HS 2015
14-16.10.2015

Quantum BV theories on manifolds with boundary

ref 1201.0290 (CMR 1, Cl. BV th. on mflds w/ bdry), 1507.01221 (CMR 2)

(1)

Motivation

· (1) Chern-Simons theory defined classically by

action $S = \int \mathrm{tr}\, \frac{1}{2} A\, dA + \frac{1}{3} A \wedge A \wedge A$,

fields $F = \mathrm{Conn}(M, P) \ni A$

(M: 3-mfld, P: triv G-bundle)

$\xrightarrow{\text{quantization}}$ Heuristically, $Z = \text{"}\int_{\mathrm{Conn}/\cancel{\mathrm{Gauge}}} \mathcal{D}A\text{"}\, e^{\frac{i}{\hbar} S(A)}$ — invariant of a 3-manifold

$\rightsquigarrow$ replace Z (formal) by a stationary phase evaluation of the $\int \mathcal{D}A$ $\rightarrow$ critical points of S (flat connections) are degenerate. $\rightarrow$

- employ gauge-fixing
- Batalin-Vilkovisky construction (evolved from Faddeev-Popov → BRST)

$$\int_N e^{\frac{i}{\hbar} f} \mu \underset{\hbar \to 0}{\sim} \sum_{\substack{\text{crit pts}\\ x_0 \in N}} e^{\frac{i}{\hbar} f(x_0)} (2\pi\hbar)^{\frac{\dim N}{2}} (\det \partial^2_{x_0} f)^{-\frac{1}{2}} e^{\frac{\pi i}{4} \operatorname{sgn} \partial^2 f(x_0)} \cdot \exp \sum_{\Gamma \text{ conn'ted},\, b_1 \geq 2} \hbar^{-\chi(\Gamma)} \Phi_\Gamma(x_0)$$

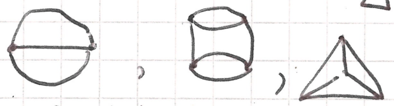
framing; cancellation of anomaly

$$\rightsquigarrow Z^{\mathrm{pert}}(M, \hbar, A_0, g) = e^{\frac{i}{\hbar} S(A_0)}\, \tau(M, E)^{1/2} \cdot e^{\frac{\pi i}{4} \psi(A_0, g)} \cdot \exp\Big(\sum_{\substack{\text{connected}\\ \text{3-valent}\\ \text{graphs } \Gamma}}^{V+E} \hbar^{-\chi(\Gamma)} \frac{1}{|\mathrm{Aut}\,\Gamma|} \Phi_\Gamma(A_0, g) \Big) \cdot e^{i\, S_{\mathrm{grav}}(g) \cdot c(\hbar)}$$

- Witten-Axelrod-Singer for M closed, Rational homology sphere
- A_0: acyclic flat conn.
- τ: R-torsion; E - loc. system assoc to A_0
- APS η-invariant of $L_- = *d_{A_0} + d_{A_0}*$ on $\Omega^{\mathrm{odd}}(M, g)$
- $\Phi_\Gamma = \int_{\mathrm{Conf}_V(M)} \prod_{e \in E} \pi^*_{e_{\mathrm{in}}, e_{\mathrm{out}}} \eta$

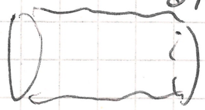
$\eta \in \Omega^2(\mathrm{Conf}_2(M), E \boxtimes E^*)$ - int. kernel for d^*_E / Δ_E

- propagator.

Γ: , ...

