13.4

# Motion in space: velocity & acceleration

9/10/2018

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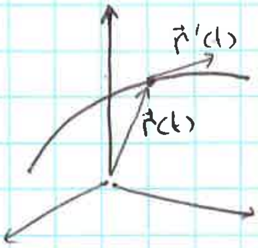


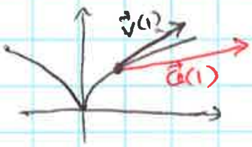

particle moving through space; position at time $t$ - $\vec r(t)$

$$\vec r\,'(t) = \underbrace{\lim_{h\to 0} \frac{\vec r(t+h) - \vec r(t)}{h}}_{\text{displacement per unit time}} = \underline{\text{velocity vector}}\quad \vec v(t) \text{ at time } t$$

<u>speed</u>: $|\vec v(t)| = |\vec r\,'(t)| = \frac{ds}{dt}$ ← arc length — rate of change of distance w.r.t. time

<u>acceleration</u>: $\vec a(t) = \vec v\,'(t) = \vec r\,''(t)$

<u>Ex</u>: On $\mathbb{R}^2$: $\vec r(t) = t^3 \vec i + t^2 \vec j$ find $\vec v(t)$, $\vec a(t)$ and $\underset{\text{speed}}{|\vec v(t)|}$ at $t=1$.




<u>Sol</u>: $\vec v(t) = \vec r\,'(t) = 3t^2 \vec i + 2t \vec j$

$\vec a(t) = \vec r\,''(t) = 6t \vec i + 2 \vec j$

$|\vec v(t)| = \sqrt{9t^4 + 4t^2}$

at $t=1$: $\vec v(1) = 3\vec i + 2\vec j$, $\vec a(1) = 6\vec i + 2\vec j$, $|\vec v(1)| = \sqrt{3^2+2^2} = \sqrt{13}$

<u>Ex</u>: moving particle starts at $\vec r(0) = \langle 1,0,0\rangle$ with initial velocity $\vec v(0) = \vec i - \vec j + \vec k$, $\vec a(t) = 4t\,\vec i + 6t\,\vec j + \vec k$. Find $\vec r(t)$, $\vec v(t)$

<u>Sol</u>: ① $\vec a(t) = \vec v\,'(t) \rightsquigarrow \vec v(t) = \int \vec a(t)\,dt = 2t^2 \vec i + 3t^2 \vec j + t\vec k + \underbrace{\vec C}_{\text{int. constant}}$

①' at $t=0$: $\vec v(0) = \vec i - \vec j + \vec k = \vec C \rightsquigarrow \boxed{\vec v(t) = (2t^2+1)\vec i + (3t^2-1)\vec j + (t+1)\vec k}$

② $\vec v(t) = \vec r\,'(t) \rightsquigarrow \vec r(t) = \int \vec v(t)\,dt = \left(\frac{2}{3}t^3 + t\right)\vec i + (t^3 - t)\vec j + \left(\frac{t^2}{2} + t\right)\vec k + \underbrace{\vec D}_{\text{constant}}$

②' at $t=0$: $\vec r(0) = \vec i = \vec D \rightsquigarrow \boxed{\vec r(t) = \left(\frac{2}{3}t^3 + t + 1\right)\vec i + (t^3 - t)\vec j + \left(\frac{t^2}{2} + t\right)\vec k}$

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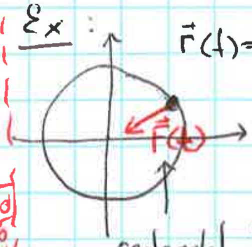
<u>Generally</u>: If we know $\vec a(t)$ and $\underbrace{\vec r(t_0), \vec v(t_0)}_{\text{initial position, velocity}}$, then: $\vec v(t) = \int_{t_0}^{t} \vec a(u)\,du + \vec v(t_0)$, $\vec r(t) = \int_{t_0}^{t} \vec v(u)\,du + \vec r(t_0)$

- if force $\vec F(t)$ acting on particle is known, then we know $\vec a(t)$ from $\boxed{\vec F(t) = m\,\vec a(t)}$ - Newton's 2nd law

<u>Ex</u>: $\vec r(t) = a\cos\omega t\,\vec i + a\sin\omega t\,\vec j$ - object of mass $m$ moves in a circular path with angular speed $\omega$. Find $\vec F(t)$

optional




<u>Sol</u>: $\vec v(t) = \vec r\,'(t) = -a\omega\sin\omega t\,\vec i + a\omega\cos\omega t\,\vec j$

