10/29/2018

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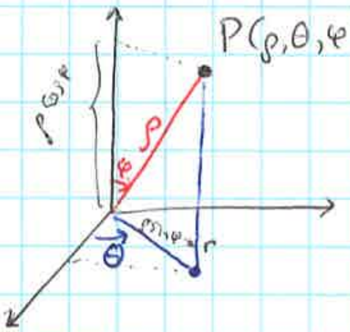
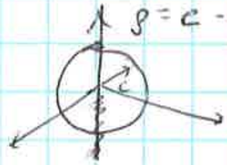
15.8 Triple integrals in spherical coordinates

spherical coordinates ρ, θ, φ — ρ: $|OP|$; θ: same as in cylindrical; φ: angle between $\vec{OP}$ and z-axis

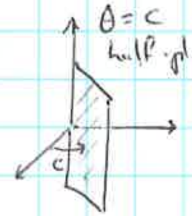
$\rho \geq 0$, $0 \leq \theta \leq 2\pi$, $\boxed{0 \leq \varphi \leq \underline{\underline{\pi}}}$



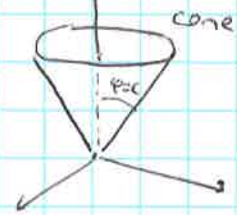
$\rho = c$ - a sphere



$\theta = c$ half-plane



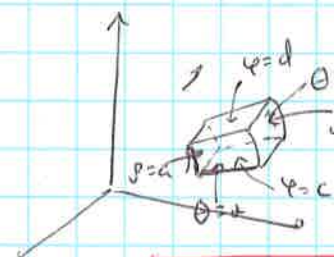
$\varphi = c$ cone



$$\boxed{\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho \cos\varphi \end{aligned}} \leftarrow \begin{aligned} z &= \rho\cos\varphi \\ \underbrace{r = \rho \sin\varphi}_{r \text{ of cylindrical}} \end{aligned}$$

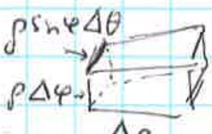
also: $\rho^2 = x^2 + y^2 + z^2$

Triple integrals $\quad E: \begin{aligned} c &\leq \varphi \leq d \\ \alpha &\leq \theta \leq \beta \\ a &\leq \rho \leq b \end{aligned}$ - spherical wedge



$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\, \boxed{\rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi}$$

- from the volume of a small spherical wedge



$\Delta V = \rho^2 \sin\varphi \, \Delta\rho \, \Delta\theta \, \Delta\varphi$

- can extend this to more general spherical regions,

$E: \quad c \leq \varphi \leq d$

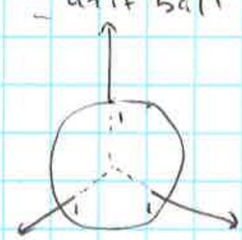
$\quad \alpha \leq \theta \leq \beta$

$\quad g_1(\theta,\varphi) \leq \rho \leq g_2(\theta, \varphi)$

when boundary of E is formed of cones and spheres - use spherical coords.

Ex: find $\displaystyle\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$

where $B = \{(x,y,z) \mid x^2+y^2+z^2 \leq 1\}$ - unit ball



Sol: $\displaystyle = \int_0^\pi \int_0^{2\pi} \int_0^1 \overbrace{e^{(\rho^2)^{3/2}} \rho^2 \sin\varphi \, d\rho}^{e^{\rho^3} \rho^2 \sin\varphi \, d\rho} \, d\theta \, d\varphi$

$B = \left\{(\rho, \theta, \varphi) \;\middle|\; \begin{aligned} 0 &\leq \varphi \leq \pi \\ 0 &\leq \theta \leq 2\pi \\ 0 &\leq \rho \leq 1 \end{aligned}\right\}$

$\displaystyle = \int_0^\pi \int_0^{2\pi} \frac{1}{3} \sin\varphi \, e^{\rho^3} \Big|_{\rho=0}^{\rho=1} d\theta \, d\varphi$

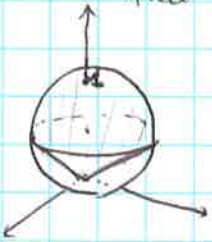
$\displaystyle = \int_0^\pi \int_0^{2\pi} \frac{1}{3} \sin\varphi \cdot (e-1)\, d\theta \, d\varphi$

$\displaystyle = \frac{2\pi}{3}(e-1) \int_0^\pi \sin\varphi \, d\varphi = \boxed{\frac{4\pi}{3}(e-1)}$

* in rectangular same integral: $\displaystyle \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}}\, dz\, dy\, dx$ ← much more cumbersome!

Ex: E - solid above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$
find the volume of E.

$$x^2+y^2+\left(z-\tfrac{1}{2}\right)^2=\tfrac{1}{4}$$



Sol: sphere: $\rho^2=\rho\cos\varphi$ or $\rho=\cos\varphi$

cone: $\rho\cos\varphi=\rho\sin\varphi \rightsquigarrow \cos\varphi=\sin\varphi \to \varphi=\pi/4$

So, E: $0\le\varphi\le\frac{\pi}{4}$

$0\le\theta\le 2\pi$

$0\le\rho\le\cos\varphi$

$$V(E)=\iiint_E dV=\int_0^{\pi/4}\int_0^{2\pi}\underbrace{\int_0^{\cos\varphi}\rho^2\sin\varphi\,d\rho}_{\frac{1}{3}\cos^3\varphi\sin\varphi}\,d\theta\,d\varphi=\frac{2\pi}{3}\underbrace{\int_0^{\pi/4}\cos^3\varphi\sin\varphi\,d\varphi}_{-\frac{1}{4}\cos^4\varphi\Big|_0^{\pi/4}=\frac{1}{4}\left(1-\frac{1}{4}\right)=\frac{3}{16}}=\frac{\pi}{8}$$