(2) Atiyah-Segal axiomatic description of QFT:

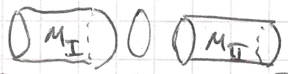
- $(n-1)$-closed mflds $\Sigma \longmapsto \mathcal{H}_\Sigma$ (vect. space) · space of states
- n-mfld M, $\partial M = \partial_{\mathrm{in}} M \sqcup \partial_{\mathrm{out}} M \longmapsto Z_M \in \mathrm{Hom}(\mathcal{H}_{\partial_{\mathrm{in}} M}, \mathcal{H}_{\partial_{\mathrm{out}} M})$



in, out

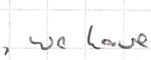
· for $M \xrightarrow{\varphi} M'$ a diffeo, we have

$$\begin{tikzcd} \Sigma_{\mathrm{in}} \arrow[r, Rightarrow, "M"] \arrow[d, "\varphi|_{\mathrm{in}}"] & \Sigma_{\mathrm{out}} \arrow[d, "\varphi|_{\mathrm{out}}"] \\ \Sigma'_{\mathrm{in}} \arrow[r, Rightarrow, "M'"] & \Sigma'_{\mathrm{out}} \end{tikzcd} \longrightarrow \begin{tikzcd} \mathcal{H}_{\mathrm{in}} \arrow[r, "Z_M"] \arrow[d, "\wr\, \rho(\varphi|_{\mathrm{in}})"] \arrow[dr, phantom, "\square"] & \mathcal{H}_{\mathrm{out}} \arrow[d, "\wr\, \rho(\varphi|_{\mathrm{out}})"] \\ \mathcal{H}'_{\mathrm{in}} \arrow[r, "Z_{M'}"] & \mathcal{H}'_{\mathrm{out}} \end{tikzcd}$$

- disjoint unions $\longmapsto \otimes$
- gluing  $\longmapsto Z_M = Z_{M_{II}} \circ Z_{M_I}$ - composition of Homs

Σ_1, Σ_2, Σ_3; $M = M_I \cup_{\Sigma_2} M_{II}$

$\mathcal{H}_{\Sigma_3} \longleftarrow \mathcal{H}_{\Sigma_2} \longleftarrow \mathcal{H}_{\Sigma_1}$

- topological case: for M closed, $Z_M \in \mathbb{C}$ - diff. invariant of M,  can be calculated from cutting M into simpler pieces. (e.g. surfaces → pairs of pants & disks)
- non-top. case (Segal): Z depends on loc. geometric data on M

$Z \in \mathrm{Hom}(\mathcal{H}_{\mathrm{in}}, \mathcal{H}_{\mathrm{out}}) \otimes \mathrm{Fun}(\mathrm{Geom}_M)$

E.g.
- Riem. metric (QM, 4D YM)
- volume form (2D YM)
- conformal structure (CFT)

The program (C-M-R): data of classical gauge theory S, F $\xrightarrow{\text{perturbative path integral}}$ (functorial) QFT in Atiyah-Segal* sense (on mflds with bdry & gluing) ← some modification of axioms

Batalin-Vilkovisky formalism

Let $V^\bullet$ - a f.d. $\mathbb{Z}$-graded v.sp. with $\omega: V^k \otimes V^{1-k} \to \mathbb{R}$ a non-deg pairing

$\mathcal{A} = \mathrm{Sym}\, V^* = \bigotimes_{k\ \text{even}} \mathrm{Sym}(V^k)^* \otimes \bigotimes_{k\ \text{odd}} \Lambda(V^k)^*$ is a $\mathbb{Z}$-graded supercom. algebra,

moreover - a BV algebra: $(\mathcal{A}^\bullet, \cdot, \{,\}, \Delta)$

$\{,\}: \mathcal{A}^i \otimes \mathcal{A}^j \to \mathcal{A}^{i+j+1}$ odd Poisson bracket

$\Delta: \mathcal{A}^i \to \mathcal{A}^{i+1}$ BV Laplacian, 2nd order derivation

s.t. $\Delta(xy) = \Delta x \cdot y + (-1)^{|x|} x \Delta y + (-1)^{|x|}\{x,y\}$, $\quad \Delta^2 = 0$

here, if (x^i, ξ_i) are Darboux coordinates on $V^\bullet$,

$$\Delta = \sum_i \frac{\partial^2}{\partial x^i \partial \xi_i}, \qquad \{f,g\} = \sum_i f\left(\frac{\overleftarrow{\partial}}{\partial x^i}\frac{\overrightarrow{\partial}}{\partial \xi_i} - \frac{\overleftarrow{\partial}}{\partial \xi_i}\frac{\overrightarrow{\partial}}{\partial x^i}\right) g$$

