11/05/2018

i

16.1, 16.2 Vector fields, line integrals

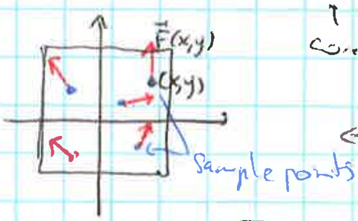
A vector field on $\mathbb{R}^2$ is a function $\vec{F}(x,y)$ assigning to each point $(x,y) \in \underbrace{D}_{\text{region in } \mathbb{R}^2}$ a 2D vector $\vec{F}(x,y)$.

on $\mathbb{R}^3$ ——— $\vec{F}(x,y,z)$ ——— $(x,y,z) \in E$ – a 3D vector $\vec{F}(x,y,z)$

2D: $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = P(x,y)\vec{i} + Q(x,y)\vec{j}$

component functions (scalar)

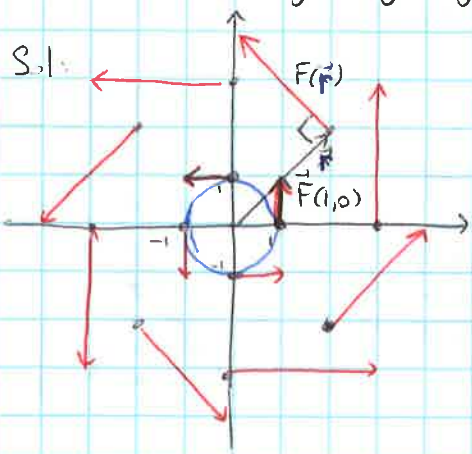
* one calls P, Q scalar fields



← picture of a vector field

Ex: $F(x,y) = -y\vec{i} + x\vec{j}$ – vector field on $\mathbb{R}^2$. Sketch it.

Sol:

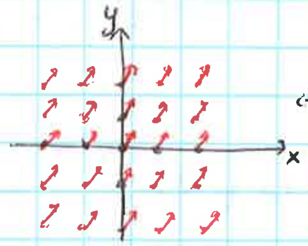


in fact: each arrow is tangent to circle centered at origin:

$$\underbrace{\vec{r}}_{\langle x,y\rangle} \cdot \underbrace{\vec{F}(\vec{r})}_{\vec{F}(x,y)} = (x\vec{i} + y\vec{j})(-y\vec{i} + x\vec{j}) = -xy + yx = 0$$

$\rightsquigarrow \vec{F}(\vec{r})$ perp. to $\vec{r}$ → $\vec{F}(\vec{r})$ tangent to the circle with center at the origin, $r = \sqrt{x^2+y^2}$.

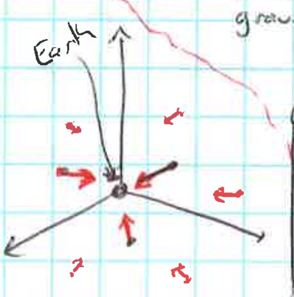
Ex: $F(x,y) = \vec{i} + 2\vec{j}$



← one can rescale all vectors in a picture by (same) factor

Ex: Gravitational force field of Earth:

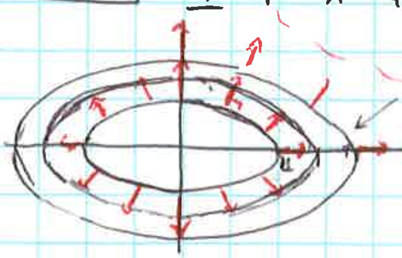
(*) $\vec{F}(\vec{r}) = \underbrace{\frac{MmG}{r^2}}_{\text{grav. pull}} \cdot \underbrace{\left(-\frac{\vec{r}}{|\vec{r}|}\right)}_{\text{unit vector pointing to the origin}} = -\frac{MmG}{|\vec{r}|^3}\vec{r} = \left\langle \frac{-MmGx}{(x^2+y^2+z^2)^{3/2}}, \frac{-MmGy}{(x^2+y^2+z^2)^{3/2}}, \frac{-MmGz}{(x^2+y^2+z^2)^{3/2}} \right\rangle$



