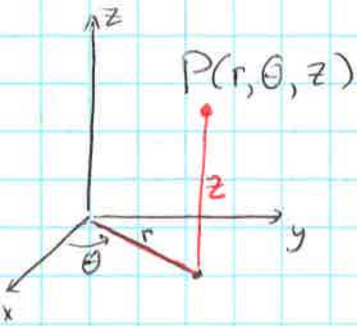
10/26/2018

15.7 Triple integrals in cylindrical coordinates



r, θ, z - cylindrical coordinates of a point P

$\underbrace{r, \theta}_{\text{polar of projection of P onto xy plane}}$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$r^2 = x^2 + y^2$

for a solid $E \begin{cases} \alpha \le \theta \le \beta \\ h_1(\theta) \le r \le h_2(\theta) \\ g_1(r,\theta) \le z \le g_2(r,\theta) \end{cases}$

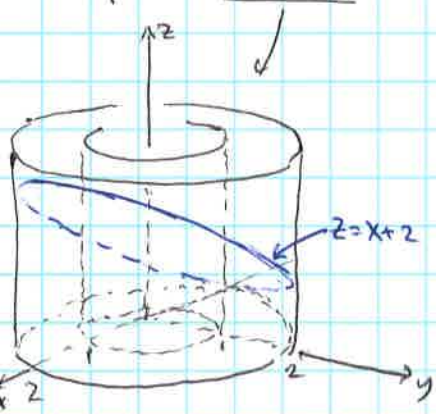
$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,\cancel{dz}\; r\,dz\,dr\,d\theta$$

Ex: compute $\bar{y}$ of the center of mass for solid region E below the plane $z = x+2$, above xy plane and between cylinders $x^2+y^2=1$ and $x^2+y^2=4$, assuming ρ = constant.

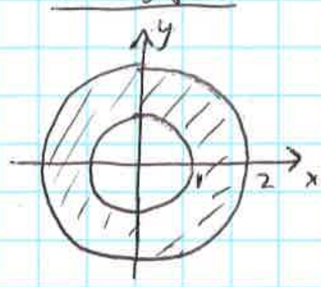
Sol recall $$\bar{y} = \frac{\iiint_E y\,\rho(x,y,z)\,dV}{\iiint_E \rho(x,y,z)\,dV} = \frac{M_{xz}}{m}$$ ← moment about xz plane; ← total mass

· since ρ = const, can assume $\rho = 1$.

Step 0: sketch E



projection D to xy plane:



Step 1: describe E in cylindrical coord.

$\ldots \le \theta \le \ldots \quad \ldots \le r \le \ldots \quad \ldots \le z \le \ldots$

in that order!

$\boxed{0 \le \theta \le 2\pi \quad 1 \le r \le 2 \quad 0 \le z \le \underbrace{r\cos\theta + 2}_{x+2}}$

So: $$M_{xz} = \iiint_E y\overset{1}{\rho}\,dV = \int_0^{2\pi}\int_1^2 \underbrace{\int_0^{r\cos\theta+2} r\sin\theta\; r\,dz}_{r^2\sin\theta\,(r\cos\theta+2)}\,dr\,d\theta =$$

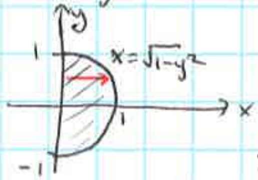
$$= \int_0^{2\pi}\int_1^2 (r^3\sin\theta\cos\theta + 2r^2\sin\theta)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{16-1}{4}\sin\theta\cos\theta + 2\,\frac{8-1}{3}\sin\theta\right)d\theta =$$

$$= \int_0^{2\pi}\left(\frac{15}{8}\sin 2\theta + \frac{14}{3}\sin\theta\right)d\theta = \underline{\underline{0}} \longrightarrow \bar{y} = 0$$

Ex convert to cylindrical

$$I = \int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz \; dz\, dx\, dy = \iiint_E xyz \; dV$$

solid E : ~~over~~ above



, with $x^2+y^2 \le z \le \sqrt{x^2+y^2}$

in ~~polar~~ cylindrical coord. : $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$0 \le r \le 1$

$r^2 \le z \le r$

$$I = \int_{-\pi/2}^{\pi/2}\int_{0}^{1}\int_{r^2}^{r} \underbrace{r\cos\theta \, r\sin\theta \, z r dz}_{\frac{1}{2} z r^3 \sin 2\theta} dr\, d\theta = \int_{-\pi/2}^{\pi/2}\int_{0}^{1} \frac{r^2-r^4}{4} r^3 \sin 2\theta \; dr\, d\theta = \int_{-\pi/2}^{\pi/2} \frac{1}{4}\underbrace{\left(\frac{1}{6}-\frac{1}{8}\right)}_{\frac{1}{24}} \sin 2\theta \; d\theta =$$

$$= \frac{1}{96} \cdot \frac{-1}{2} \cos 2\theta \Big|_{-\pi/2}^{\pi/2} = \frac{1}{96}$$