Diff. geom. version:

$\mathcal{N}$ - $\mathbb{Z}$-graded supermanifold, $\omega \in \Omega^2(\mathcal{N})_{-1}$ - degree -1 symplectic form ($\delta\omega = 0$, $T\mathcal{N} \xrightarrow[\sim]{\omega^\#} T^*[-1]\mathcal{N}$ - iso)

$\mathcal{A} = C^\infty(\mathcal{N})$,

$f \longmapsto \check{f} \in \mathfrak{X}(\mathcal{N})$ s.t. $\iota_{\check{f}}\,\omega = \delta f$

$\{f,g\} := \check{f}(g)$ - makes $(\mathcal{A}, \cdot, \{,\})$ a Gerstenhaber algebra

Let $\mu \in \mathrm{Dens}(\mathcal{N})$ a volume element, set

$\Delta_\mu f := \frac{1}{2}\,\mathrm{div}_\mu \check{f}$ $\quad$ (Rem: $\int_{\mathcal{N}} \mu\, X(g) = -\int_{\mathcal{N}} \mu\, g \cdot \mathrm{div}_\mu X$ defines div_μ)

Assume that $\Delta^2 = 0$, i.e. μ and ω are "compatible".

$(\mathcal{A}, \cdot, \{,\}, \Delta)$ is again a BV algebra.

$S \in \mathcal{A}^0$ satisfies "classical master equation" (CME) if $\{S,S\} = 0$

- then $\check{S} =: Q \in \mathfrak{X}(\mathcal{N})_1$ satisfies $Q^2 = 0$. such vector fields are called "cohomological"

$S_\hbar = S^{(0)} + \hbar S^{(1)} + \hbar^2 S^{(2)} + \ldots \in \mathcal{A}^0[[\hbar]]$ satisfies "quantum master equation" (QME)

if $\boxed{\frac{1}{2}\{S_\hbar, S_\hbar\} - i\hbar\, \Delta_\mu S_\hbar = 0} \iff \boxed{\Delta_\mu e^{\frac{i}{\hbar} S_\hbar} = 0}$

Rem One can introduce "half-densities" $\mathrm{Dens}^{1/2}(\mathcal{N}) \ni \alpha$

locally, in a chart $\{z^i\}$ $\quad \alpha = f(z) \cdot \prod_i (Dz^i)^{1/2}$

in an overlapping chart $\{z'^i\}$ $\quad \alpha = f'(z') \cdot \prod_i (Dz'^i)^{1/2}$,

$$f' = f \circ (\varphi_U \varphi_V^{-1}) \cdot |\mathrm{Ber}\, d(\varphi_U \varphi_V^{-1})|^{1/2}$$

$U \subset \mathcal{N} \supset V$, $\varphi_U : U \to \mathbb{R}^{n|m} \ni z$, $\varphi_V : V \to \mathbb{R}^{n|m} \ni z'$ (Global)

Lin. alg. case: $\mathrm{Dens}^{1/2}_{\mathrm{const}}(V^\bullet) \simeq (\underbrace{\mathrm{Det}\, V^\bullet}_{\bigotimes_k (\Lambda^{top} V^k)^{(-1)^k}})^{\otimes -\frac{1}{2}}$

[Ševera's definition: $\mathrm{Dens}^{1/2}(\mathcal{N}) = H^\bullet_{\omega\wedge}(\Omega^\bullet(\mathcal{N}))$]

$f' = (f \circ \varphi) \cdot |\det \varphi|^{1/2}$

Khudaverdian, Ševera: One has the "canonical" BV Laplacian $\Delta : \mathrm{Dens}^{1/2}(\mathcal{N}) \circlearrowleft$

defined locally by (in Darboux chart)

$$\Delta : f(x,\xi) \cdot \prod_i (Dx^i D\xi_i)^{1/2} \longmapsto \Big(\sum_i \frac{\partial^2 f}{\partial x^i \partial \xi_i}\Big) \cdot \prod_i (Dx^i D\xi_i)^{1/2}$$

