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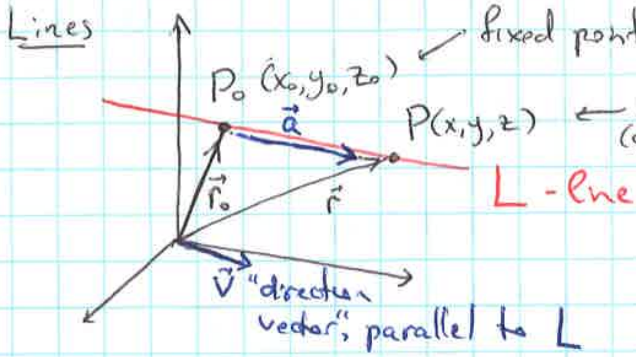
12.5 Lines and planes

Lines

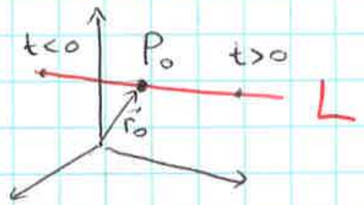


$$\underset{\substack{\text{pos. vector}\\ \text{of } P}}{\vec{r}} = \underset{\substack{\text{pos. vect}\\ \text{of } P_0}}{\vec{r}_0} + \underset{\text{parallel to } \vec{v}}{\vec{a}}, \qquad \vec{a} = \underbrace{t}_{\text{scalar}}\vec{v}$$

$$\boxed{\vec{r} = \vec{r}_0 + t\vec{v}}$$ - vector equation of L

in components: $\langle x,y,z\rangle = \langle x_0+ta,\, y_0+tb,\, z_0+tc\rangle$

for $\vec{v} = \langle a,b,c\rangle$



So: (*) $\boxed{\begin{aligned} x &= x_0+ta \\ y &= y_0+tb \\ z &= z_0+tc \end{aligned}}$ parametric equations of the line L through point $P_0(x_0,y_0,z_0)$ and parallel to the direction vector $\vec{v}\ \langle a,b,c\rangle$

$t \in \mathbb{R}$ parameter

Ex: $P_0(1,0,-1)$, $\vec{v} = \langle 1,2,3\rangle$

line through ↑ parallel to ↑ is: $\vec{r} = \underbrace{\vec{i} - \vec{k}}_{\vec{r}_0} + t(\vec{i}+2\vec{j}+3\vec{k}) = (1+t)\vec{i} + 2t\vec{j} + (-1+3t)\vec{k}$ - vector eq.

or: $\boxed{x = 1+t,\ y = 2t,\ z = -1+3t}$ - param eq.

- find two points on L other than P_0: Exg. $t=1 \rightsquigarrow (2,2,2)$
  $t=-1 \rightsquigarrow (0,-2,-4)$

* Vector & param equations of L are not unique!

E.g. we could take $P_0(2,2,2)$ ← another point on L → $x = 2+t,\ y = 2+2t,\ z = 2+3t$ - same line

or could take $\langle 2,4,6\rangle$ as dir. vector → $x = 1+2t,\ y = 4t,\ z = -1+6t$ - same line.

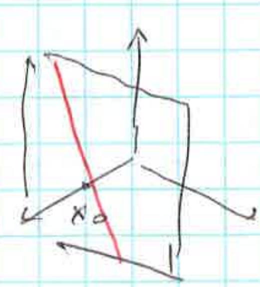
components of $\vec{v} = \langle a,b,c\rangle$ - "direction numbers" of L. [any triple proportional to a, b, c could be used]

Eliminate t from (*):

$$(t=)\quad \boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}}$$ - symmetric equations of L.

