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12.4 Cross product

for $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$ two vectors, the vector $\vec{a}\times\vec{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$ is the cross-product or (vector product) - works only in $\mathbb{R}^3$!

- $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ - determinant of order 2

Ex: $\begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix} = 2\cdot 5 - 3(-1) = 13$

- determinant of order 3: $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \overset{!}{-} a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$

Ex: $\begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 3 \\ 3 & -1 & 5 \end{vmatrix} = 2 \underbrace{\begin{vmatrix} 2 & 3 \\ -1 & 5 \end{vmatrix}}_{13} - 1 \cdot \underbrace{\begin{vmatrix} 0 & 3 \\ 3 & 5 \end{vmatrix}}_{0\cdot5-3\cdot3=-9} + 0 \underbrace{\begin{vmatrix} 0 & 2 \\ 3 & -1 \end{vmatrix}}_{-6} = 26 + 9 = 35$

- Cross-product: $\vec{a}\times\vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} + \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k} = \boxed{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$

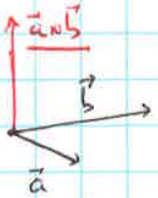
Ex ① $\vec{a} = \langle 1, 2, 3 \rangle$, $\vec{b} = \langle 2, 0, -5 \rangle$

$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 0 & -5 \end{vmatrix} = (-10)\vec{i} + (-11)\vec{j} + (-4)\vec{k} = \langle -10, 11, -4 \rangle$

- $(\vec{a}\times\vec{b}) \cdot \vec{a} = (-10)(1) + (11)(2) + (-4)(3) = -10 + 22 - 12 = 0$
  $(\vec{a}\times\vec{b}) \cdot \vec{b} = (-10)(2) + (11)(0) + (-4)(-5) = -20 + 0 + 20 = 0$

$\Rightarrow$ $\vec{a}\times\vec{b}$ ~~perp~~ orthogonal to ~~both~~ both $\vec{a}$ and $\vec{b}$!

It holds generally: **Thm** $\vec{a}\times\vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

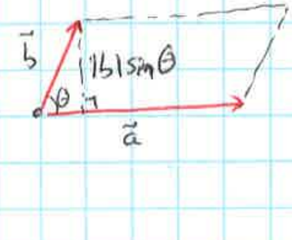


direction of $\vec{a}\times\vec{b}$ is given by right hand rule.

Thm $|\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}| \sin\theta$

θ: angle between $\vec{a}, \vec{b}$.

area of the parallelogram



- $\vec{a}$ and $\vec{b}$ are parallel iff $\vec{a}\times\vec{b} = \vec{0}$

- $\vec{a}\times\vec{a} = \vec{0}$

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Ex*: $P(-1,2,5)$, $Q(0,4,8)$, $R(1,2,0)$

find a vector perpendicular to the plane passing through P, Q, R

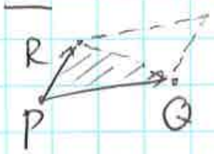
Sol: $\vec{PQ} \times \vec{PR}$ works! $\quad \vec{PQ} = \langle 1,2,3\rangle$, $\vec{PR} = \langle 2,0,-5\rangle$

$\vec{PQ} \times \vec{PR} = \langle -10, 11, -4\rangle$ $\quad \leftarrow$ we computed it in Ex ①

any nonzero multiple of $\vec{PQ}\times\vec{PR}$ works too!

Ex find the area of the triangle PQR.

Sol area $= \frac{1}{2}$ area(parallelogram) $= \frac{1}{2}|\vec{PQ}\times\vec{PR}| = \frac{1}{2}\sqrt{(-10)^2+(11)^2+(-4)^2} = \boxed{\frac{1}{2}\sqrt{237}}$

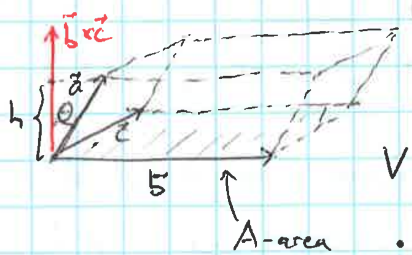


properties of $\times$:

- $\vec{i}\times\vec{j} = \vec{k}$, $\vec{j}\times\vec{k} = \vec{i}$, $\vec{k}\times\vec{i} = \vec{j}$
  $\vec{j}\times\vec{i} = -\vec{k}$, $\vec{k}\times\vec{j} = -\vec{i}$, $\vec{i}\times\vec{k} = -\vec{j}$.
- $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ $\qquad$ $\times$ is not associative, e.g. $(a\times b)\times c \neq a\times(b\times c)$, e.g.,
  $i\times(\underbrace{i\times j}_{k}) = -j$ while $(\underbrace{i\times i}_{0})\times j = 0$
- $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

Triple product

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$ – (scalar) triple product of $\vec{a}, \vec{b}, \vec{c}$



volume of the parallelepiped:

$V = A h = |\vec{b}\times\vec{c}|(|\vec{a}||\cos\theta|) = \boxed{|\vec{a}\cdot(\vec{b}\times\vec{c})|}$

- If $V=0$ then $\vec{a},\vec{b},\vec{c}$ lie in the same plane (are co-planar)

Ex: $\vec{a} = \langle 1,2,3\rangle$ $\vec{b} = \langle 4,5,6\rangle$ $\vec{c} = \langle 7,8,9\rangle$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\underbrace{(45-48)}_{-3} - 2\underbrace{(36-42)}_{-6} + 3\underbrace{(32-35)}_{-3} = 0 \Rightarrow \vec{a},\vec{b},c \text{ co-planar!}$$

(• Torque (physics))

Ex: $P(-1,2,5)$, $Q(0,4,8)$, $R(1,2,0)$, $S(0,3,7)$

find volume of parallelepiped with adjacent edges $\vec{PQ}, \vec{PR}, \vec{PS}$.

Sol: $\vec{a} = \vec{PQ} = \langle 1,2,3\rangle$ $\quad \vec{b} = \vec{PR} = \langle 2,0,-5\rangle$ $\quad \vec{c} = \vec{PS} = \langle 1,1,2\rangle$

$\text{Vol} = |\vec{a}\cdot(\vec{b}\times\vec{c})| = |\underbrace{(\vec{a}\times\vec{b})}_{\langle -10,11,-4\rangle \text{ from Ex}^*}\cdot\vec{c}| = |(-10)1+(11)1+(-4)2| = |-7| = 7$