We have

$$\begin{array}{ccc} C^\infty(\mathcal{N}) & \xrightarrow{\Delta_\mu} & C^\infty(\mathcal{N}) \\ \downarrow \cdot \mu^{1/2} & \square & \downarrow \cdot \mu^{1/2} \\ \mathrm{Dens}^{1/2}(\mathcal{N}) & \xrightarrow[\Delta]{} & \mathrm{Dens}^{1/2}(\mathcal{N}) \end{array}$$

- for $\mathcal{L} \subset \mathcal{N}$ a Lagrangian submanifold, one has a restriction map

$$\mathrm{Dens}^{1/2}\mathcal{N} \longrightarrow \mathrm{Dens}^1 \mathcal{L}$$

locally if $\mathcal{L}$ is given by $\xi = 0$: $\quad f(x,\xi) \cdot \prod (Dx^i D\xi_i)^{1/2} \longmapsto f(x,0) \cdot \prod_i Dx^i$

In lin. alg. $\mathrm{Det}(\underbrace{L \oplus L^*[-1]}_{V \text{ odd-symp}}) \cong (\mathrm{Det}\, L)^{\otimes 2}$.

BV integrals – integrals of form $\int_{\mathcal{L} \subset \mathcal{N}\ \mathrm{Lagr}} \alpha|_{\mathcal{L}}$, $\alpha \in \mathrm{Dens}^{1/2}\mathcal{N}$, $\Delta\alpha = 0$

- "BV-Stokes' theorem"

(1) $\int_{\mathcal{L}} \Delta\beta = 0$

(2) if $\{\mathcal{L}_t\}$ a smooth family of Lagrangians,

$\int_{\mathcal{L}_0} \alpha = \int_{\mathcal{L}_1} \alpha$ if $\Delta\alpha = 0$.

So: BV-integral is a pairing between H_Δ and classes of Lagrangians up to Lagr. homotopy.

Application: "Gauge-fixing". $\mathcal{N} = T^*[-1]\mathcal{B}$ ("BV fields", "BRST fields"), $\alpha = e^{\frac{i}{\hbar}S(x,\xi)}\mu^{1/2}$, where S solves QME

$Z = \int e^{\frac{i}{\hbar}S(x,\xi)} \mu^{1/2}\Big|_{\mathcal{L}_0}$ can have degenerate crit. points;

$\mathcal{L}_0$ = zero section of $T^* \subset \mathcal{N}$

choose $\Psi \in C^\infty(\mathcal{B})_{-1}$ – "gauge-fixing fermion"

and replace $Z \longmapsto Z_\Psi = \int_{\text{graph } d\Psi} e^{\frac{i}{\hbar}S}\mu^{1/2} = \int e^{\frac{i}{\hbar}S(x,\xi = \frac{\partial\Psi}{\partial x})} Dx$

$\xi_i = \frac{\partial \Psi}{\partial x^i}$

– will have isolated crit points (for a good choice of Ψ).

Example: G (group) $\curvearrowright F$, $S_{cl} \in C^\infty(F)^G$, $\nu \in \Omega^{top}(F)^G$

$\mathcal{B} = \underset{x}{F} \times \underset{c}{\mathfrak{g}[1]} \times \underset{\lambda}{\mathfrak{g}^*} \times \underset{\bar{c}}{\mathfrak{g}^*[-1]}$

$S(x, c, x^+, c^+) = S_{cl}(x) + x_i^+ \underbrace{v_a^i(x)}_{\text{fund vector fields of } G\text{-action}} c^a + \frac{1}{2} f^a_{bc} c^+_a c^b c^c + \bar{c}^+_a \lambda^a$ $\quad \Psi = \sum_a \bar{c}_a \varphi^a(x)$

then $Z_\Psi = \mathrm{Vol}\, G \cdot \int_{F/G} e^{\frac{i}{\hbar}S_{cl}} \nu$, but

