - $B_1 \xrightarrow{\Sigma} B_2$, $B_1' \xrightarrow{\Sigma'} B_2'$ $\rightsquigarrow$ $Z_{\Sigma \sqcup \Sigma'} = Z_\Sigma \otimes Z_{\Sigma'} : \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_1'} \to \mathcal{H}_{B_2} \otimes \mathcal{H}_{B_2'}$



$$\Downarrow$$

$$\mathcal{H}_{B_1} \otimes \mathcal{H}_{B_1'} \xrightarrow{Z_\Sigma \otimes Z_{\Sigma'}} \mathcal{H}_{B_2} \otimes \mathcal{H}_{B_2'}$$

- Symmetry under diffeomorphisms.

A correction to the picture: in fact, $\mathcal{H}_B$, Z_Σ depend on the choice of local geometric data on B, Σ: $\gamma_B \in \mathrm{Geom}(B)$, $\gamma_\Sigma \in \mathrm{Geom}(\Sigma)$ spaces of admissible geom. data on bdry component / cobordism <depends on particular QFT>

Examples of local geometric data:

- (Pseudo-) Riemannian metric
- volume form
- conformal structure
- flat connection
- nothing

We assume that geom. data can be sewn, restricted to a bdry component and is contravariant w.r.t. diffeomorphisms

So, value of QFT on a cobordism is

$$Z = Z_{\Sigma, \gamma_\Sigma} : \mathcal{H}_{B_1, \gamma_{B_1}} \longrightarrow \mathcal{H}_{B_2, \gamma_{B_2}}$$

Additional data in QFT:

Symmetry data: diffeomorphisms of B act on $\mathcal{H}_B$ by linear maps

i.e. for a diffeo $m: B \longrightarrow B'$ we have $(m^{-1})^*: \mathrm{Geom}(B) \longrightarrow \mathrm{Geom}(B')$, $\gamma_B \longmapsto \gamma_{B'}$

and a linear map $\rho(m): \mathcal{H}_{B, \gamma_B} \longrightarrow \mathcal{H}_{B', \gamma_{B'}}$

So that for a composition of diffeo $B \xrightarrow{m} B' \xrightarrow{m'} B''$ we have
$\rho(m' \circ m) = \rho(m') \circ \rho(m)$

Naturality (equivariance) axiom:

For a diffeo of cobordisms

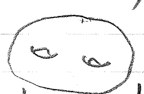
$$\begin{tikzcd}
(B_1, \gamma_{B_1}) \arrow[rr, "{(\Sigma, \gamma_\Sigma)}"] \arrow[d, "m_1"'] & {} \arrow[d, phantom, "\Downarrow m"] & (B_2, \gamma_{B_2}) \arrow[d, "m_2"] \\
(B_1', \gamma_{B_1'}) \arrow[rr, "{(\Sigma', \gamma_{\Sigma'})}"'] & {} & (B_2', \gamma_{B_2'})
\end{tikzcd}$$

we have a commutative diagram

$$\begin{tikzcd}
\mathcal{H}_{B_1, \gamma_{B_1}} \arrow[r, "Z_{\Sigma, \gamma_\Sigma}"] \arrow[d, "\rho(m_1)"'] & \mathcal{H}_{B_2, \gamma_{B_2}} \arrow[d, "\rho(m_2)"] \\
\mathcal{H}_{B_1', \gamma_{B_1'}} \arrow[r, "Z_{\Sigma', \gamma_{\Sigma'}}"] & \mathcal{H}_{B_2', \gamma_{B_2'}}
\end{tikzcd}$$

(i.e. partition function transforms equivariantly w.r.t. diffeomorphisms)

Remarks ① for a closed D-mfd $\varnothing \xrightarrow{\Sigma} \varnothing$ the partition function



$Z_\Sigma : \mathbb{C} \to \mathbb{C}$ is a multiplication by some number; by abuse of notation, this number is also called the partition function and denoted Z_Σ

② One says: a QFT is a functor of symmetric monoidal categories

$$(\mathcal{H}, Z) : \left\{ \text{space-time category} \;\middle|\; \begin{array}{l} \text{Ob: } (B, \gamma_B) \\ \text{Hom: } (\Sigma, \gamma_\Sigma) \\ \text{composition} = \text{sewing} \\ \text{tensor product} = \sqcup \\ \text{unit} = \varnothing \end{array} \right\} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$$

Naturality axiom says that diffeomorphism act on $(\mathcal{H}, Z)$ by natural transformations

(Alternatively, one can understand QFT as a functor of 2-categories with diffeomorphisms being the 2-morphisms of space-time category)

Reference: N. Reshetikhin "Lectures on quantization of gauge systems", arXiv: 1008.1411

③ It is interesting to restrict the naturality axiom to $\mathrm{Sym}_{\Sigma, \gamma_\Sigma} \subset \mathrm{Diff}(\Sigma)$ - subgroup of diffeo, preserving chosen geom. data $\gamma_\Sigma \in \mathrm{Geom}(\Sigma$

