16.3 Fundamental theorem of line integrals

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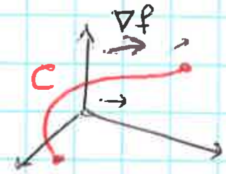
Recall Gradient vector field $\vec{F}(x,y) = \nabla f(x,y)$

"potential" of $\vec{F}$
"function"

a vector field with a potential is called "conservative"

Recall fund. THM of Calculus:

$$\int_a^b \underbrace{F'(x)\,dx}_{\text{rate of change}} = \underbrace{F(b) - F(a)}_{\text{net change}}$$



Fundamental THM of line integrals

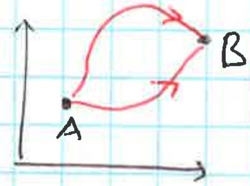
Let C - curve given by $\vec{r}(t)$, $a \le t \le b$. Then
(on a plane or in space)

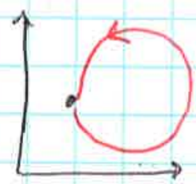
$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$$

Ex: Gravitational field is $\vec{F}(\vec{r}) = -\frac{k\vec{r}}{|\vec{r}|^3} = \nabla \underbrace{\frac{k}{|\vec{r}|}}_{f}$. Find the work done by grav. field when moving a particle from $(3,4,12)$ to $(2,2,0)$ along some curve C.

Sol: $W = \int_C \vec{F}(\vec{r})\cdot d\vec{r} = \underbrace{f(2,2,0)}_{\text{end-point}} - \underbrace{f(3,4,12)}_{\text{beginning of motion}} = \frac{k}{\sqrt{8}} - \frac{k}{\sqrt{\underbrace{3^2+4^2+12^2}_{169}}} = \boxed{k\left(\frac{1}{2\sqrt{2}} - \frac{1}{13}\right)}$

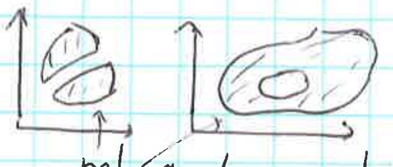
- For $\vec{F}$ a vector field, the following are equivalent:
  (1) $\vec{F}$ is conservative, i.e. $\vec{F} = \nabla f$
  (2) $\int_C \vec{F}\cdot d\vec{r}$ is the same for all curves (paths) with initial point A and final point B
  "independent of path".
  (3) $\int_C \vec{F}\cdot d\vec{r} = 0$ for any closed curve (starting and ending at same point)



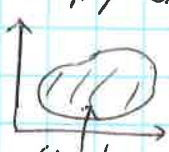


- Let $\vec{F}(x,y)$ be a vector field in a simply-connected region D
  $= P(x,y)\vec{i} + Q(x,y)\vec{j}$ <single-piece, without holes>

THEN: $\vec{F}(x,y)$ is conservative iff $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$



not simply-connected



simply-connected

Idea $(\rightarrow)$: $F = \nabla f \rightarrow P = f_x, Q = f_y \rightarrow \frac{\partial P}{\partial y} = f_{xy}$, $\frac{\partial Q}{\partial y} = f_{yx}$.

Ex: $\vec{F}(x,y) = \langle \underbrace{x-y}_{P}, \underbrace{x-2}_{Q} \rangle$. Is it conservative?

Sol: $P_y = -1$, $Q_x = 1 \rightarrow P_y \neq Q_x \rightarrow$ NO!

*Ex**: $\vec{F}(x,y) = \langle \underbrace{3+2xy}_{P}, \underbrace{x^2-3y^2}_{Q} \rangle$. Is it conservative?

Sol: $P_y = 2x$, $Q_x = 2x \rightarrow$ YES!

<u>In space</u>: $\vec{F}(x,y,z)=\langle P,Q,R\rangle$ is conservative iff $\boxed{P_y=Q_x,\ P_z=R_x,\ Q_z=R_y}$

<u>Ex</u> $\vec{F}(x,y)=\langle 3+2xy,\ x^2-3y^2\rangle$ - from Ex*

(a) find f such that $\nabla f=\vec{F}$

(b) find $\int_C \vec{F}\cdot d\vec{r}$ for the curve C: $\vec{r}(t)=\langle e^t\sin t, e^t\cos t\rangle$, $0\le t\le\pi$

<u>Sol</u>: (a) want f such that:

(1) $f_x=3+2xy \xrightarrow{\text{integrate w.r.t. } x} f(x,y)=3x+x^2y+g(y)$

(2) $f_y=x^2-3y^2 \xrightarrow{\text{substitute in (2)}} f_y=x^2+g'(y)=x^2-3y^2 \rightarrow g'(y)=-3y^2$

$\rightarrow g(y)=-y^3+\underbrace{K}_{\text{constant}}$

thus: $\boxed{f(x,y)=3x+x^2y-y^3+K}$

(b) curve C starts at $\vec{r}(0)=\langle 0,1\rangle$ and ends at $\vec{r}(\pi)=\langle 0,-e^{\pi}\rangle$

So: $\int_C \vec{F}\cdot d\vec{r}=\int_C \nabla f\cdot d\vec{r}=f(0,-e^{\pi})-f(0,1)=(e^{3\pi}+K)-(-1+K)=\boxed{e^{3\pi}+1}$

<u>Ex</u>: $\vec{F}(x,y,z)=\langle y^2,\ 2xy+e^{3z},\ 3ye^{3z}\rangle$. Find f such that $\vec{F}=\nabla f$

<u>Sol</u> (1) $f_x=y^2 \xrightarrow{\int} f=xy^2+g(y,z)$

(2) $f_y=2xy+e^{3z} \longrightarrow 2xy+g_y=2xy+e^{3z}\rightarrow g_y=e^{3z}\xrightarrow{\int} g=ye^{3z}+h(z)\rightarrow f=xy^2+ye^{3z}+h(z)$

(3) $f_z=3ye^{3z} \longrightarrow 3ye^{3z}+h_z(z)=3ye^{3z}\rightarrow h_z=0\rightarrow h=\underbrace{K}_{\text{constant}} \rightarrow \boxed{f(x,y,z)=xy^2+ye^{3z}+K}$