Z_Ψ has non-deg cr. points and can be evaluated via stat. phase

BV pushforward

Let $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ – as odd-symp manifolds, $\Delta = \Delta_1 \otimes id_2 + id_1 \otimes \Delta_2$

If $\mathcal{L} \underset{\text{Lagr}}{\subset} \mathcal{N}_2$, we have $P_*: \mathrm{Dens}^{1/2}\mathcal{N} \longrightarrow \mathrm{Dens}^{1/2}\mathcal{N}_1$

$$P: \mathcal{N} \twoheadrightarrow \mathcal{N}_1 \qquad \alpha \longmapsto \int_{\mathcal{L}\subset \mathcal{N}_2} \alpha$$

$$\mathrm{Dens}^{1/2}\mathcal{N} \simeq \mathrm{Dens}^{1/2}\mathcal{N}_1 \hat{\otimes} \mathrm{Dens}^{1/2}\mathcal{N}_2 \xrightarrow[id\otimes \int_{\mathcal{L}}]{} \mathrm{Dens}^{1/2}\mathcal{N}_1$$

– BV pushforward, or fiber BV integral

Properties – P_* is a chain map wrt Δ, Δ_1. In particular, sends solutions of QME on $\mathcal{N}$ to solutions of QME on $\mathcal{N}_1$ via $e^{\frac{i}{\hbar}S}\mu^{1/2} \longmapsto e^{\frac{i}{\hbar}S_1}\mu_1^{1/2}$.
S_1 is the "effective BV action" induced on $\mathcal{N}_1$ (~"slow fields")

– If $\mathcal{L}, \mathcal{L}'$ "homotopic" in $\mathcal{N}_2$,
$$P_*^{\mathcal{L}'}\alpha - P_*^{\mathcal{L}}\alpha = \Delta_1(\dots)$$

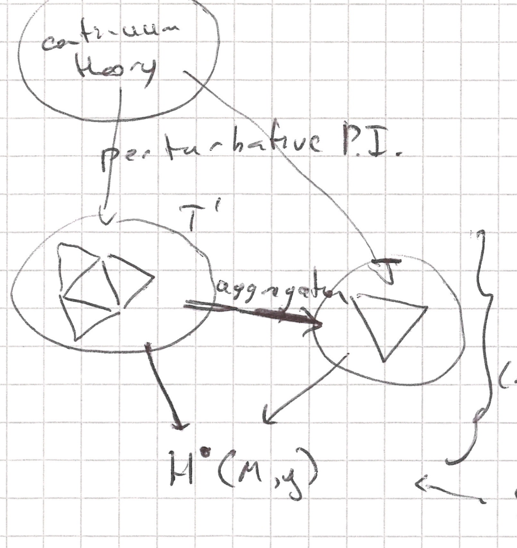
– we call solutions S, S' of QME equivalent, if
$$e^{\frac{i}{\hbar}S'}\mu^{1/2} - e^{\frac{i}{\hbar}S}\mu^{1/2} = \Delta(\dots)$$

Thus, BV pushforward gives a map
$$\mathrm{SolQME}(\mathcal{N})/\sim \;\xrightarrow{P_*^{[\mathcal{L}]}}\; \mathrm{SolQME}(\mathcal{N}_1)/\sim$$
– depends on the class of $\mathcal{L}$.

If $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2 \times \dots \times \mathcal{N}_N$,
one has a model for Wilson's renormalization flow,
$$S \longrightarrow S_{\leq N-1} \longrightarrow S_{\leq N-2} \longrightarrow \dots \dots \longrightarrow S_1$$

Example (P.M.) – "simplicial BF theory". $\mathcal{F}_{cont} = \underset{A}{\Omega^\bullet(M,\mathfrak{g})[1]} \oplus \underset{B}{\Omega^\bullet(M,\mathfrak{g}^*)[n-2]}$, $S = \int_M \langle B, dA + \frac{1}{2}[A,A]\rangle$

triangulation T of M $\rightsquigarrow$ $\mathcal{F}_T = C^\bullet(T,\mathfrak{g})[1] \oplus C_\bullet(T,\mathfrak{g}^*)[-2]$,
$$S_T = \sum_{\sigma\in T} S_\sigma(A_T|_\sigma, B_T|_\sigma; \hbar)$$
univ. local building blocks, depending only on dim σ.