Then the axiom yields symmetries of $Z_{\Sigma, \gamma_\Sigma}$ (the "Ward identities")

4 Examples of QFTs:

- $D=1$, $\mathrm{Geom} = \{\text{Riemannian metrics}\}$

for an interval $\Sigma = \bullet\!\!-\!\!-\!\!-\!\!-\!\!\bullet$ we have

$$Z_{\bullet\!-\!\bullet} \in \mathrm{End}(\mathcal{H}_{pt}) \otimes \underbrace{\mathrm{Fun}(\mathrm{RiemMet}_{\bullet\!-\!\bullet})}_{}{}^{\mathrm{Diff}(\bullet\!-\!\bullet)}$$

due to naturality axiom

dependence on geom. data on the interval

i.e. $$Z_{\bullet\!-\!\bullet} \in \mathrm{End}(\mathcal{H}_{pt}) \otimes \mathrm{Fun}(\mathbb{R}_+)$$

length of the interval

Sewing axiom: $\overset{t_1}{\bullet\!-\!-\!\bullet}\overset{t_2}{\!-\!-\!\bullet}$

$$Z(t_1+t_2) = Z(t_2)\circ Z(t_1) \quad \text{- semi-group law}$$

if we assume in addition the asymptotics $Z(t) \sim \mathrm{id} + \hat{H}\cdot t + O(t^2)$, we recover

$$Z(t) = e^{\hat{H}t} \quad \text{- evolution operator of quantum mechanics!}$$

- $D=2$, $\mathrm{Geom} = \left\{ \begin{matrix} \text{volume forms} \\ \text{on } \Sigma \end{matrix} \right\}$, $\mathrm{Sym} = \left\{ \begin{matrix} \text{volume-preserving} \\ \text{diffeomorphisms} \end{matrix} \right\}$, $\mathrm{Geom}/_{\mathrm{Diff}} = \mathbb{R}_+$ - volume of Σ

  - 2D Yang-Mills theory (Migdal '75)

  reference: S. Cordes, G. Moore, S. Ramgoolam, "Lectures on 2D Yang-Mills theory ...", arXiv: hep-th/9411210

$$\mathcal{H}_{S^1} = \mathrm{Span}_{\mathbb{C}} \left\{ \begin{matrix} \text{irrep. of} \\ \text{a compact Lie group } G \end{matrix} \right\}, \quad Z_\Sigma \text{ known explicitly and depend on volume of } \Sigma$$

- TQFT (Atiyah '89)

  $\mathrm{Geom} = \varnothing$, $\mathrm{Sym} = \{\text{all diffeomorphisms}\}$

  if $\partial\Sigma = \varnothing$, then $Z_\Sigma \in \mathbb{C}$ is a topological invariant

  e.g. 3D Chern-Simons theory (Witten '89)

  and 2D BF theory (="topological sector" of 2D Yang-Mills: limit volume $\to 0$)

- $D=4$, $\mathrm{Geom} = \left\{ \begin{matrix} \text{metrics of signature} \\ (3,1) \text{ on } \Sigma \end{matrix} \right\}$

  - the zoo of physically-relevant QFTs like QED, 4D Yang-Mills etc.

Conformal field theories:

- $D=2$, Geom = {conformal structures on Σ}, Sym = {conformal maps} ⑤

  $\mathrm{Geom}/\mathrm{Diff} = \left\{\begin{array}{l}\text{moduli space of}\\ \text{conf. structures on } \Sigma\end{array}\right\}$ – finite-dim variety ! (specifically for $D=2$)

  $\mathcal{H}_{S^1}$ is a representation of "local conformal algebra"

⑤ Classical field theory also admits a "pre-definition" as a functor, but with a different target category:

Ob: symplectic mfds ("phase spaces")

Hom: canonical relations ⟨a problem may arise with smoothness of the composition⟩

⑥ What has to be corrected in these axioms of QFT:

- definition of objects of the space-time category has to be modified
  (closed mfds with geom. data) $\to$ $\left(\begin{array}{l}\text{germs of collars endowed with}\\ \text{germs of geom. data}\end{array}\right)$

- Z is a projective functor, ρ is a projective representation of diffeomorphisms, naturality axiom holds in projective sense

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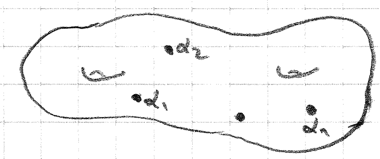
CFT as a set of correlators

Typically one studies CFT in a more restrictive setting



boundary circles $\longrightarrow$ punctures = "infinitesimal circles"

partition function $\longrightarrow$ n-point correlator (n = number of punctures)

$$Z \in \mathcal{H}^{\otimes n} \otimes \mathrm{Fun}(\underbrace{\mathcal{M}_{g,n}}_{})$$

moduli space of conformal (complex) structures

One denotes the correlator evaluated on states (= "local quantum fields") $\alpha_1, \ldots, \alpha_n \in \mathcal{H}$ inserted in punctures as $\langle \alpha_1 \ldots \alpha_n \rangle \in \mathrm{Fun}(\mathcal{M}_{g,n})$

