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Ex* $A(1,0,2)$ $B(2,3,4)$

a) find param & sym eq. of the line L through A, B

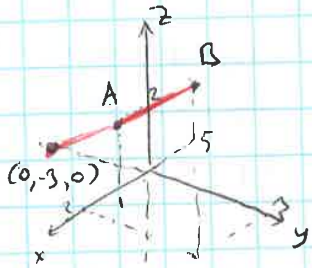
b) where does L intersect xy-plane?

Sol a) can use $\vec{v} = \vec{AB} = \langle 1,3,2\rangle$, use $P_0 = A$ $\rightsquigarrow$ $\left.\begin{array}{l} x = 1+t \\ y = 3t \\ z = 2+2t \end{array}\right\}$ param. eq. (**)

- parall to L

$$\frac{x-1}{1} = \frac{y}{3} = \frac{z-2}{2}$$ - sym. eq.

b) xy-plane: $z=0$. From ↑ : $\frac{x-1}{1} = \frac{y}{3} = \underbrace{\frac{0-2}{2}}_{-1}$ $\Rightarrow$ $\begin{array}{l} x=0 \\ y=-3 \end{array}$ intersection point: $(0,-3,0)$



• Line through $P_0(x_0,y_0,z_0)$, $P_1(x_1,y_1,z_1)$ has direction numbers $x_1-x_0,\ y_1-y_0,\ z_1-z_0$

$\rightsquigarrow$ sym. eq. $\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0}$

Ex: line segment from $A(1,0,2)$ to $B(2,3,4)$ is given by

(**) $x = 1+t$, $y = 3t$, $z = 2+2t$, with $\underline{0 \le t \le 1}$

• Line segment from $\vec{r}_0$ to $\vec{r}_1$ is given by

$\vec{r} = \vec{r}_0 + t(\vec{r}_1 - \vec{r}_0) = \boxed{(1-t)\vec{r}_0 + t\vec{r}_1}$ with $0 \le t \le 1$

Ex: L_1: $x = 1+t$, $y = -2+3t$, $z = 4-t$
L_2: $x = 2s$, $y = 3+s$, $z = -3+4s$

- show that these lines are skew, i.e., they do not intersect and are not parallel (thus don't lie in the same plane)

Sol: dir. vectors $\vec{v}_1 = \langle 1,3,-1\rangle$, $\vec{v}_2 = \langle 2,1,4\rangle$ are not proportional $\Rightarrow$ not parallel

look for an intersection point: - need (t,s) s.t.

$$\left.\begin{array}{l} 1+t = 2s \\ -2+3t = 3+s \\ 4-t = -3+4s \end{array}\right\} \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

$(1)-2(2)$: $5-5t = -6$ $\Rightarrow$ $t = \frac{11}{5}$

from (1): $s = \frac{8}{5}$

check (3): $\underbrace{4-\frac{11}{5}}_{\frac{9}{5}} \ne \underbrace{-3+4\cdot\frac{8}{5}}_{\cancel{\frac{32}{5}}\ \frac{17}{5}}$

equations (1), (2), (3) are incompatible!

$\Rightarrow$ no intersection. $\Rightarrow$ L_1, L_2 are skew

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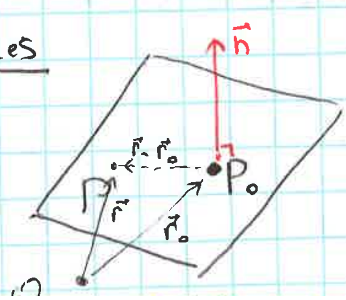
Planes



a plane is defined by a point P_0 and an orthogonal vector $\vec{n}$ ("the normal vector")

then for P any point on the plane,

$\vec{P_0P} \cdot \vec{n} = 0$, or

$\boxed{\vec{n}\cdot(\vec{r}-\vec{r}_0) = 0}$, or $\boxed{\vec{n}\cdot\vec{r} = \vec{n}\cdot\vec{r}_0}$ - vector eq. of the plane