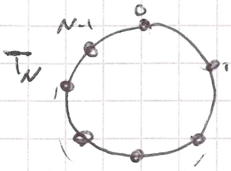
BV pushforwards given by f.d. (measure-theoretic) integral

– "Wilson's renorm. flow along the poset of triangulations" = "realizations" of the theory.

$S_{H^\bullet}$ is a gen. function for uL_∞ structure on $H^\bullet(M,\mathfrak{g})$ an contains information on rational homotopy type of M and R-torsion, in nbhd of $[0] \in MFC(M,G)$.

hep-th/0610326 ; 0809.1160

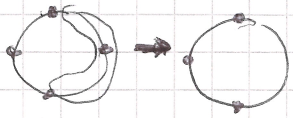
Example: 1D BF



T_N ← triangulation of S^1

$$S_{T_N} = \sum_{k=0}^{N-1} \langle \underbrace{B_k}_{gh=-2}, \tfrac{1}{2}[\underbrace{A_k}_{gh=1}, A_k]\rangle + \sum_{k=0}^{N-1} \langle B_{k,k+1}, [A_{k,k+1}, \tfrac{A_k + A_{k+1}}{2}] + \left(\tfrac{ad_{A_{k,k+1}}}{2} \coth \tfrac{ad_{A_{k,k+1}}}{2}\right)\circ (A_{k+1} - A_k)\rangle - i\hbar \sum_{k=0}^{N-1} \log \det_{\mathfrak{g}} \frac{\sinh \frac{ad_{A_{k,k+1}}}{2}}{\frac{ad_{A_{k,k+1}}}{2}}$$

merging of edges



⇝ BV pushforward for S

$$P_*: S_{T_N} \longrightarrow S_{T_{N-1}}$$

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Classical field theory with boundary (Tulczyjew - Fock - C-M-R) 1207.0239

M - mfld with bdry $\quad F_M = \Gamma(M, \mathcal{F}) \ni \varphi$ ← a sheaf $\quad S_M = \int_M \mathcal{L}(\varphi, \partial\varphi, \partial^2\varphi, \ldots)$

$$\delta S_M = \int_M (\ldots\ldots)\,\delta\varphi + \underbrace{\int_{\partial M} \ldots}$$

↑ Euler-Lagrange equations

CS: $\delta S = \int_M F_A \delta A + \int_\partial \mathrm{tr}\, \frac{1}{2} A \delta A$

$\alpha_\partial^{pre} \in \Omega^1(\Phi_{\partial M}^{pre})$ ← normal jets of order $N \gg 1$ of fields at ∂M, $\quad \omega_\partial^{pre} = \delta \alpha_\partial^{pre} \in \Omega^2(\Phi_{\partial M}^{pre})$

assume ω_∂^{pre} is pre-symplectic, i.e. $\ker(\omega_\partial^{pre})^\# \subset T\Phi_\partial^{pre}$ a subbundle,

assume pre-symp reduction $\Phi_{\partial M} = \Phi_{\partial M}^{pre} / \ker \omega_\partial^{pre}$ exists; $\underline{\omega_\partial^{pre}} =: \omega_\partial = \delta\alpha_\partial$ - symplectic structure on $\Phi_{\partial M}$

$EL \subset F$ - space of solutions of EL equations

$C_\partial \subset \Phi_\partial$ (coiso) - space of boundary fields extendable to $[0,\varepsilon) \times \partial M$ as sol. of EL.

