Fall 2017
MATH 70330 "Intermediate Geometry and Topology"
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Detailed plan.

I. Characteristic classes.
   (a) Fiber/vector/principal bundles. Examples. Connections, curvature. Riemannian case. **Ref:** [MT97].
   (b) (2 classes.) Stiefel-Whitney, Chern and Pontryagin classes – properties, axiomatic definition. Classifying map and classifying bundle. $\mathbb{R}P^\infty, \mathbb{C}P^\infty$, infinite Grassmanians $Gr(n,\infty)$, CW structure, cohomology ring (universal characteristic classes). Characteristic classes as obstructions (e.g. for embeddability of projective spaces). Euler class. Also: Chern character, splitting principle, Chern roots. Also: BG for a finite group/Lie group, group cohomology. **Ref:** [MS74]; also: [BT82, Hat98, LM98, May99].
   (c) Chern-Weil homomorphism. Example: Chern-Gauss-Bonnet formula. Chern-Simons forms. **Ref:** Appendix C in [MS74]; [AM05, Dup78, MT97].
   (d) Equivariant cohomology. Borel model (homotopy quotient), algebraic version – Cartan and Weil models. **Ref:** [GS99, Mei06, Tu13].

II. Bits of symplectic geometry.
   (a) Symplectic linear algebra, Lagrangian Grassmanian, Maslov class. **Ref:** [BW97, Ran, MS17].
   (b) Symplectic manifolds, Darboux theorem (proof via Moser's trick). Distinguished submanifolds (isotropic, coisotropic, Lagrangian). Examples. Constructions of Lagrangians in T^*M (graph, conormal bundle). **Ref:** [DS00].
   (c) Hamiltonian group actions, moment maps, symplectic reduction. **Ref:** [Jef] (lectures 2–4,7), [DS00].
   (d) (Optional) Convexity theorem (Atyiah-Guillemin-Sternberg) for the moment map of a Hamiltonian torus action. Toric varieties as symplectic reductions, their moment polytopes, Delzant's theorem. **Ref:** [Pra99, Sch], Part XI in [DS00], lectures 5,6 in [Jef].
   (e) Localization theorems: Duistermaat-Heckman and Atiyah-Bott. **Ref:** [Mei06, Tu13], [Jef] (lecture 10).
   (f) (Optional) Classical field theory via Lagrangian correspondences. Chern-Simons theory and Atiyah-Bott symplectic structure on the moduli space of flat connections on a surface; line bundle on the moduli space. **Ref:** [Fre95, CMR12, Mne17].
   (g) Introduction to Floer homology. **Ref:** [MS04].
   (h) (Optional) Fukaya category. Explicit example of homological mirror symmetry for an elliptic curve. **Ref:** Polishchuk-Zaslow paper [PZ98].
   (i) (Optional) Geometric quantization of symplectic (and in particular Kähler) manifolds. Examples: quantization of coadjoint orbits (Borel-Weil-Bott theorem / Kirillov's orbit method), quantization of the moduli space of flat connections on a surface (Axelrod - Della Pietra - Witten). **Ref:** [BW97], Lecture 12 in [Jef].

III. Index theory.
Ref: [Roe99, BGV03, Fre87, LM16]; a short review and review talk slides: [HK09, Maz02]; original papers: [AS68b, AS68a].
(a) Clifford algebra, Spin and Pin groups. Laplacian and Dirac operator on a flat space. Differential-geometric setting: spin structure, (2nd Stiefel-Whitney class as obstruction), Dirac operator.
(b) Elliptic operators and elliptic complexes. Analytic index. Examples.
(c) Multiplicative sequences, genera. Cobordism ring and Thom's theorem. Some important genera: Todd genus, $\hat{A}$, L.
(d) (2 classes?) Topological index. Statement of Atiyah-Singer index theorem. Examples: Riemann-Roch-Hirzebruch, Hirzebruch signature theorem, Rochlin's theorem.
(e) (2 classes?) Topological K theory - basic definitions. K-theoretic statement of the index theorem.
(f) Atiyah-Patodi-Singer approach to index theorems via the heat equation. Index theorem on manifolds with boundary, eta invariant. **Ref:** [APS75].
(g) (Optional) Applications of index theorems. E.g. application of Riemann-Roch-Hirzebruch to geometric quantization; dimension of Verlinde space (geometric quantization of the moduli space of flat connections on a surface).

IV. Topological quantum field theory – Atiyah's definition and examples.
(a) Atiyah's definition of topological quantum field theory. Classification theorems in dimensions $1, 2$. **Ref:** [Ati88].
(b) Dijkgraaf-Witten theory – count of coverings twisted by a group cocycle. **Ref:** [DW90].
(c) (Optional) Turaev-Viro invariants of 3-manifolds. **Ref:** [TV92].

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