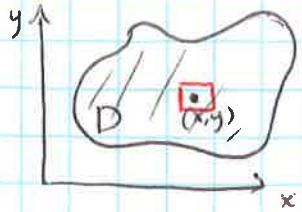
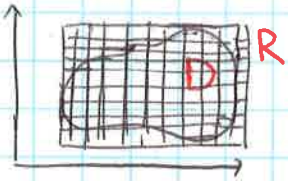
15.4 Applications of double integrals: mass, center of mass, moments 10/12/2018



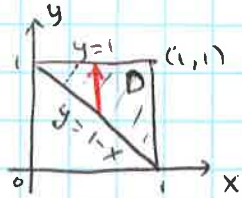
lamina (plate), density $\rho(x,y) = \lim \frac{\Delta m}{\Delta A}$ ← mass in a tiny rectangle about (x,y); ← area of the rectangle

(in units of mass per unit area)



total mass: $m = \lim_{k,l\to\infty} \sum_{i=1}^{k}\sum_{j=1}^{l} \underbrace{\rho(x_{ij}^*, y_{ij}^*)\Delta A}_{\approx \text{ mass in } ij\text{-th rectangle}} = \boxed{\iint_D \rho(x,y)\,dA}$

Ex

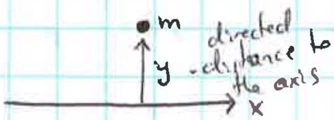


density: $\rho(x,y) = x+y$ (kg/m^2); find the total mass

Sol: $m = \iint_D (x+y)\,dA = \int_0^1 \underbrace{\int_{1-x}^{1} (x+y)\,dy}_{} \,dx = \int_0^1 \left(x + \frac{x^2}{2}\right) dx = \frac{x^2}{2} + \frac{x^3}{6}\Big|_0^1 = \frac{2}{3}$

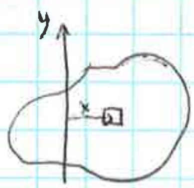
$xy + \frac{y^2}{2}\Big|_{y=1-x}^{y=1} = x + \frac{1}{2} - x(1-x) - \frac{(1-x)^2}{2} = \cancel{x} + \cancel{\tfrac{1}{2}} - \cancel{x} + x^2 - \cancel{\tfrac{1}{2}} + x - \frac{x^2}{2} = x + \frac{x^2}{2}$

moment of a particle about ~~x~~-axis: my



moment of a lamina about x-axis:

$M_x = \lim \sum_i\sum_j y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \boxed{\iint_D y\,\rho(x,y)\,dA}$



similarly: $M_y = \boxed{\iint_D x\,\rho(x,y)\,dA}$

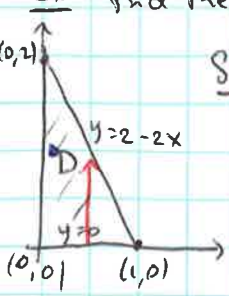
center of mass $(\bar{x}, \bar{y})$: $m\bar{x} = M_y \qquad m\bar{y} = M_x$

- lamina balances when supported at the center of mass

So, coordinates of center of mass: $\bar{x} = \frac{M_y}{m} = \frac{\iint_D x\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}$, ~~$\bar{y} = \frac{M_x}{m} = \iint$~~

$\bar{y} = \frac{M_x}{m} = \frac{\iint_D y\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}$ ← m →

Ex Find the mass and center of mass of a triangular lamina with vertices (0,0), (1,0), (0,2) with density $\rho(x,y) = y$ ~~$\rho(x,y) = 1+3x+y$~~



Sol: ~~$m = \int_0^1 \int_0^{2-2x} (1+3x+y)\,dy\,dx = \int_0^1 (4-4x^2$~~

~~$y + 3xy + \frac{y^2}{2}\Big|_{y=0}^{y=2-2x} = 4-4x^2$~~

$$m = \int_0^1 \int_0^{2-2x} y \, dy \, dx = 2\int_0^1 (1-x)^2 \, dx = -2\frac{(1-x)^3}{3}\Big|_0^1 = \frac{2}{3} \leftarrow \text{total mass}$$

$$\underbrace{\frac{y^2}{2}\Big|_{y=0}^{y=2-2x}}_{} = 2(1-x)^2$$

$$M_y = \int_0^1 \underbrace{\int_0^{2-2x} xy \, dy}_{2x(1-x)^2} \, dx = 2\int_0^1 (x - 2x^2 + x^3) \, dx = 2\left(\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right)\Big|_0^1 = 2\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{2}{12} = \frac{1}{6}$$

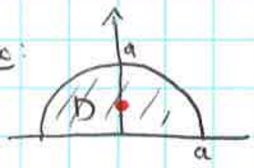
$$\Rightarrow \bar{x} = \frac{M_y}{m} = \frac{1/6}{2/3} = \frac{1}{4}$$

$$M_x = \int_0^1 \int_0^{2-2x} y^2 \, dy \, dx = \frac{8}{3}\int_0^1 (1-x)^3 \, dx = -\frac{8}{3}\frac{(1-x)^4}{4}\Big|_0^1 = \frac{8}{3}\cdot\frac{1}{4} = \frac{2}{3}$$

$$\underbrace{\frac{y^3}{3}\Big|_0^{y=2-2x}}_{} = \frac{8}{3}(1-x)^3$$

$$\Rightarrow \bar{y} = \frac{M_x}{m} = 1$$

Thus, Center of mass: $\boxed{(\bar{x}, \bar{y}) = \left(\frac{1}{4}, 1\right)}$



Ex: semicircular lamina $x^2+y^2 \le a^2$, $y \ge a$, density proportional to the center of the circle, $\rho(x,y) = K\sqrt{x^2+y^2}$.

Find the center of mass.

Sol $m = \iint_D K\sqrt{x^2+y^2} \, dA \underset{\text{polar coord.}}{=} \int_0^{\pi} \underbrace{\int_0^a Kr \cdot r \, dr}_{\frac{K}{3}a^3} d\theta = \frac{\pi}{3} K a^3$ - mass

By symmetry of D and ρ wrt y-axis, $M_y = 0$, $\bar{x} = 0$

$$M_x = \iint_D K\sqrt{x^2+y^2} \cdot y \, dA \underset{\text{polar}}{=} \int_0^{\pi} \underbrace{\int_0^a Kr \, r\sin\theta \, r \, dr}_{\frac{K}{4}a^4 \sin\theta} d\theta =$$

$$= \frac{K}{4}a^4 \int_0^{\pi} \sin\theta \, d\theta = \frac{K}{4}a^4 \left[-\cos\theta\right]_0^{\pi} = \frac{K}{2}a^4 \quad \Rightarrow \bar{y} = \frac{\frac{K}{2}a^4}{\frac{\pi}{3}Ka^3} = \frac{3}{2\pi}a$$

Thus, center of mass: $\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{3}{2\pi}a\right)}$