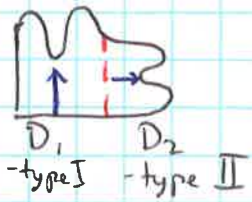
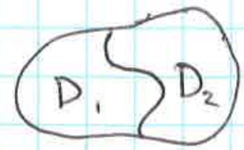
10/10/2018

Properties of double integrals:

① for $D = D_1 \cup D_2$, $\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$

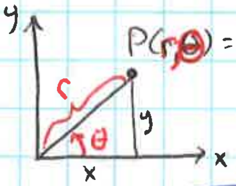
not overlapping except possibly over boundaries





② $\iint_D 1\,dA = \text{Area}(D)$.

15.3 Double integrals in polar coordinates



$P(r,\theta) = P(x,y)$ $\quad x = r\cos\theta$

$y = r\sin\theta$

polar rectangle $R = \left\{(r,\theta) \,\middle|\, \begin{matrix} a \le r \le b \\ \alpha \le \theta \le \beta \end{matrix}\right\}$

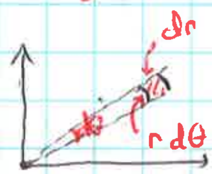
(assume: $0 \le a \le b$, $0 \le \beta - \alpha \le 2\pi$)

change to polar coords. in a double integral

do not forget this factor!

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, \underbrace{r\,dr\,d\theta}_{\text{area of an infinitesimal polar rectangle}}}$$

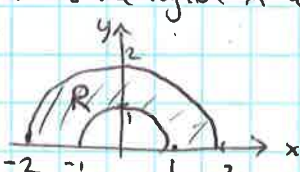
R - polar rectangle



Ex find $\iint_R (3x + 4y^2)\,dA$ where R is the region in upper half-plane bounded by circles $x^2+y^2=1$ and $x^2+y^2=4$

Sol i.e. $R = \left\{(x,y) \,\middle|\, \begin{matrix} y \ge 0 \\ 1 \le x^2+y^2 \le 4 \end{matrix}\right\}$

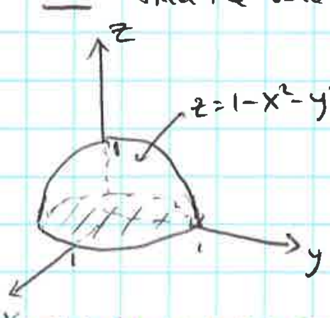
$= \left\{(r,\theta) \,\middle|\, \begin{matrix} 1 \le r \le 2 \\ 0 \le \theta \le \pi \end{matrix}\right\}$ - polar rectangle



So, $\iint_R (3x + 4y^2)\,dA = \int_0^\pi \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\,r\,dr\,d\theta = \int_0^\pi \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\,dr\,d\theta$

$= \int_0^\pi \left| r^3\cos\theta + r^4\sin^2\theta \right|_{r=1}^{r=2} d\theta = \int_0^\pi (7\cos\theta + 15\underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}})\,d\theta = \left| 7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin 2\theta \right|_0^\pi = \frac{15}{2}\pi$

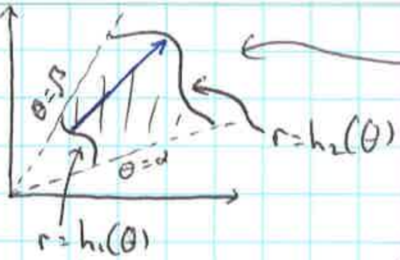
Ex find the volume V of the solid bounded by the plane $z=0$ and paraboloid $z = 1-x^2-y^2$



Sol: solid lies over the disk D $x^2+y^2 \le 1$ in xy plane and below surface $z = 1-x^2-y^2$

So: $V = \iint_D (1-x^2-y^2)\,dA \underset{\text{polar}}{=} \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 (r - r^3)\,dr\,d\theta$

$= \int_0^{2\pi} \left| \frac{r^2}{2} - \frac{r^4}{4} \right|_0^1 d\theta = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$



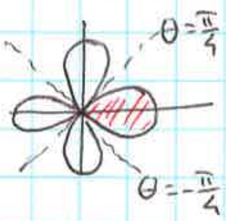
for D like that: $D=\left\{(r,\theta)\ \middle|\ \begin{array}{l}\alpha\le\theta\le\beta\\ h_1(\theta)\le r\le h_2(\theta)\end{array}\right\}$,

$$\iint_D f(x,y)\,dA=\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta,\ r\sin\theta)\ r\,dr\,d\theta$$

Special case: $f(x,y)=1$, $h_1(\theta)=0$, $h_2(\theta)=h(\theta)$

(*) $$\underset{=\text{Area}(D)}{\iint_D 1\,dA}=\int_\alpha^\beta\underbrace{\int_0^{h(\theta)} r\,dr}_{\left.\frac{r^2}{2}\right|_0^{h(\theta)}}\,d\theta=\int_\alpha^\beta\frac{h(\theta)^2}{2}\,d\theta$$

Ex: find the area enclosed by one loop of the "four-leaved rose" $r=\cos 2\theta$

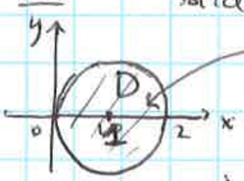


$$D=\left\{(r,\theta)\ \middle|\ \begin{array}{l}-\frac{\pi}{4}\le\theta\le\frac{\pi}{4}\\ 0\le r\le\cos 2\theta\end{array}\right\}$$

$$\text{Area}(D)\overset{(*)}{=}\int_{-\frac{\pi}{4}}^{\pi/4}\frac{1}{2}\underbrace{\cos^2 2\theta}_{\frac{1+\cos 4\theta}{2}}\,d\theta=\frac{1}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1+\cos 4\theta)\,d\theta=\frac{1}{4}\left|\theta+\frac{1}{4}\sin 4\theta\right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}=\frac{\pi}{8}$$

Ex find the volume V of the solid under $z=x^2+y^2$, above xy plane and inside the cylinder $\underbrace{x^2+y^2=2x}_{(x-1)^2+y^2=1}$.

Sol: solid lies above the disk D; boundary circle in polar coordinates:



$$r^2=2r\cos\theta\ \rightsquigarrow\ r=2\cos\theta$$

So, $D=\left\{(r,\theta)\ \middle|\ \begin{array}{l}-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\\ 0\le r\le 2\cos\theta\end{array}\right\}$

$$V=\iint_D(x^2+y^2)\,dA=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\underbrace{\int_0^{2\cos\theta} r^2\,r\,dr}_{\left.\frac{r^4}{4}\right|_{r=0}^{r=2\cos\theta}}\,d\theta=4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\underbrace{\cos^4\theta}\,d\theta=$$

$$\left(\frac{1+\cos 2\theta}{2}\right)^2=\frac{1}{4}+\frac{1}{2}\cos 2\theta+\frac{1}{4}\underbrace{\cos^2 2\theta}_{\frac{1+\cos 4\theta}{2}}$$

$$=\frac{3}{8}+\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta$$

$$=4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{3}{8}+\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta\right)d\theta$$

$$=4\left|\frac{3}{8}\theta+\frac{1}{4}\sin 2\theta+\frac{1}{32}\sin 4\theta\right|_{-\pi/2}^{\pi/2}=\frac{3}{2}\pi$$