Gradient vector field: $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

or $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$

Ex: $f = x^2 + 4y^2$ $\qquad \nabla f = \langle 2x, 8y \rangle$

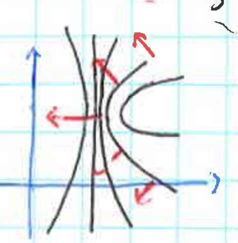


level curves

gradient vectors are perpendicular to level ~~surfaces~~ curves

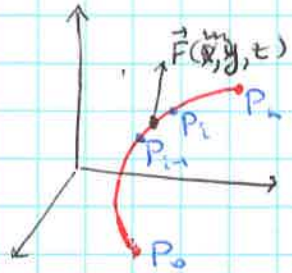
grad. vectors are long where level curves are ~~narrowly spaced~~ close to each other

Generally:



Ex: (*) is a gradient vector field!

$\vec{F}(x,y,z) = \nabla f(x,y,z)$ with $f(x,y,z) = \frac{MmG}{r}$.



C - curve , $\vec{F}_{(x,y,z)}$ - force field

work W done by force $\vec{F}$ while moving a particle along the curve C :

$$W = \lim_{n\to\infty} \sum \vec{F}(x_i^*, y_i^*, z_i^*) \cdot \Delta s_i \, \underbrace{\vec{T}(t_i^*)}_{\text{unit tangent vector to } C} = \int_C \vec{F}(x,y,z) \cdot \underbrace{\vec{T}(x,y,z)}_{\frac{\vec{r}'(t)}{|\vec{r}'(t)|}} \, \underbrace{ds}_{|\vec{r}'(t)|\,dt}$$

$$W = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt.$$

Line Integral of a vector field $\vec{F}$ along a curve C given by $\vec{r}(t)$, $a \le t \le b$:

$$\int_C \vec{F} \cdot d\vec{r} = \boxed{\int_a^b \underbrace{\vec{F}(\vec{r}(t))}_{\vec{F}(x(t),y(t),z(t))} \cdot \vec{r}\,'(t)\, dt} = \int_C \vec{F} \cdot \vec{T}\, ds$$

Ex: find the work W done by force field $\vec{F}(x,y) = x^2 \vec{i} - xy \vec{j}$ in moving a particle along quarter-circle $\vec{r}(t) = \cos t\, \vec{i} + \sin t\, \vec{j}$, $0 \le t \le \frac{\pi}{2}$

$$W = \int_0^{\pi/2} \underbrace{\vec{F}(\vec{r}(t))}_{\langle \cos^2 t, -\cos t \sin t \rangle} \cdot \underbrace{\vec{r}\,'(t)}_{\langle -\sin t, \cos t \rangle} \, dt = \int_0^{\pi/2} (-\cos^2 t \sin t - \cos^2 t \sin t)\, dt = -2 \int_0^{\pi/2} \cos^2 t \sin t\, dt$$

$$= 2 \frac{\cos^3 t}{3} \Big|_0^{\pi/2} = -\frac{2}{3}$$

- $\int_C \underbrace{\vec{F}}_{\langle P,Q,R \rangle} \cdot d\vec{r} = \int_C P(x,y,z)\, dx + \int_C Q(x,y,z)\, dy + \int_C R(x,y,z)\, dz$

Ex Find $\underbrace{\int_C \vec{F} \cdot d\vec{r}}_{W}$ where $\vec{F}(x,y,z) = \langle xy, yz, zx \rangle$ and C is given by $x = t$, $y = t^2$, $z = t^3$, $0 \le t \le 1$

Sol: $\vec{r}(t) = \langle t, t^2, t^3 \rangle$, $\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle$, $\vec{F}(\vec{r}(t)) = \langle t^3, t^5, t^5 \rangle$

So, $W = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^1 (t^3 + 2t^6 + 3t^6)\, dt = \frac{1}{4} + \frac{5}{7} = \frac{27}{28}$