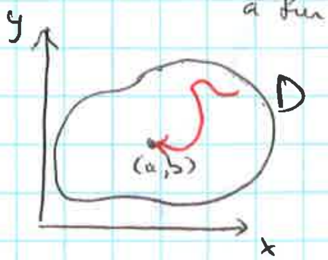
14.2 Limits, continuity

$\lim_{(x,y)\to(a,b)} \underbrace{f(x,y)}_{\text{a fun. of 2 vars}} = L$ · if values of $f(x,y)$ approach L as (x,y) approaches (a,b) along any path in the domain D

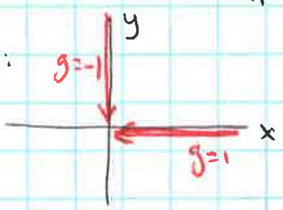
· or: can make $f(x,y)$ as close to L as we want by taking (x,y) sufficiently close to (a,b)



Ex 1: $f(x,y) = \frac{\sin(x^2+y^2)}{x^2+y^2}$ ~~Pr~~ ~~Sin~~ $\lim_{(x,y)\to(0,0)} f(x,y) = 1$

Ex 2 $g(x,y) = \frac{x^2-y^2}{x^2+y^2}$ $\lim_{(x,y)\to(0,0)} g(x,y) = ?$

Sol:



approaching along x-axis, ~~limit is~~ g stays 1
approaching along y-axis, g stays −1
$\Rightarrow$ $\lim_{(x,y)\to(0,0)}$ does not exist!

· $f(x,y)$ is <u>continuous</u> at (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

f is cont. on D if f is cont. ~~for all point~~ at each point (a,b) in D.

Ex: a <u>polynomial</u> in two vars $f(x,y) = x^2 + xy + y^5$ is cont. on $\mathbb{R}^2$

a <u>rational function</u> $g(x) = \frac{x+y^2}{x^2+y^2}$ - cont. in <u>its domain</u> ~~$(\mathbb{R}^2 \setminus \{(0,0)\})$~~ $\mathbb{R}^2 \setminus \{(0,0)\}$

14.3 Partial derivatives

$f(x,y)$ partial derivative w.r.t. x at (a,b): set $g(x) = f(\underset{\text{variable}}{x}, \underbrace{b}_{\text{constant}})$, — fun. of single variable

$f_x(a,b) = g'(a)$

likewise, part. der. wrt. y: set $h(y) = \cancel{g}f(a,y)$.

$f_y(a,b) = h'(b)$.

I.e. to find $f_x(x,y)$, view y as constant, differentiate wrt. x

— $f_y(x,y)$ — x ——— y

Ex $f(x,y) = x^3 + x^2y^3 - 2y^2$. Find $f_x(2,1)$, $f_y(2,1)$

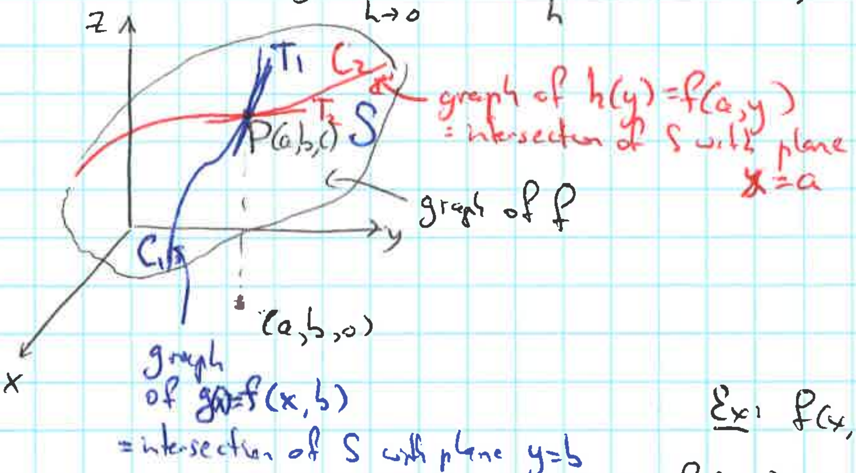
Sol $f_x(x,y) = 3x^2 + 2xy^3$ $f_x(2,1) = 3\cdot 2^2 + 2\cdot 2\cdot 1^3 = 16$

