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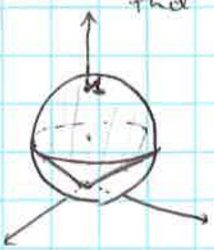
Ex: E – solid above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=z$
find the volume of E.

$$x^2+y^2+(z-\tfrac{1}{2})^2=\tfrac{1}{4}$$



Sol: sphere: $\rho^2=\rho\cos\varphi$ or $\rho=\cos\varphi$

cone: $\rho\cos\varphi=\rho\sin\varphi \rightsquigarrow \cos\varphi=\sin\varphi \to \varphi=\pi/4$

So, E: $0\le\varphi\le\frac{\pi}{4}$

$0\le\theta\le 2\pi$

$0\le\rho\le\cos\varphi$

$$V(E)=\iiint_E dV=\int_0^{\pi/4}\int_0^{2\pi}\underbrace{\int_0^{\cos\varphi}\rho^2\sin\varphi\, d\rho}_{\frac{1}{3}\cos^3\varphi\sin\varphi}\, d\theta\, d\varphi = \frac{2\pi}{3}\underbrace{\int_0^{\pi/4}\cos^3\varphi\sin\varphi\, d\varphi}_{-\frac{1}{4}\cos^4\varphi\Big|_0^{\pi/4}=\frac{1}{4}(1-\frac{1}{4})=\frac{3}{16}} = \frac{\pi}{8}$$

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15.9. Change of variables in multiple integrals [substitution]

in 1D integrals: $\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du = \int_c^d f(x(u))\frac{dx}{du}\,du$

substitution $x=g(u)$
with $a=g(c)$
$b=g(d)$

in 2D integrals [example] rectangular → polar

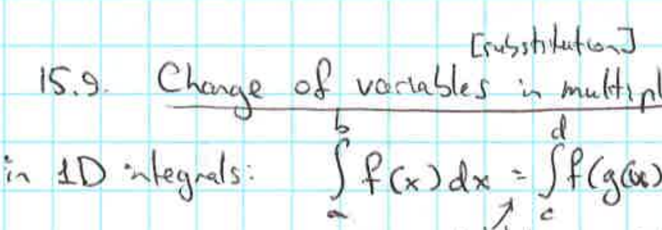
$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$x=r\cos\theta$
$y=r\sin\theta$

S ← region in $r\theta$-plane corresponding to R (region in xy-plane)

Generally: T – transformation from uv-plane to xy-plane, given by $x=g(u,v)$, $y=h(u,v)$ (*)

$T(u,v)=(x,y)$



$T(u_1,v_1)=\underbrace{(x_1,y_1)}_{\text{image of }(u_1,v_1)}$

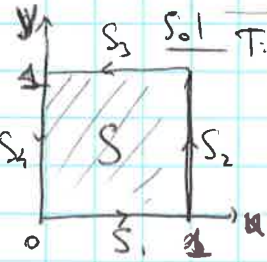
R = image of S
(images of all points in S)

if no two points in S have same image, T is "one-to-one"

if T one-to-one, there is an inverse transformation $u=G(x,y)$, $v=H(x,y)$ – from solving (*).

Ex(1) $T: (u,v) \longmapsto (x = u^2 - v^2,\ y = 2uv)$

find the image R of the square $S = \{(u,v) \mid {0 \le u \le 1 \atop 0 \le v \le 1}\}$



Sol T: boundary of $S \longrightarrow$ boundary of R

$S_1 \ni (u, 0) \overset{T}{\longmapsto} (u^2, 0)$

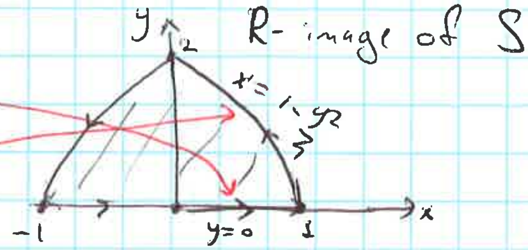
$S_2 \ni (1, v) \overset{T}{\longmapsto} (\overset{x=}{1-v^2}, \overset{y=}{2v})$

thus, $x = 1 - \frac{y^2}{4}$

$S_3 \ni (u, 1) \overset{T}{\longmapsto} (\overset{x=}{u^2-1}, \overset{y=}{2u})$