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if $\vec{n} = \langle a, b, c\rangle$, $\vec{r} = \langle x, y, z\rangle$, $\vec{r}_0 = \langle x_0, y_0, z_0\rangle$, then eq. of the plane is:

$\langle a, b, c\rangle \cdot \langle x-x_0, y-y_0, z-z_0\rangle = 0$

or

$$\boxed{a(x-x_0) + b(y-y_0) + c(z-z_0) = 0} \quad (**)$$

– scalar eq of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c\rangle$

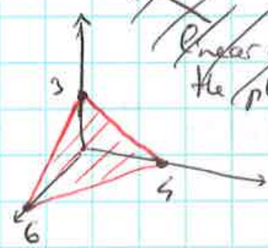
Ex: find an eq. of the plane through $(2, 4, -1)$, with $\vec{n} = \langle 2, 3, 4\rangle$ normal vector; find intercepts.

Sol: $2(x-2) + 3(y-4) + 4(z+1) = 0$ or $2x + 3y + 4z = 12$ $(*)$ ← linear eq. of the plane

x-intercept: set $y = z = 0$ in $(*)$ $\rightsquigarrow 2x = 12 \rightarrow x = 6$

y-intercept: $x = z = 0 \rightsquigarrow 3y = 12 \rightarrow y = 4$

z-intercept: $x = y = 0 \rightsquigarrow 4z = 12 \rightsquigarrow z = 3$



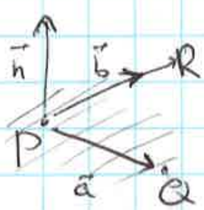
Eq. $(**)$ in the form $ax + by + cz + d = 0$ – linear equation of the plane.

$d = -(ax_0 + by_0 + cz_0)$

Ex: $P(-1, 2, 5)$, $Q(0, 4, 8)$, $R(1, 2, 0)$. Find an eq. of the plane through P, Q, R.

Sol: $\vec{a} = \overrightarrow{PQ} = \langle 1, 2, 3\rangle$, $\vec{b} = \overrightarrow{PR} = \langle 2, 0, -5\rangle$

normal vector: $\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 0 & -5 \end{vmatrix} = \langle -10, 11, -4\rangle$



$\rightsquigarrow$ eq. of the plane with $P_0 = P$: $\boxed{-10(x+1) + 11(y-2) - 4(z-5) = 0}$

or $\boxed{-10x + 11y - 4z - 12 = 0}$ $(*)$

Ex find the intersection point of plane with the line $x = -t$, $y = t - 1$, $z = 6t - 2$. $(**)$

Sol: $(*) \rightsquigarrow$ plugging in the param eq of the line

$-10(-t) + 11(t-1) - 4(6t-2) - 12 = 0$

$-3t - 3 - 12 = -3t - 15$ So: $t = -5$

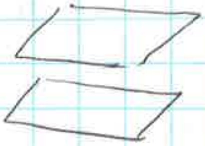
$(**) \xrightarrow{t=-5}$ $(x, y, z) = \cancel{\langle -8, 4, 28\rangle}$ $\boxed{(5, -6, -32)}$ – intersection point

Check: $\underbrace{-10(5) + 11(-6)}_{-116} \underbrace{-4(-32)}_{+128} - 12 = 0$ ✓

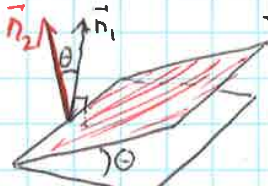
- Two planes are parallel if their normal vectors are parallel

Ex: $(*)$ and $20x - 22y + 8z + 1 = 0$ are parallel

$\vec{n}_1 = \langle -10, 11, -4\rangle$ $\vec{n}_2 = \langle 20, -22, 8\rangle$



parallel planes



two non-parall. planes intersect in a line; angle between planes = angle between normal vectors ~~θ_1, θ_2~~ $\vec n_1, \vec n_2$

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Ex: two planes $x+y+2z=0$ ①, $-y+z=1$ ②

