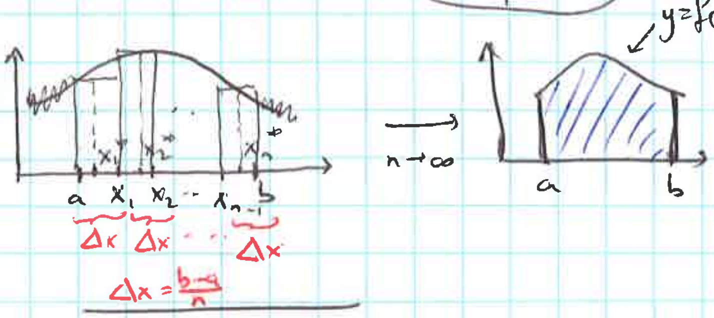
10/5/2018

15.1 Double integrals over rectangles

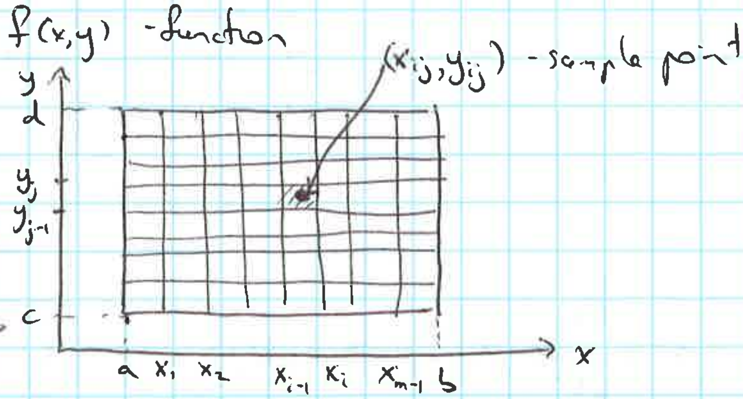
Recall: $\displaystyle\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i^*)\,\Delta x}_{\text{area of a rectangle}}$ — sample point x_i^* in $[x_{i-1}, x_i]$

$= $ area of the region $\left\{(x,y) \,\middle|\, \begin{array}{l} a\le x\le b \\ 0\le y\le f(x) \end{array}\right\}$ for $f \geq 0$



$R = \left\{(x,y)\in\mathbb{R}^2 \,\middle|\, \begin{array}{l} a\le x\le b \\ c\le y\le d \end{array}\right\}$ - rectangle, $f(x,y)$ - function
$=[a,b]\times[c,d]$

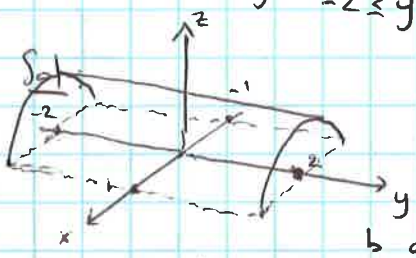
$$\iint_R f(x,y)\,dA = \lim_{\substack{m\to\infty \\ n\to\infty}} \underbrace{\underbrace{f(x_{ij}^*, y_{ij}^*)}_{\text{sample point in } [x_{i-1},x_i]\times[y_{j-1},y_j]} \underbrace{\Delta A}_{\Delta x\Delta y}}_{\text{volume of a column}}$$



- If $f \geq 0$, volume V of a solid that lies above the rectangle R and below the surface $z = f(x,y)$ is:
$$\boxed{V = \iint_R f(x,y)\,dA}$$

Ex: $R = \left\{(x,y) \,\middle|\, \begin{array}{l} -1\le x\le 1 \\ -2\le y\le 2 \end{array}\right\}$, find $\displaystyle\iint_R \underbrace{\sqrt{1-x^2}}_{f(x,y)}\,dA$

Sol



solid = half-cylinder, $V = \frac{1}{2}\pi r^2 L = \frac{1}{2}\pi\, 1^2 \cdot 4 = 2\pi$

Iterated integrals: $\displaystyle\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_a^b \underbrace{\left(\int_c^d f(x,y)\,dy\right)}_{A(x)} dx$ ← working from inside out

$A(x)$ - partial integration of f w.r.t. y, regarding x as a constant

Similarly: $\displaystyle\int_c^d\int_a^b f(x,y)\,dx\,dy = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$

Ex find (a) $\displaystyle\int_0^3\int_1^2 x^2y\,dy\,dx$, (b) $\displaystyle\int_1^2\int_0^3 x^2y\,dx\,dy$

Sol (a) $\displaystyle\int_0^3\underbrace{\left(\int_1^2 x^2y\,dy\right)}_{x^2\cdot\frac{y^2}{2}\Big|_{y=1}^{y=2} = \frac{3}{2}x^2}dx = \int_0^3 \frac{3}{2}x^2\,dx = \frac{3}{2}\cdot\frac{1}{3}x^3\Big|_0^3 = \frac{27}{2}$

