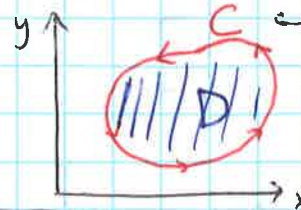
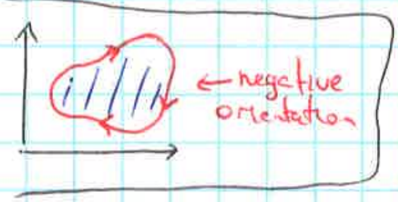
16.4 11/9/2018

Green's Theorem



C ← simple, closed curve with positive (counterclockwise) orientation, bounding the region D.
<no crossing itself>

[region is always on the left as we go around C]

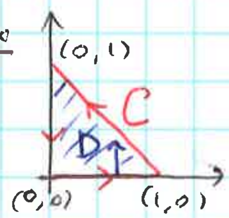


Green's THM:

$$\underbrace{\int_C}_{\text{positively oriented}} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

* one also ~~uses~~ can use notation $\oint_C P\,dx + Q\,dy$ for a line integral over a closed curve.

Ex

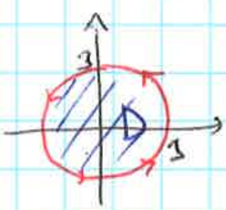


Find the integral $\int_C x^4 dx + xy\,dy$ over the triangular curve C, using Green's theorem

Sol: $\int_C \underbrace{x^4}_{P} dx + \underbrace{xy}_{Q} dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \underbrace{\int_0^{1-x} (y - 0)\, dy}_{\frac{(1-x)^2}{2}}\, dx$

$= -\frac{(1-x)^3}{6}\Big|_0^1 = \boxed{\frac{1}{6}}$

Ex



Find $I = \oint_C \underbrace{(3y - e^{\sin x})}_{P} dx + \underbrace{(7x + \sqrt{y^4+1})}_{Q} dy$ where C is the circle $x^2+y^2=9$

Sol: $I \underset{\text{Green's THM}}{=} \iint_{D \leftarrow \text{disk}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (7-3)\, dA = 4\int_0^{2\pi}\int_0^3 r\,dr\,d\theta =$

$= 4 \cdot 2\pi \cdot \frac{9}{2} = 36\pi$

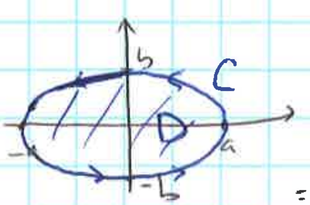
* can use Green's THM in the other direction:
convert a double integral to a line integral.

e.g. $\frac{1}{2}\oint_C \underbrace{x}_{Q} dy - \underbrace{y}_{P} dx \overset{(*)}{=} \frac{1}{2}\iint_D (1-(-1))\, dA = \iint_D dA$ = area of D

$= \oint_C x\,dy$ $= -\oint_C y\,dx$

so, can calculate area as a line integral (*) over the boundary

Ex: D enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the area.



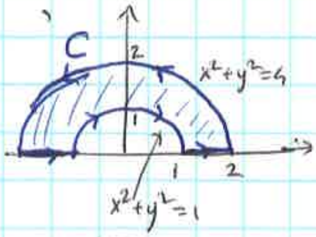
Sol: area $= \frac{1}{2}\oint_C x\,dy - y\,dx \underset{\text{parametrize } C:}{=} \frac{1}{2}\int_0^{2\pi} a\cos t \underbrace{b\cos t\, dt}_{\frac{dy}{dt}} - b\sin t \underbrace{(-a\sin t)\, dt}_{\frac{dx}{dt}} =$

$x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$

$= \frac{1}{2}\int_0^{2\pi} ab\, dt = \pi ab$

11/9/2018
2

Ex



find $I = \oint_C \overbrace{y^2}^{P}\, dx + \overbrace{3xy}^{Q}\, dy$

C: boundary of semi-annular region

Sol $I = \iint_D \underbrace{(3y - 2y)}_{y}\, dA \underset{\text{polar}}{=} \int_0^{\pi} \underbrace{\int_1^2 r\sin\theta \, r\, dr\, d\theta}_{\left.\frac{r^3 \sin\theta}{3}\right|_{r=1}^{r=2} = \frac{7}{3}\sin\theta} = \frac{7}{3}\left[-\cos\theta\right]_0^{\pi} = \frac{14}{3}$

* For a negatively oriented curve C:

$$\oint_C P\,dx + Q\,dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$