- If e.g. $a=0$, can still eliminate t: $x = x_0$, $\dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$

L lies in vert. plane $x = x_0$



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Ex* $A(1,0,2)$ $B(2,3,4)$

a) find param & sym eq. of the line L through A, B

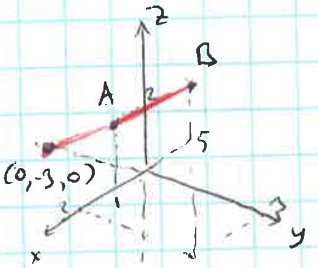
b) where does L intersect xy-plane?

Sol a) can use $\vec{v} = \vec{AB} = \langle 1,3,2\rangle$, use $P_0 = A$ $\rightsquigarrow$ $\left.\begin{array}{l} x = 1+t \\ y = 3t \\ z = 2+2t \end{array}\right\}$ param. eq. (**)

- parall to L

$\dfrac{x-1}{1} = \dfrac{y}{3} = \dfrac{z-2}{2}$ - syn. eq.

b) xy-plane: $z=0$. From : $\dfrac{x-1}{1} = \dfrac{y}{3} = \underbrace{\dfrac{0-2}{2}}_{-1} \Rightarrow \begin{array}{l} x=0 \\ y=-3 \end{array}$ intersection point: $(0,-3,0)$



• line through $P_0(x_0,y_0,z_0)$, $P_1(x_1,y_1,z_1)$
has direction numbers $x_1-x_0,\ y_1-y_0,\ z_1-z_0$

$\rightsquigarrow$ sym. eq. $\dfrac{x-x_0}{x_1-x_0} = \dfrac{y-y_0}{y_1-y_0} = \dfrac{z-z_0}{z_1-z_0}$

Ex: line segment from $A(1,0,2)$ to $B(2,3,4)$ is given by

(**) $x = 1+t$, $y = 3t$, $z = 2+2t$, with $0 \le t \le 1$

• Line segment from $\vec{r}_0$ to $\vec{r}_1$ is given by

$\vec{r} = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0) = \boxed{(1-t)\vec{r}_0 + t\vec{r}_1}$ with $0 \le t \le 1$

Ex: L_1: $x = 1+t$, $y = -2+3t$, $z = 4-t$
L_2: $x = 2s$, $y = 3+s$, $z = -3+4s$

- show that these lines are skew, i.e, they do not intersect and are not parallel (they don't lie in the same plane)

Sol: dir. vectors $\vec{v}_1 = \langle 1,3,-1\rangle$, $\vec{v}_2 = \langle 2,1,4\rangle$ are not proportional $\Rightarrow$ not parallel

look for an intersection point: - need (t,s) s.t. $\left.\begin{array}{l} 1+t = 2s \\ -2+3t = 3+s \\ 4-t = -3+4s \end{array}\right\}\begin{array}{l}(1)\\(2)\\(3)\end{array}$

$(1)-2(2)$: $5-5t = -6 \Rightarrow t = \frac{11}{5}$

from (1): $s = \frac{8}{5}$

check (3): $\underbrace{4-\frac{11}{5}}_{\frac{9}{5}} \ne \underbrace{-3+4\cdot\frac{8}{5}}_{\frac{17}{5}}$

equations (1), (2), (3) are incompatible!

$\Rightarrow$ no intersection. $\Rightarrow$ L_1, L_2 are skew

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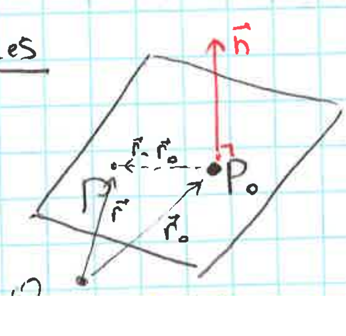
Planes



a plane is defined by a point P_0 and an orthogonal vector $\vec{n}$ ("the normal vector")

then for P any point on the plane,

$\vec{P_0P} \cdot \vec{n} = 0$, or

$\boxed{\vec{n}\cdot(\vec{r}-\vec{r}_0) = 0}$, or $\boxed{\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r}_0}$ - vector eq. of the plane