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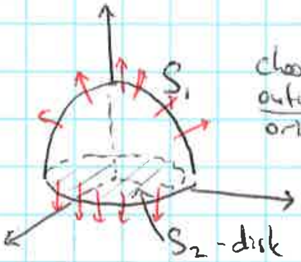
Ex: S = boundary of the solid enclosed by $z = 1 - x^2 - y^2$ and $z = 0$,

$\vec{F} = \langle y, x, z \rangle$ Find the flux.



choose outward (positive) orientation

Sol: flux through S_1:

$$\iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_{x^2+y^2\le 1} \langle y, x, \underbrace{z}_{1-x^2-y^2} \rangle \cdot \langle 2x, 2y, 1\rangle \, dA = \iint_{x^2+y^2\le 1} (1 - x^2 - y^2 + 4xy)\, dA =$$

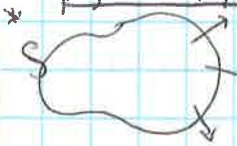
- express in terms of parameters

$$\underset{\text{polar}}{=} \int_0^{2\pi}\int_0^1 (1 - r^2 - \cancel{4}r^2 \underbrace{\cancel{2}\sin\theta\cos\theta}_{2\sin 2\theta}) r\,dr\,d\theta = \int_0^{2\pi}\left(\frac{1}{2} - \frac{1}{4} - \frac{1}{2}\sin 2\theta\right) d\theta = \frac{\pi}{2}$$

$$\iint_{S_2} \vec{F}\cdot \underbrace{d\vec{S}}_{-\vec{k}\,dS} = \iint_{x^2+y^2\le 1} \underbrace{\langle y, x, 0\rangle \cdot \langle 0, 0, -1\rangle}_{0} \, dA = 0.$$

Thus: total flux $\displaystyle\iint_S \vec{F}\cdot d\vec{S} = \frac{\pi}{2}$

* physical interpretation



$\vec{v}$ - velocity field of the fluid of density ρ

$\displaystyle\iint_S \underset{\text{density}}{\rho}\, \vec{v}\cdot d\vec{S}$ - flow of the fluid (outward) through S.

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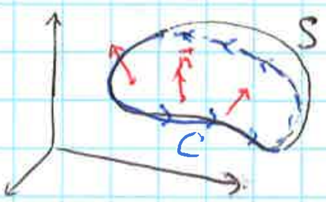
16.8 Stokes' theorem



S - oriented surface (choice of $\vec{n}$), bounded by a closed curve C

C has an "induced positive orientation"

<walking along C with your head in the direction of $\vec{n}$, you should see S on your left>

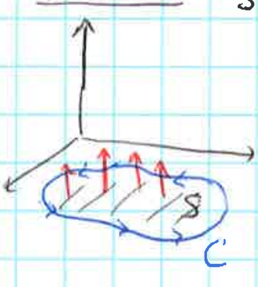
Stokes' Theorem:

$$\boxed{\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}}$$

C = "∂S" - notation

- line integral of (tangential component) of $\vec{F}$ along the boundary equals the flux of curl through the surface.

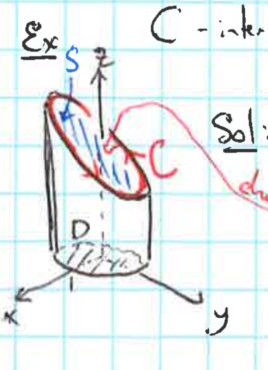
* special case: S is flat and lies in xy-plane, $\vec{n} = \vec{k}$, we get



$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \underbrace{\operatorname{curl}\vec{F}\cdot\vec{k}}_{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}\, dS$$

we recovered - Green's theorem!

Ex C - intersection of plane $y + z = 2$, cylinder $x^2 + y^2 = 1$ find $\oint_C \vec{F}\cdot d\vec{r}$ for $\vec{F}(x,y,z) = \langle -y^2, x, z^2\rangle$



Sol: use Stokes'.

$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0, 0, 1+2y\rangle$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_{\substack{S \\ z = 2 - y = g(x,y)}} \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_D \langle 0, 0, 1+2y\rangle \cdot \underbrace{\langle 0, 1, 1\rangle}_{\langle -g_x, -g_y, 1\rangle} \, dA = \iint_D (1+2y)\, dA$$

$$\iint_D (1+2y)\,dA \underset{\text{polar}}{=} \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\,r\,dr\,d\theta = \int_0^{2\pi}\left(\tfrac{1}{2}+\tfrac{2}{3}\sin\theta\right)d\theta = \pi .$$

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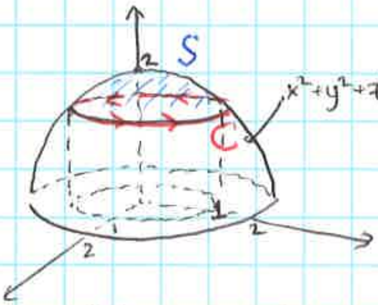
Ex: S = part of the sphere $x^2+y^2+z^2=4$ inside the cylinder $x^2+y^2=1$, above xy-plane



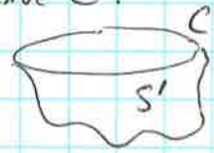
find $\iint_S \operatorname{curl}\vec F\cdot d\vec S$ where $\vec F(x,y,z)=\langle xz,\ yz,\ xy\rangle$

Sol: Use Stokes! Boundary of S is the curve C: $\begin{cases} x^2+y^2+z^2=4 \\ x^2+y^2=1\end{cases} \to \begin{cases} z=\sqrt3 \\ x^2+y^2=1\end{cases}$ - circle

parameterize C: $\vec r(t)=\langle\cos t,\ \sin t,\ \sqrt3\rangle$, $0\le t\le 2\pi$

$$\iint_S \operatorname{curl}\vec F\cdot d\vec S \underset{\text{Stokes'}}{=} \oint_C \vec F\cdot d\vec r = \int_0^{2\pi} \overbrace{\langle\sqrt3\cos t,\ \sqrt3\sin t,\ \sin t\cos t\rangle\cdot\underbrace{\langle-\sin t,\cos t,0\rangle}_{\frac{d\vec r}{dt}}}^{0}\,dt = \boxed{0}.$$

note: the answer depends only on values of $\vec F$ on the boundary curve C!
we could have taken another surface S'



with same boundary C, and $\iint_{S'} \operatorname{curl}\vec F\cdot d\vec S$ would be the same!

* interpretation of $\operatorname{curl}\vec v$: $\vec v$ = velocity field of a fluid

tiny disk about P_0 of radius a

$$\underbrace{\iint_S \operatorname{curl}\vec v\cdot\vec n\,dS}_{\approx \pi a^2 \operatorname{curl}\vec v(P_0)\cdot\vec n} = \underbrace{\oint_C \vec v\cdot d\vec r}_{\text{"circulation" of }\vec v\text{ around }C}$$

- tendency of fluid to move around C

$$\Longrightarrow \boxed{\operatorname{curl}\vec v(P_0)\cdot\vec n = \lim_{a\to0}\frac{1}{\pi a^2}\oint_{C_a}\vec v\cdot d\vec r}$$