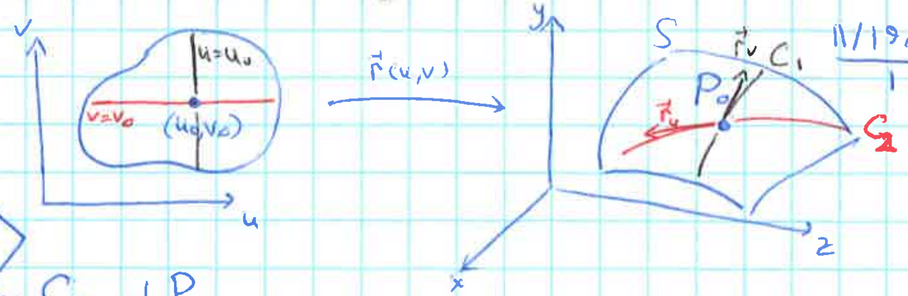
16.6 Cont'd

11/19/2018

* Tangent plane to a parametric surface



$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0) \right\rangle$ - ~~velocity~~ tangent vector for C_2 at P_0

$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \Big|_{u_0,v_0}$ - tangent vector to C_1 at P_0

assume $\vec{r}_u \times \vec{r}_v \neq 0$ (then S is "smooth") $\to$ tangent plane to S at P_0 has normal vector $\vec{r}_u \times \vec{r}_v$

Ex: $\left.\begin{array}{l} x = u^2 \\ y = v^2 \\ z = u + 2v \end{array}\right\}$ - parametric surface. Find the tangent plane at $P_0(1,1,3)$

Sol: P_0 corresponds to $u_0 = 1, v_0 = 1$. $\vec{r}(u,v) = \langle u^2, v^2, u+2v \rangle$

$\vec{r}_u = \langle 2u, 0, 1 \rangle$

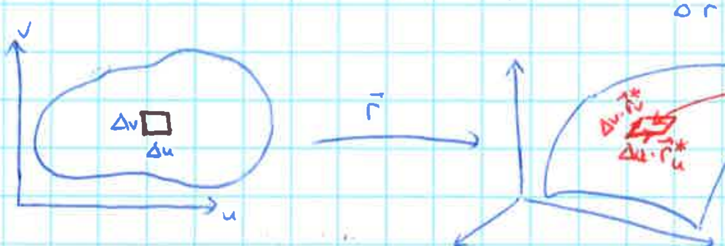
$\vec{r}_v = \langle 0, 2v, 2 \rangle$

at $u_0 = v_0 = 1$: $\vec{r}_u = \langle 2, 0, 1 \rangle$, $\vec{r}_v = \langle 0, 2, 2 \rangle$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 0 & 2 & 2 \end{vmatrix} = \langle -2, -4, 4 \rangle$ $\rightsquigarrow$ eq. of tangent plane: $-2(x-1) - 4(y-1) + 4(z-3) = 0$

or: $x + 2y - 2z + 3 = 0$

Surface area:



area: ~~Δu Δv~~ $|(\Delta u\, \vec{r}_u^{\,*}) \times (\Delta v\, \vec{r}_v^{\,*})|$

surface area of S (a smooth parametric surface):

$$\boxed{S(A) = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA}$$

, with $\vec{r}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle$, $\vec{r}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$

We assume that S is covered once when (u,v) ranges throughout D

Ex: Find the surface area of a sphere of radius a.

Sol: parametrize: $x = a\sin\varphi\cos\theta$, $y = a\sin\varphi\sin\theta$, $z = a\cos\varphi$

$D = \left\{ (\varphi, \theta) \,\middle|\, \begin{array}{l} 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{array} \right\}$

$\vec{r}_\varphi = \langle a\cos\varphi\cos\theta, a\cos\varphi\sin\theta, -a\sin\varphi \rangle$

$\vec{r}_\theta = \langle -a\sin\varphi\sin\theta, +a\sin\varphi\cos\theta, 0 \rangle$

$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos\varphi\cos\theta & a\cos\varphi\sin\theta & -a\sin\varphi \\ -a\sin\varphi\sin\theta & a\sin\varphi\cos\theta & 0 \end{vmatrix} = \langle a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\sin\varphi\cos\varphi \underbrace{(\cos^2\theta + \sin^2\theta)}_{1} \rangle$

$|\vec{r}_\varphi \times \vec{r}_\theta| = a^2\sin\varphi \sqrt{\underbrace{(\sin\varphi\cos\theta)^2 + (\sin\varphi\sin\theta)^2}_{\sin^2\varphi} + (\cos\varphi)^2} = a^2\sin\varphi$

So, $A(S) = \int_0^{\pi}\int_0^{2\pi} a^2\sin\varphi\, d\theta\, d\varphi = 2\pi a^2 \int_0^{\pi} \sin\varphi\, d\varphi = 4\pi a^2$

we don't have to include any additional factor here!

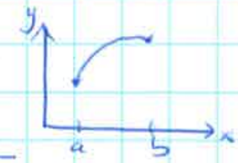
Surface area of the graph of a function

$S: z = f(x,y)$ or $\vec{r}(x,y) = \langle x, y, f(x,y)\rangle \rightsquigarrow \vec{r}_x = \langle 1, 0, \frac{\partial f}{\partial x}\rangle$, $\vec{r}_y = \langle 0, 1, \frac{\partial f}{\partial y}\rangle$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right\rangle, \quad |\vec{r}_x \times \vec{r}_y| = \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$$

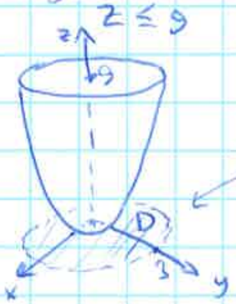
$$\rightarrow A(S) = \boxed{\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dA}$$

cf. arc length formula



$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Ex $S: z = x^2+y^2$, $z \le 9$. Find $A(S)$



Sol. $$A(S) = \iint_D \sqrt{1+\underbrace{(2x)}_{\frac{\partial z}{\partial x}}{}^2+\underbrace{(2y)}_{\frac{\partial z}{\partial y}}{}^2}\, dA = \iint_D \sqrt{1+4(x^2+y^2)}\, dA \underset{\text{polar}}{=} \int_0^{2\pi}\int_0^3 \sqrt{1+4\underbrace{r^2}_{u}}\, \underbrace{r\,dr}_{\frac{1}{2}du}\, d\theta =$$

$r^2 = u$

$$= \int_0^{2\pi} \underbrace{\int_0^9 \sqrt{1+4u}\, \frac{1}{2}du}_{\frac{1}{8}\cdot\frac{2}{3}(1+4u)^{3/2}\Big|_{u=0}^{u=9} = \frac{1}{12}(37^{3/2}-1)}\, d\theta = 2\pi \cdot \frac{1}{12}(37\sqrt{37}-1) = \boxed{\frac{\pi}{6}(37\sqrt{37}-1)}$$