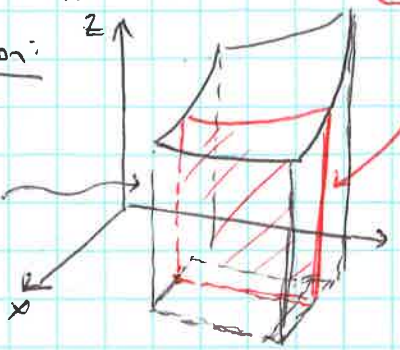
(b) $\displaystyle\int_1^2\underbrace{\left(\int_0^3 x^2y\,dx\right)}_{y\frac{x^3}{3}\Big|_{x=0}^{x=3} = 9y}dy = \frac{9y^2}{2}\Big|_1^2 = \frac{9}{2}(4-1) = \frac{27}{2}$

Fubini's theorem if $f(x,y)$ continuous on a rectangle $R: \begin{matrix} a \le x \le b \\ c \le y \le d \end{matrix}$,

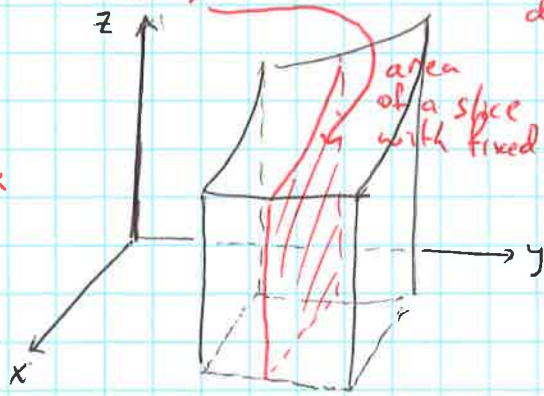
then: $\iint_R f(x,y)\,dA = \underbrace{\int_a^b \int_c^d f(x,y)\,dy\,dx}_{} = \underbrace{\int_c^d \int_a^b f(x,y)\,dx\,dy}_{}$ ← allows us to effectively calculate double integrals / volumes of solids

Intuition:

solid



area of the slice with fixed x



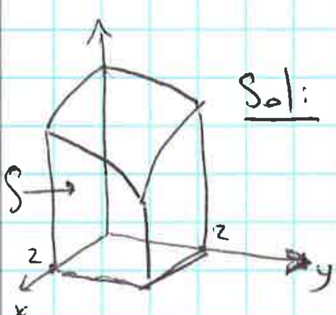
area of a slice with fixed y

Ex: find $\iint_R y \sin(xy)\,dA$ where $R = [1,2] \times [0,\pi]$

Sol: $\iint_R y\sin(xy)\,dA \underset{\text{Fubini}}{=} \int_0^\pi \Big(\underbrace{\int_1^2 y \sin(xy)\,dx}_{-y\frac{1}{y}\cos(xy)\big|_{x=1}^{x=2} = \cos y - \cos 2y} \Big) dy = \int_0^\pi (\cos y - \cos 2y)\,dy = \sin y - \frac{1}{2}\sin 2y \Big|_0^\pi = \boxed{0}$

note: we could do instead $\int_1^2 \Big(\int_0^\pi y \sin(xy)\,dy \Big) dx$, but it is more complicated (need to integrate by parts twice)

Ex: find the volume of the solid S bounded by $x^2 + 2y^2 + z = 16$ (elliptic paraboloid), planes $x = 2$, $y = 2$ and xy, xz, yz planes



Sol: $V = \iint_{R=[0,2]\times[0,2]} \overbrace{(16 - x^2 - 2y^2)}^{z \text{ from}}\,dA = \int_0^2 \Big(\underbrace{\int_0^2 (16 - x^2 - 2y^2)\,dx}_{16x - \frac{x^3}{3} - 2y^2x \big|_{x=0}^{x=2} = 32 - \frac{8}{3} - 4y^2 = \frac{88}{3} - 4y^2} \Big) dy$

$= \int_0^2 \left(\frac{88}{3} - 4y^2 \right) dy = \frac{88}{3}y - \frac{4}{3}y^3 \Big|_0^2 = \frac{88 \cdot 2 - 4 \cdot 8}{3} = \frac{176 - 32}{3} = \frac{144}{3} = 48$

• special case: $f(x,y) = g(x)h(y)$ (factorizes)

then: $\iint_R g(x)h(y)\,dx\,dy = \left(\int_a^b g(x)\,dx \right) \left(\int_c^d h(y)\,dy \right)$

Ex: $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$, $\iint_R \sin x \cos y\,dx\,dy = \Big(\underbrace{\int_0^{\pi/2} \sin x\,dx}_{-\cos x \big|_0^{\pi/2} = 1} \Big) \Big(\underbrace{\int_0^{\pi/2} \cos y\,dy}_{\sin y \big|_0^{\pi/2} = 1} \Big) = 1 \cdot 1 = 1$