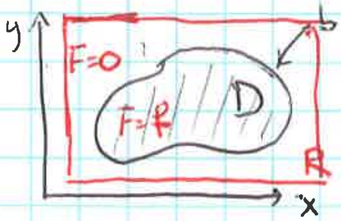
15.2

Double integrals over general regions

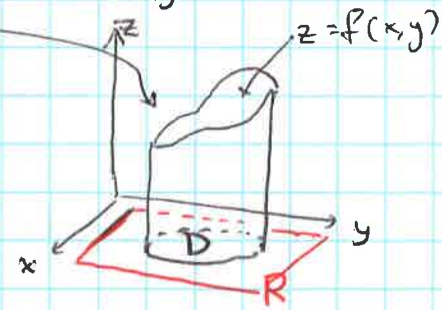


bounded region in $\mathbb{R}^2$, $f(x,y)$ - function on D

$$\iint_D f(x,y)\,dA \overset{\text{def}}{=} \iint_R F(x,y)\,dA \quad , F(x,y)=\begin{cases} f(x,y) & \text{if } (x,y) \text{ in } D \\ 0 & \text{if } (x,y) \text{ in } R\setminus D \end{cases}$$

R - rectangle containing D

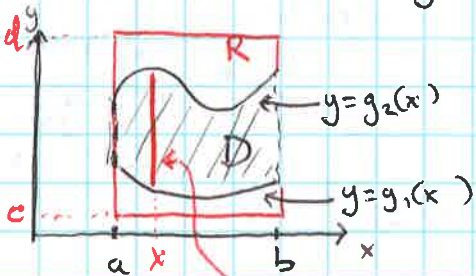
for $f \geq 0$, $\iint_D f(x,y)\,dA$ = volume of the solid lying above D and below $z=f(x,y)$



* plane region D of type I:

$$D=\left\{(x,y)\;\middle|\; \begin{array}{l} a \leq x \leq b \\ g_1(x) \leq y \leq g_2(x) \end{array}\right\}$$

g_1, g_2 - cont. functions on $[a,b]$



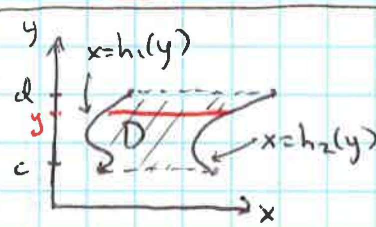
$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b\int_c^d F(x,y)\,dy\,dx$$

$$= \boxed{\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx}$$

- iterated integral (limits for the inner integral depend on x!)

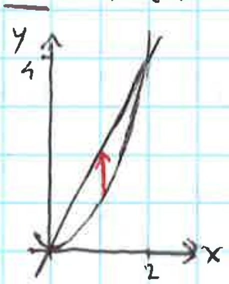
* region of type II:

$$D=\left\{(x,y)\;\middle|\; \begin{array}{l} c \leq y \leq d \\ h_1(y) \leq x \leq h_2(y) \end{array}\right\}$$



$$\iint_D f(x,y)\,dA = \boxed{\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy}$$

Ex Find the volume V of the solid under $z=\underbrace{x^2+y^2}_{f(x,y)}$ and above the region D in xy plane bounded by $y=2x$ and $y=x^2$



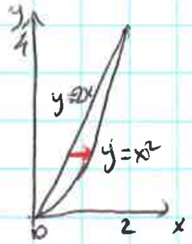
Sol: $D:\; 0 \leq x \leq 2$, $\underbrace{x^2}_{g_1(x)} \leq y \leq \underbrace{2x}_{g_2(x)}$

$$V=\int_0^2\int_{x^2}^{2x}(x^2+y^2)\,dy\,dx = \quad \left(=\iint_D (x^2+y^2)\,dA\right)$$

$$=\int_0^2 \left(x^2y+\frac{y^3}{3}\Big|_{y=x^2}^{y=2x} \right) dx = \int_0^2 \Big(\underbrace{2x^3+\frac{8}{3}x^3-x^4-\frac{x^6}{3}}_{-\frac{x^6}{3}-x^4+\frac{14}{3}x^3}\Big)dx = -\frac{x^7}{21}-\frac{x^5}{5}+\frac{7}{6}x^4\Big|_{x=0}^{x=2}$$

$$=-\frac{128}{21}-\frac{32}{5}+\frac{56}{3} = \boxed{\frac{216}{35}}$$

Sol 2:



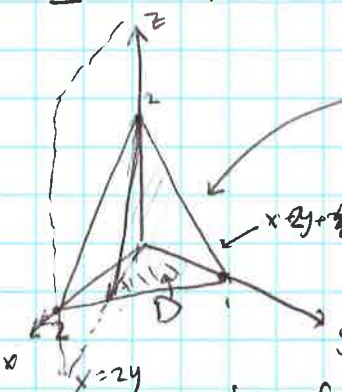
Same D can be seen as a type II region:

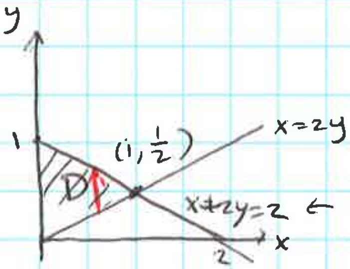
$0 \le y \le 4$, $\frac{y}{2} \le x \le \sqrt{y}$

$$V = \int_0^4 \int_{y/2}^{\sqrt{y}} (x^2+y^2)\,dx\,dy = \int_0^4 \left| \frac{x^3}{3} + xy^2 \right|_{x=\frac{y}{2}}^{x=\sqrt{y}} dy = \int_0^4 \left(\frac{y^{3/2}}{3} + y^{5/2} \underbrace{- \frac{y^3}{24} - \frac{y^3}{2}}_{-\frac{13}{24}y^3} \right) dy$$

$$= \frac{2}{15} y^{5/2} + \frac{2}{7} y^{7/2} - \frac{13}{96} y^4 \Big|_{y=0}^{y=4} = \dots = \boxed{\frac{216}{35}}$$

Ex find the volume of the tetrahedron bounded by planes $x+2y+z=2$ (1), $x=2y$, $x=0$, $z=0$





← intersection of (1) with xy plane

(1) → $z = 2-x-2y$

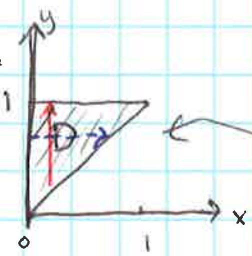
$$V = \iint_D (2-x-2y)\,dA = \int_0^1 \int_{\frac{x}{2}}^{1-\frac{x}{2}} (2-x-2y)\,dy\,dx = \int_0^1 \left| 2y - xy - y^2 \right|_{y=\frac{x}{2}}^{y=1-\frac{x}{2}} dx =$$

$$= \int_0^1 \left((2-x)(1-x) - \left(1 - x + \cancel{\frac{x^2}{4}}\right) + \cancel{\frac{x^2}{4}} \right) dx = \int_0^1 (1-2x+x^2)\,dx = \left(x - x^2 + \frac{x^3}{3}\right) \Big|_{x=0}^{x=1} = \frac{1}{3}$$

Ex find $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$

reverse the order of the iterated integral

Sol:



$$\underbrace{\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx}_{\text{hard!}} = \iint_D \sin(y^2)\,dA = \int_0^1 \underbrace{\int_0^y \sin(y^2)\,dx}_{y\sin(y^2)}\,dy = \int_0^1 y \sin(y^2)\,dy$$

view D as type II region

$$= -\frac{1}{2}\cos(y^2) \Big|_{y=0}^{y=1} = \boxed{\frac{1}{2}(1-\cos 1)}$$

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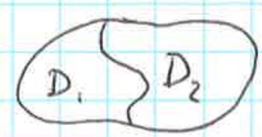
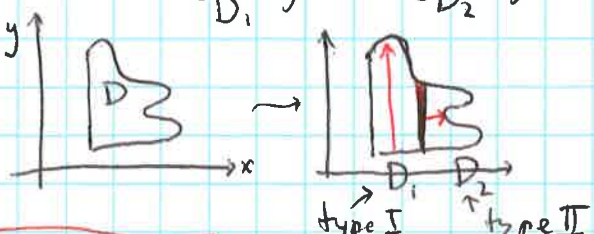
* if $D = D_1 \cup D_2$ with D_1, D_2 not overlapping except perhaps on boundaries,



then $\int_D f(x,y)\,dA = \int_{D_1} f(x,y)\,dA + \int_{D_2} f(x,y)\,dA$

Ex



* $\boxed{\iint_D 1\,dA = \text{Area}(D)}$

* average value of f on D : $\boxed{f_{avg} = \frac{1}{\text{Area}(D)} \iint_D f(x,y)\,dA}$