$\vec a(t) = \vec v\,'(t) = -a\omega^2 \sin\omega t\,\vec i - a\omega^2\cos\omega t\,\vec j$

Thus: $\vec F(t) = m\,\vec a(t) = -m\omega^2(a\sin\omega t\,\vec i + a\cos\omega t\,\vec j) = -m\omega^2\,\vec r(t)$

<u>centripetal</u> force - points towards the origin

Projectile motion

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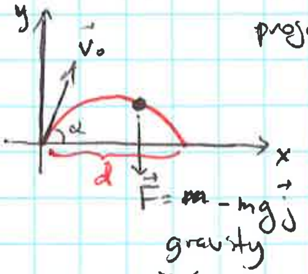



$g = 9.8\, m/s^2$

projectile is fired at angle of elevation $\alpha$, init. velocity $\vec{v}_0$

find $\vec{r}(t)$, what value of $\alpha$ maximizes range $d$? (no air resistance)

Sol: ① $\vec{F} = m\vec{a} = -mg\vec{j} \rightsquigarrow \vec{a} = -g\vec{j}$

② $\rightsquigarrow \vec{v}\,'(t) = -g\vec{j} \xrightarrow{\int} \vec{v}(t) = -gt\vec{j} + \vec{C}$

②' $\vec{v}(0) = \vec{v}_0 = v_0\cos\alpha\, \vec{i} + v_0\sin\alpha\, \vec{j} = \vec{C} \rightsquigarrow \vec{v}(t) = -gt\vec{j} + \vec{v}_0$

$= v_0\cos\alpha\,\vec{i} + (v_0\sin\alpha - gt)\vec{j}$

③ $\vec{r}\,'(t) = \underbrace{-gt\vec{j} + \vec{v}_0}_{\vec{v}(t)} \overset{\int}{\rightsquigarrow} \vec{r}(t) = -\frac{gt^2}{2}\vec{j} + \vec{v}_0 t + \vec{D}$

③' $\vec{r}(0) = 0 = \vec{D} \rightsquigarrow \vec{r}(t) = -\frac{gt^2}{2}\vec{j} + \vec{v}_0 t = v_0\cos\alpha\, t\,\vec{i} + (v_0\sin\alpha\, t - \frac{gt^2}{2})\vec{j}$

④ param. eq. of trajectory: $x = (v_0\cos\alpha)t$

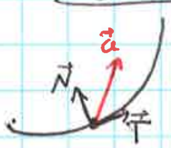
$y = (v_0\sin\alpha)t - \frac{gt^2}{2}$

$d$ = value of $x$ when $y = 0$. $\rightsquigarrow$ $t = \frac{2}{g}v_0\sin\alpha$ $\rightarrow d = x = \frac{2}{g}v_0^2\sin\alpha\cos\alpha = \frac{v_0^2}{g}\sin 2\alpha$

$y = 0$ if
or
$t = 0$

So, $d$ is maximal if $2\alpha = \frac{\pi}{2}$ or $\alpha = \frac{\pi}{4}$.

tangential and normal components of acceleration




$\vec{a} = \underbrace{a_T}_{\text{tangential component}}\vec{T} + \underbrace{a_N}_{\text{normal component}}\vec{N}$

$a_T = \vec{a}\cdot\underbrace{\vec{T}}_{\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}} = \frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{|\vec{r}\,'(t)|}$

$a_N = \vec{a}\cdot\vec{N} = \boxed{\frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|}}$

Ex: $\vec{r}(t) = \langle t^2, t^2, t^3\rangle$ find $a_T$, $a_N$

Sol: $\vec{r}\,'(t) = \langle 2t, 2t, 3t^2\rangle$, $\vec{r}\,''(t) = \langle 2, 2, 6t\rangle$, $|\vec{r}\,'(t)| = \sqrt{(2t)^2 + (2t)^2 + (3t^2)^2}$

$= \sqrt{8t^2 + 9t^4}$

$a_T = \frac{\vec{r}\,'(t)\cdot\vec{r}\,''(t)}{|\vec{r}\,'(t)|} = \frac{8t + 18t^3}{\sqrt{8t^2 + 9t^4}}$

$\vec{r}\,'(t)\times\vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2t & 2t & 3t^2 \\ 2 & 2 & 6t \end{vmatrix} = 6t^2\vec{i} - 6t^2\vec{j} \rightsquigarrow a_N = \frac{|\vec{r}\,'\times\vec{r}\,''|}{|\vec{r}\,'|} = \frac{6t^2\sqrt{2}}{\sqrt{8t^2+9t^4}}$