thus $x = \frac{y^2}{4} - 1$

$S_4 \ni (0, v) \overset{T}{\longmapsto} (-v^2, 0)$



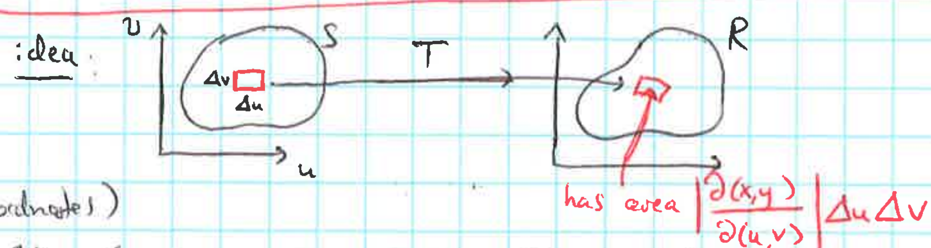
Jacobian of T: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$

change of variables in a double integral: suppose T maps S in uv-plane to R in xy-plane and is differentiable and 1-1 except perhaps on boundary

suppose Jacobian $\ne 0$ everywhere inside S

Then: $$\iint_R f(x,y)\, dA = \iint_S f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

idea:



(polar coordinates)

Ex(2) $T: (r,\theta) \to (x,y)$ $\quad x = r\cos\theta,\ y = r\sin\theta$ $\quad \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

$\rightsquigarrow \iint_R f(x,y)\, dA = \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$

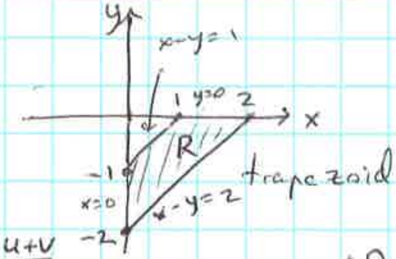
Ex(3) ~~for~~ $\iint_R y\, dA$ from Ex(1) $\to R$ — use the change of var. $x = u^2 - v^2$, $y = 2uv$

Sol: $R = T(S)$ (square from Ex(1)), $\quad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4u^2 + 4v^2 > 0$ in S

So, $\iint_R y\, dA = \iint_S \underbrace{y}_{2uv}(4u^2+4v^2)\, du\, dv = 8\int_0^1\int_0^1 (u^3 v + u v^3)\, du\, dv = 8\int_0^1 \left(\frac{v}{4} + \frac{v^3}{2}\right) dv = 8\left(\frac{1}{8} + \frac{1}{8}\right) = 2$

Ex: Find $I=\iint_R e^{\frac{x+y}{x-y}}\,dA$

suggested by

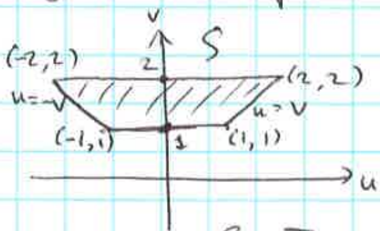


Sol: make a change $x+y=u$, $x-y=v$ $\rightsquigarrow$ $x=\frac{u+v}{2}$, $y=\frac{u-v}{2}$

Jacobian: $\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac12 & \frac12\\ \frac12 & -\frac12\end{vmatrix}=-\frac12$

region S in uv-plane corresponding to R: sides of R:

$x-y=1 \longrightarrow v=1$
$y=0 \longrightarrow v=u$
$x-y=2 \longrightarrow v=2$
$x=0 \longrightarrow v=-u$



$S=\left\{(u,v)\,\middle|\, \begin{matrix}1\le v\le 2\\ -v\le u\le v\end{matrix}\right\}$

So $I=\iint_S e^{\frac{u}{v}}\underbrace{\left|-\frac12\right|}_{\text{Jacobian}}du\,dv=\int_1^2\underbrace{\int_{-v}^{v} e^{\frac{u}{v}}\frac12\,du}_{\frac{v}{2}e^{\frac{u}{v}}\Big|_{u=-v}^{u=v}=\frac{e-e^{-1}}{2}v}dv=\int_1^2\frac{e-e^{-1}}{2}v\,dv=\frac{e-e^{-1}}{2}\,\frac{v^2}{2}\Big|_1^2=\boxed{\frac34(e-e^{-1})}$