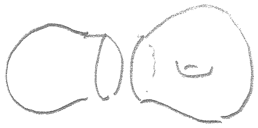
Some special fields: • identity field $\mathbb{1}$ (= "vacuum") effectively forgets the puncture:

$$\langle \mathbb{1}\, \alpha_2 \ldots \alpha_n \rangle = \langle \alpha_2 \ldots \alpha_n \rangle$$

- Stress-energy tensor T : $\langle T \alpha_1 \ldots \alpha_n \rangle$ gives a connection on $\mathcal{M}_{g,n}$ w.r.t. which $\langle \alpha_1 \ldots \alpha_n \rangle$ is a covar. constant section

- primary fields: their correlators "transform nicely" w.r.t. conformal maps which do not move the punctures

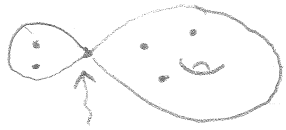
Instead of sewing



one has degenerations

merging of punctures

or equivalently



neck

close to degenerations (on $\mathcal{M}_{g,n}$)
the corr. functions are singular and described by the Operator product expansion (OPE):

$$\langle \alpha(z_1)\beta(z_2)\underbrace{\cdots}\rangle = \sum_{\gamma\in\mathcal{H}} C^{\gamma}_{\alpha\beta}(z_1,z_2)\cdot\langle\gamma(z_2)\underbrace{\cdots}\rangle$$

fields at other punctures

The idea in CFT
is to recover correlators from the singular part of functions $C^{\gamma}_{\alpha\beta}(z_1,z_2)$ at $z_1\to z_2$
(like recovering a meromorphic function from singular parts of Laurent expansions at poles)

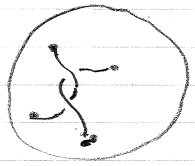
In CFT one studies:
- the space $\mathcal{H}$ (as a rep. of local conformal algebra)
- OPE
- correlators

Why care about CFT?

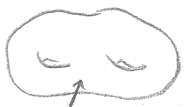
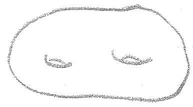
① As a toy QFT with lots of explicit answers, e.g. corr. functions given explicitly in terms of hypergeometric functions <in some CFTs> and partition functions expressed in terms of modular functions (Dedekind η, Jacobi ϑ etc.)

② Due to relation to critical phenomena in statistical physics
E.g. Ising model on 2D lattice (in the limit of infinite lattice) at the point of 2^{nd} order phase transition attains conformal symmetry
$\rightsquigarrow$ one is interested in "critical exponents", e.g. in the spin-spin correlator $\langle\sigma(x)\sigma(y)\rangle\sim\frac{1}{|x-y|^{\alpha}}$
<Reference: BPZ'84>

③ Relation 2D CFT $\longleftrightarrow$ 3D TQFT
correlator of a 3D TQFT on a 3-mfd with bdry and a tangle inserted gives a correlator of certain CFT on the boundary.



More technically: $\left(\begin{array}{l}\text{space of}\\ \text{states of a}\\ \text{3-TQFT for}\\ \text{a closed 2-mfd } B\end{array}\right) = \left(\begin{array}{l}\text{"space of conformal blocks"}\\ \text{for some 2D CFT}\\ \text{for } B\end{array}\right)$
<maybe with punctures>

4 Relation to string theory:
as a quantization of
a Lagrangian field theory on
the space of maps  $\longrightarrow$ target $\mathbb{R}^N$ $\longrightarrow$ N quantum scalar fields on 

2D surface (worldsheet)

5 CFT $\longleftarrow$ Rep. theory of Virasoro algebra, current algebras

6 Mirror symmetry

Tentative plan of the course

- Conformal symmetry
  (- in $\mathbb{R}^{p,q}$, exceptional cases: $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^{1,1}$)
- Symmetry in field theory
  (- Noether thm, Stress-energy tensor, Ward identities)
- Implications of conformal symmetry
  (- Primary fields, global conf. symmetry - restrictions on correlators, conf. Ward identities, OPE)
- Free boson, free fermion, bc system (mode expansions, Wick's thm, correlators, OPE, stress-energy tensor, vertex operators)
- Central charge
- Representations of Virasoro algebra
- Minimal models of CFT (+ relation to Ising)
- WZW and relation to Chern-Simons theory
- mathematical approaches to CFT $\nearrow$ axiomatics of Segal, Moore-Seiberg
  $\searrow$ vertex operator algebras

Main reference (for the most part of the course):
P. Ginsparg, "Applied Conformal Field Theory", arXiv:hep-th/9108028