$f_y(x,y) = 3x^2y^2 - 4y$ $f_y(2,1) = 3\cdot 2^2\cdot 1^2 - 4\cdot 1 = 8$

$\dfrac{9/13/2018}{2}$

Notation: $\dfrac{\partial f}{\partial x} = f_x(x,y)$, $\dfrac{\partial f}{\partial y} = f_y(x,y)$

def. $f_x(x,y) = \lim\limits_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$, $f_y(x,y) = \lim\limits_{h\to 0} \dfrac{f(x,y+h) - f(x,y)}{h}$

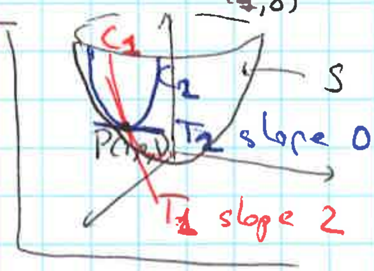


So: $f_x(a,b)$ = slope of tangent line T_1 to C_1 at P,

$f_y(a,b)$ = slope of tan. line T_2 to C_2 at P.

Ex: $f(x,y) = x^2 + y^2$

$f_x(1,0) = 2x\big|_{(1,0)} = 2$ $\quad$ $f_y(1,0) = +2y\big|_{(1,0)} = 0$



Ex: $f(x,y) = \sin\left(\dfrac{x}{1+y}\right)$ find f_x, f_y

Sol: $\dfrac{\partial f}{\partial x} = \cos\left(\dfrac{x}{1+y}\right)\cdot\dfrac{\partial}{\partial x}\left(\dfrac{x}{1+y}\right) = \cos\left(\dfrac{x}{1+y}\right)\cdot\dfrac{1}{1+y}$

Chain Rule!

$\dfrac{\partial f}{\partial y} = \cos\left(\dfrac{x}{1+y}\right)\cdot\dfrac{\partial}{\partial y}\left(\dfrac{x}{1+y}\right) = -\cos\left(\dfrac{x}{1+y}\right)\dfrac{x}{(1+y)^2}$

Higher derivatives $\quad f_{xx} = (f_x)_x = \dfrac{\partial^2 f}{\partial x^2}$, $f_{xy} = (f_x)_y = \dfrac{\partial^2 f}{\partial x \partial y}$, f_{yx} , f_{yy} ← 2nd order derivatives

Ex: $f(x,y) = x^3 + x^2y^3 - 2y^2$

$f_x = 3x^2 + 2xy^3$ $\quad \rightsquigarrow \quad$ $f_{xx} = 6x + 2y^3$ $\quad$ $f_{xy} = 6xy^2$

$f_y = 3x^2y^2 - 4y$ $\quad$ $f_{yx} = 6xy^2$ $\quad$ $f_{yy} = 6x^2y - 4$

Note: $f_{xy} = f_{yx}$ – this is true for any f! – doesn't matter, in which order we are taking partial derivatives

Ex: z defined implicitly by $x^3 + y^3 + z^3 + 6xyz = 1$ (*) find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$

[Ex 5 from K.S]

Sol: apply $\dfrac{\partial}{\partial x}$ to (*), keeping y constant: $3x^2 + 3z^2\dfrac{\partial z}{\partial x} + 6xy\dfrac{\partial z}{\partial x} + 6yz = 0$

$\rightsquigarrow \dfrac{\partial z}{\partial x} = -\dfrac{x^2}{z^2+2y}$

$\boxed{\dfrac{\partial z}{\partial x} = -\dfrac{x^2+2yz}{z^2+2xy}}$

applying $\dfrac{\partial}{\partial y}$ to (*) keeping x const: $3y^2 + 3z^2\dfrac{\partial z}{\partial y} + 6xy\dfrac{\partial z}{\partial y} + 6xz = 0$

$\rightsquigarrow \boxed{\dfrac{\partial z}{\partial y} = -\dfrac{y^2+xz}{z^2+2xy}}$