16.2 Line integrals of functions.

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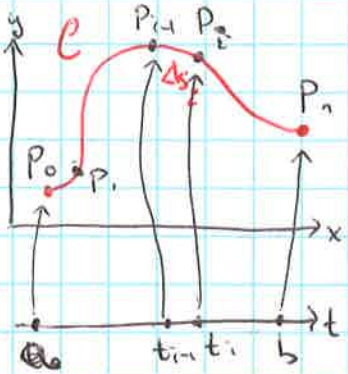
· C - plane curve $\begin{matrix} x = x(t) \\ y = y(t) \end{matrix}$, $a \le t \le b$



Line integral of $f(x,y)$ along C: [curve]

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i^*, y_i^*)}_{\substack{\text{sample}\\ \text{point in}\\ i\text{-th sub-arc}}} \underbrace{\Delta s_i}_{\text{length of } i^{\text{th}} \text{ sub-arc}}$$

Recall: the length of the curve: $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

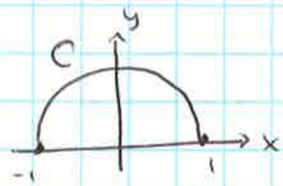
Similarly, the length integral is:

$$\boxed{\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t)) \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}_{(*)}\,dt}$$

- does not depend on the parametrization of C! (assuming that C is traversed exactly once as t goes from a to b)

optional: if C is parameterized by the arc length $P_0 \overset{s}{\frown} P$, then $(*) = 1$.

$\underline{Ex}$: find $\int_C (2 + x^2 y)\,ds$, C - upper semi-circle $\begin{matrix} x^2 + y^2 = 1 \\ y \ge 0 \end{matrix}$



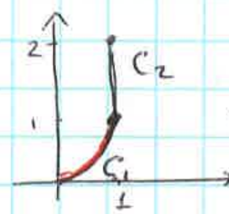
$\underline{Sol}$ parameterize C by $x = \cos t$, $y = \sin t$, $0 \le t \le \pi$

$$\int_C (2 + x^2 y)\,ds = \int_0^{\pi} (2 + \cos^2 t \sin t) \underbrace{\sqrt{(x')^2 + (y')^2}}_{\sqrt{\sin^2 t + \cos^2 t} = 1}\,dt = 2t - \frac{1}{3}\cos^3 t \Big|_0^{\pi} = 2\pi - \frac{1}{3}(-1-1) = \boxed{2\pi + \frac{2}{3}}$$

$\underline{Ex}$: find $\int_C 2x\,ds$, $C = C_1 \cup C_2$

C_1 - arc of parabola $y = x^2$ from $(0,0)$ to $(1,1)$

C_2 - line segment from $(1,1)$ to $(1,2)$



← piecewise-smooth curve!

$\underline{Sol}$: $\int_C 2x\,ds = \int_{C_1} 2x\,ds + \int_{C_2} 2x\,ds$

C_1: $x = x$, $y = x^2$, $0 \le x \le 1$ - parameter

C_2: $x = 1$, $y = y$, $1 \le y \le 2$ - parameter

$$\int_{C_1} 2x\,ds = \int_0^1 2x\sqrt{1 + 4x^2}\,dx = \frac{1}{4}\cdot\frac{2}{3}(1+4x^2)^{3/2}\Big|_0^1 = \frac{5\sqrt{5} - 1}{6}$$

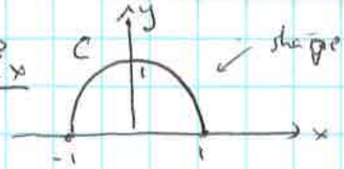
$$\int_{C_2} 2x\,ds = \int_1^2 2\cancel{x}\sqrt{0 + 1}\,dy = 2$$

Thus: $\int_C 2x\,ds = \boxed{\frac{5\sqrt{5} - 1}{6} + 2}$

· if C is the form of a thin $\underline{\text{wire}}$ with linear density $\rho(x,y)$, then $m = \int_C \rho(x,y)\,ds$ - $\underline{\text{mass}}$

$$\bar{x} = \frac{1}{m}\int_C x\rho(x,y)\,ds, \quad \bar{y} = \frac{1}{m}\int_C y\rho(x,y)\,ds$$

$(\bar{x}, \bar{y})$ - center of mass.