We have

$$\begin{array}{ccc} EL_M & \subset & F_M \\ \swarrow & & \downarrow \pi \text{ (pullback to bdry)} \\ \pi(EL_M) & \subset & C_\partial \subset \Phi_\partial \\ \| & & \text{coiso} \\ L_{M,\partial M} & & \end{array}$$

$L_{M,\partial M}$ - Lagrangian in Φ_∂

reduction:

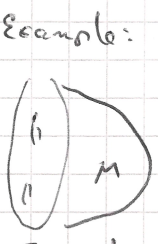
$$\begin{array}{c} EL_M / Gauge_M \quad \leftarrow \text{gauge symmetry} \\ \downarrow \pi_* \\ \underset{\|\, \mathrm{im}\,\pi_*}{L_{M,\partial M}} \underset{Lagr}{\subset} \underline{C_\partial}, \underline{\omega_\partial} \end{array}$$

- reduced phase space

Axiomatization:

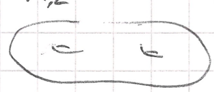
Cl.F.T. is a functor $(Cob_{n,\omega,\Vert}^{(Geom)}) \longrightarrow$
- Ob: presymp manifolds C_Σ
- Mor: Lagrangian ("canonical") relations $L_M \subset \overline{C_{\Sigma_{in}}} \times C_{out}$ [or hyper-relations $EL_M \to \overline{C_{\Sigma_{in}}} \times C_{\Sigma_{out}}$]
- $\circ$: set-theoretic composition of relations

Example: Chern-Simons, M - 3-dim, $\partial M = \Sigma$ - surface



$$\begin{array}{ccc} EL = FlatConn_M & \subset & F = Conn_M \\ & & \downarrow \pi = (\iota_\partial)^* \\ \Sigma \quad L_{M,\Sigma} \subset FlatConn_\Sigma & \subset & \Phi_\Sigma = Conn_\Sigma, \quad \omega = \frac{1}{2}\int_\Sigma \mathrm{tr}\, \delta A_\Sigma \delta A_\Sigma \\ \| & & \| \\ \{A_\Sigma \text{ s.t. } \exists\, A \in FlatConn_M \text{ with } A|_\partial = A_\Sigma\} & & C_\Sigma \end{array}$$

Theorem: $L_{M,\Sigma}$ is always Lagrangian. (cf. "half" of cycles in $\pi_1(\Sigma)$ are contractible in M)



reduction: $\mathcal{M}_M$ ← moduli of flat conn.

$$\mathcal{L} \overset{\pi_*}{\to} \mathcal{M}_\Sigma$$

BV-BFV formalism for classical (gauge) field theories.

Reminder: classical BV (for field theory on M closed):

$M \longrightarrow \mathcal{F}$, ω, Q, S, $\quad \left.\begin{matrix} Q^2=0 \\ \iota_Q\omega = \delta S \end{matrix}\right\} \Leftrightarrow \overset{\text{CME}}{\{S,S\}=0}$

- $\mathcal{F}$: $\mathbb{Z}$-graded S-manifold = space of fields
- ω: gh $=-1$ symp form
- Q: cohom. v.field on $\mathcal{F}$
- S: action = Hamiltonian for Q

Ex. 1: CS ($\dim M = 3$), $\mathcal{F} = \Omega^\bullet(M,\mathfrak{g})[1] \ni \mathcal{A} = c + A + A^+ + c^+$, $\quad \omega = \int_M \mathrm{tr}\,\frac{1}{2}\delta\mathcal{A}\wedge\delta\mathcal{A} = \int_M \delta A \wedge \delta A^+ + \delta c\wedge\delta c^+$

$$Q = \int_M \left\langle d\mathcal{A} + \frac{1}{2}[\mathcal{A},\mathcal{A}], \frac{\delta}{\delta\mathcal{A}}\right\rangle, \qquad \begin{aligned} Qc &= \tfrac{1}{2}[c,c] \\ QA &= d_A c \\ QA^+ &= F_A + [c, A^+] \\ Qc^+ &= d_A A^+ + [c, c^+] \end{aligned}$$

$$S = \int_M \mathrm{tr}\left(\frac{1}{2}\mathcal{A}\wedge d\mathcal{A} + \frac{1}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)$$
$$= \underbrace{\int_M \mathrm{tr}\left(\frac{1}{2}A\,dA + \frac{1}{3}A\wedge A\wedge A\right)}_{S_{cl}(A)} + \int \mathrm{tr}\; A^+ d_A c + \frac{1}{2}c^+[c,c]$$