(a) find the angle θ between them

Sol: $\vec n_1 = \langle 1,1,2\rangle$ $\vec n_2 = \langle 0,-1,+1\rangle$ $\cos\theta = \dfrac{\vec n_1 \cdot \vec n_2}{|\vec n_1|\,|\vec n_2|} = \dfrac{+1}{\sqrt6\,\sqrt2} = +\dfrac{1}{\sqrt{12}}$

$\theta = \arccos\left(+\frac{1}{\sqrt{12}}\right) \approx \cancel{1.86}\ 1.28$

$\approx \cancel{107^\circ}\ 73^\circ$

(b) find the symmetric equations of the intersection line L

need a point P_0 on L e.g. set $z=0$ ① $\to x+y=0$, ② $\to -y=1$ $\Rightarrow$ $x=+1$, $y=-1$ $\to P_0(+1,-1,0)$

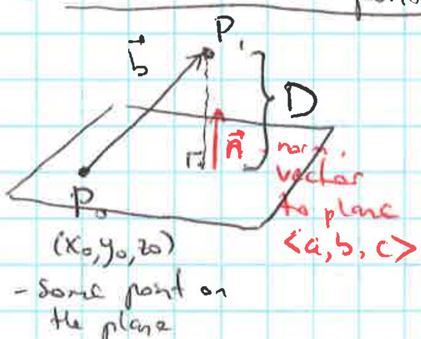
direction vector $\vec v$ - perp. to $\vec n_1$ and $\vec n_2$

- take $\vec v = \vec n_1 \times \vec n_2 = \begin{vmatrix} \vec i & \vec j & \vec k \\ 1 & 1 & 2 \\ 0 & -1 & +1 \end{vmatrix} = +3\vec i - \vec j - \vec k = \langle +3,-1,-1\rangle$

Sym. equations for L: $\boxed{\dfrac{x-1}{+3} = \dfrac{y+1}{-1} = \dfrac{z}{-1}}$

Note: we can obtain same line as intersection of planes $\dfrac{x-1}{+3} = \dfrac{y+1}{-1}$ and $\dfrac{y+1}{-1} = \dfrac{z}{-1}$

- Distance from a point $P_1(x_1,y_1,z_1)$ to a plane



distance $D = |\mathrm{comp}_{\vec n} \underbrace{\vec{P_0P_1}}_{\vec b}| = \dfrac{|\vec n \cdot \vec b|}{|\vec n|} =$

$= \dfrac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}} = \boxed{\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}}$

if $ax+by+cz=d$ - the eq. of the plane

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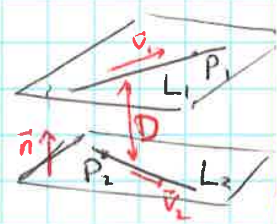
Ex find the distance from the point $(1,1,1)$ to the plane $x+y+2z=0$

Sol: $D = \dfrac{|1\cdot1+1\cdot1+2\cdot1|}{\sqrt{1^2+1^2+2^2}} = \boxed{\dfrac{4}{\sqrt6}}$

optional

* **Ex[le]** skew lines $L_1: x=1+t,\ y=-2+3t,\ z=4-t$ find distance between L_1, L_2

$L_2: x=2s,\ y=3+s,\ z=-3+4s$



common normal vector to the two parall planes: $\vec n = \underbrace{\langle 1,3,-1\rangle}_{\vec v_1} \times \underbrace{\langle 2,1,4\rangle}_{\vec v_2} = \langle 13,-6,-5\rangle$

$s=0 \to P_2=(0,3,-3) \to$ plane is: $13x-6(y-3)-5(z+3)=0$ or $13x-6y-5z+3=0$

$t=0 \to P_1=(1,-2,4) \to D = \dfrac{|13\cdot1-6(-2)-5\cdot4+3|}{\sqrt{13^2+(-6)^2+(-5)^2}} = \dfrac{8}{\sqrt{230}} \approx 0.53$