Ex  shape of the wire - semi-circle $x^2+y^2=1$, $y \geq 0$

density $\rho = \underbrace{k}_{\text{constant}}(1-y)$. Find the center of mass $(\bar{x},\bar{y})$.

Sol: $m = \int_C k(1-y)\,ds = \int_0^{\pi} k(1-\sin t)\,dt = k(\pi-2)$

$x=\cos t$
$y=\sin t$

$\bar{y} = \frac{1}{m}\int_C k(1-y)y\,ds = \frac{1}{\pi-2}\int_0^{\pi}(1-\sin t)\sin t\,dt = \frac{1}{\pi-2}\int_0^{\pi}\left(\sin t - \frac{1-\cos 2t}{2}\right)dt = \frac{1}{\pi-2}\left|-\cos t - \frac{t}{2} + \frac{1}{4}\sin 2t\right|_0^{\pi}$

$= \frac{1}{\pi-2}\left(2-\frac{\pi}{2}\right) = \frac{4-\pi}{2(\pi-2)}$ $\quad \bar{x}=0$ by symmetry, so $(\bar{x},\bar{y}) = \left(0, \frac{4-\pi}{2(\pi-2)}\right)$ - center of mass

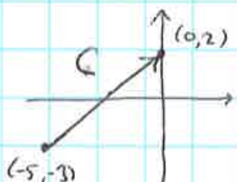
≈ 0.38

Line integrals with respect to x, y:

$\int_C f(x,y)\,dx = \lim_{n\to\infty}\sum f(x_i^*,y_i^*)\Delta x_i = \int_a^b f(x(t),y(t))\,x'(t)\,dt$

$\int_C f(x,y)\,dy = \lim_{n\to\infty} f(x_i^*,y_i^*)\Delta y_i = \int_a^b f(x(t),y(t))\,y'(t)\,dt$

Ex find $\int_C y^2\,dx + x\,dy$ $:= \int_C y^2\,dx + \int_C x\,dy$



- line segment from $(-5,-3)$ to $(0,2)$

Sol: parameterize C: $\vec{r}(t) = \underbrace{\vec{r}_0}_{\langle -5,-3\rangle}(1-t) + \underbrace{\vec{r}_1}_{\langle 0,2\rangle} t$ or $\begin{array}{l} x=-5+5t \\ y=-3+5t \end{array}$, $0\leq t\leq 1$

$0\leq t\leq 1$ $\quad \rightarrow dx = 5\,dt$, $dy = 5\,dt$

$\int_C y^2\,dx + x\,dy = \int_0^1((-3+5t)^2\cdot 5 + (-5+5t)\cdot 5)\,dt = 5\int_0^1(25t^2-25t+4)\,dt = 5\left(\underbrace{\frac{25}{3}-\frac{25}{2}}_{-25/6}+4\right) = \boxed{-\frac{5}{6}}$

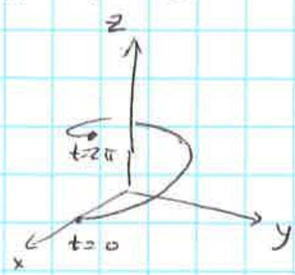
note: if C were from (0,2) to (-5,-3) (opposite orientation), we would get $+\frac{5}{6}$.

Generally: $\int_{-C} f(x,y)\,ds = -\int_C f(x,y)\,ds$

$-C$: C travelled in opposite direction

In space: $\boxed{ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt}$

Ex: find $\int_C y\sin z\,ds$, $\quad$ C - circular helix $\begin{array}{l} x=\cos t \\ y=\sin t \\ z=t \end{array}$, $0\leq t\leq 2\pi$



Sol: $\int_C y\sin z\,ds = \int_0^{2\pi}\sin t\cdot\sin t\underbrace{\sqrt{(x')^2+(y')^2+(z')^2}}_{\sqrt{2}}\,dt =$

$= \sqrt{2}\int_0^{2\pi}\frac{1-\cos 2t}{2}\,dt = \sqrt{2}\left|\frac{t}{2}-\frac{\sin 2t}{4}\right|_0^{2\pi} = \pi\sqrt{2}$

- $\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = \int_C P\,x'(t)\,dt + Q\,y'(t)\,dt + R\,z'(t)\,dt$

express in terms of t