Ex 2 BF ($\dim M = n$), $\mathcal{F} = \Omega^\bullet(M,\mathfrak{g})[1] \oplus \Omega^\bullet(M,\mathfrak{g}^*)[n-2]$ $\hookleftarrow$ $F = \mathcal{F}^0 = \underset{A}{\Omega^1(M,\mathfrak{g})} \oplus \underset{B}{\Omega^{n-2}(M,\mathfrak{g}^*)}$

$\omega = \int \langle \delta\mathcal{B}, \delta\mathcal{A}\rangle$

$$\mathcal{A} = \underset{1}{c} + \underset{0}{A} + \underset{-1}{B^+} + \underset{-2}{\tau_1^+} + \dots + \underset{1-n}{\tau_{n-2}^+} \quad \leftarrow \text{gh}$$

$$\mathcal{B} = \underset{n-2}{\tau_{n-2}} + \dots + \tau_2 + \underset{1}{\tau_1} + \underset{0}{B} + \underset{-1}{A^+} + \underset{-2}{c^+} \quad \leftarrow \text{gh}$$

$$Q = \int_M \left\langle d\mathcal{A} + \frac{1}{2}[\mathcal{A},\mathcal{A}], \frac{\delta}{\delta\mathcal{A}}\right\rangle + \left\langle d\mathcal{B} + [\mathcal{A},\mathcal{B}], \frac{\delta}{\delta\mathcal{B}}\right\rangle$$

$$S = \int_M \left\langle \mathcal{B}, d\mathcal{A} + \frac{1}{2}[\mathcal{A},\mathcal{A}]\right\rangle$$

$S_{cl} = \int_M \langle B \overset{\wedge}{,} F_A\rangle$

E-L eq: $F_A = 0$, $d_A B = 0$

Gauge symmetry:

(1) $A \mapsto A + d_A\alpha$, $B \mapsto B + [B,\alpha]$, $\alpha \in \Omega^0(M,\mathfrak{g})$

(2) $A \mapsto A$, $B \mapsto B + d_A\beta_1$, $\beta_1 \in \Omega^{n-3}(M,\mathfrak{g}^*)$

stabilizer, if E-L satisfied

$\beta_1 \mapsto \beta_1 + d_A\beta_2$, $\beta_2 \in \Omega^{n-4}(M,\mathfrak{g}^*)$

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reduction: $\mathcal{M} = \dfrac{\mathrm{zero}(Q) =: \mathcal{EL} \supset EL = \mathcal{EL}_0}{\underbrace{\text{distribution induced by } Q}_{=\,\mathrm{zero}(Q)}}$ = "moduli space" of dg manifold $(\mathcal{F}, Q)$

a derived -1-symplectic stack (?)

Ex: ab. C-S, $\mathcal{M} = H^\bullet(M)[1]$

ab. BF $\quad \mathcal{M} = H^\bullet(M)[1] \oplus H^\bullet(M)[n-2]$

BV-BFV formalism (case with boundary)

$(n-1)$-dim closed mfld $\Sigma \longmapsto$ an exact BFV manifold ("BRU phase space")

$(\Phi_\Sigma, \omega_\Sigma = \delta\alpha_\Sigma, Q_\Sigma, S_\Sigma)$

- Φ_Σ: graded mfld
- ω_Σ: gh $=0$ symp form
- Q_Σ: coh. v.f. gh $=1$
- S_Σ: gh $=1$ Hamiltonian for Q_Σ ("BFV charge")

n-mfld M with bdry $\partial M = \Sigma \longmapsto$ "BV-BFV manifold" of fields

$(\mathcal{F}_M, \underset{\text{gh}=-1}{\omega_M}, \underset{\substack{\text{gh}=1 \\ \text{coh v.f.}}}{Q_M}, \underset{\text{gh}=0}{S_M}, \pi: \mathcal{F}_M \twoheadrightarrow \Phi_{\partial M})$