Higher partial derivatives $\qquad$ recall: $\boxed{f_{xy}(a,b) = f_{yx}(a,b)}$ – Clairaut's theorem

9/19/2018
1

Ex: $f(x,y,z) = \sin(3x+yz)$. Calculate $f_{xxyz} = \frac{\partial^4 f}{\partial x^2 \partial y \partial z}$

Sol $f_x = 3\cos(3x+yz)$

$f_{xx} = -9\sin(3x+yz)$

$f_{xxy} = -9z\cos(3x+yz)$

$f_{xxyz} = -9\cos(3x+yz) + 9zy\sin(3x+yz)$

Laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (*) – partial differential equation

Ex: check that $u(x,y) = e^x \sin y$ is a solution

$u_x = e^x \sin y \rightsquigarrow u_{xx} = e^x \sin y$

$u_y = e^x \cos y \rightsquigarrow u_{yy} = -e^x \sin y$

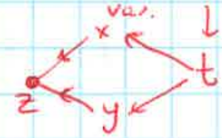
So: $\boxed{u_{xx} + u_{yy} = 0}$

Chain rule reminder

Case 0
$y = f(x)$, $x = g(t)$ then y indirectly a function of t

and $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = f'(x)\, g'(t) = f'(g(t))\, g'(t)$

intermed. var. $\qquad$ indep var



Case 1: $z = f(x,y)$, $x = g(t)$, $y = h(t)$

then $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ $\qquad$ or: $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

Ex: $z = \sqrt{1+xy}$, $x = \sin t$, $y = t^2$ $\quad$ Find $\frac{dz}{dt}$

Sol: $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{y}{2\sqrt{1+xy}}\cos t + \frac{x}{2\sqrt{1+xy}}2t$ (*)

$= \frac{\frac{t^2}{2}\cos t + t\sin t}{\sqrt{1+t^2\sin t}}$

substitute f-las for x, y in terms of t.

- find $\frac{dz}{dt}$ at $t=\pi$ $\qquad$ Sol: $t=\pi \to x=0, y=\pi^2$ $\qquad$ $(*)\big|_{t=\pi} = -\frac{\pi^2}{2}$

optional

Ex: $z = xy^3 - x^2y$, $x = t^2+1$, $y = t^2-1$ $\quad$ find $\frac{dz}{dt}$

Sol $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (y^3-2xy)2t + (3xy^2-x^2)2t$

Case 2 $\quad z = f(x,y)$ and $\begin{array}{l} x = g(s,t) \\ y = h(s,t) \end{array}$

then $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$ and $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$

Ex $\quad z = (x-y)^5 \quad x = s^2 t \quad y = st^2 \quad$ find $\dfrac{\partial z}{\partial s}, \dfrac{\partial z}{\partial t}$

Sol: $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s} = 5(x-y)^4\, 2st - 5(x-y)^4\, t^2 = 5t(s^2t - st^2)^4(2s-t)$

$\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t} = 5(x-y)^4 s^2 - 5(x-y)^4\, 2st = 5s(s^2t - st^2)^4(s-2t)$

General case of chain rule $\quad u = u(\underbrace{x_1 \cdots x_n}_{n \text{ variables}})$ and for each j, $x_j = x_j(\underbrace{t_1 \cdots t_m}_{m \text{ variables}})$

then $\dfrac{\partial u}{\partial t_i} = \underbrace{\dfrac{\partial u}{\partial x_1}\dfrac{\partial x_1}{\partial t_i} + \cdots + \dfrac{\partial u}{\partial x_n}\dfrac{\partial x_n}{\partial t_i}}_{n \text{ terms}}$

Ex: $u = x + yz, \quad x = t, \quad y = \ln t, \quad z = \sin t$

$\dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial u}{\partial z}\dfrac{dz}{dt} = 1 \cdot 1 + z\dfrac{1}{t} + y\cos t = \boxed{1 + \dfrac{\sin t}{t} + \ln t \cos t}$

Implicit differentiation $\quad$ if $y = f(x)$ is defined implicitly, by $F(x, \underset{=f(x)}{y}) = 0 \quad (*)$

can find $\dfrac{dy}{dx}$ via $0 = \dfrac{d}{dx} F(x,y) \underset{\text{chain rule}}{=} \dfrac{\partial F}{\partial x}\underbrace{\dfrac{dx}{dx}}_{1} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = -\dfrac{\partial F/\partial x}{\partial F/\partial y} = -\dfrac{F_x}{F_y}$

Ex: y defined by $x^2 + y^3 = 1$; find y'.

Sol: $F = x^2 + y^3 - 1 \quad \dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{2x}{3y^2}$

Ex: $z = z(x,y)$ defined by $x^2 + y^5 + z^3 = 0 \quad (*)$ find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

Sol: $\dfrac{\partial}{\partial x}(*)$: $\quad 2x + 0 + 3z^2\dfrac{\partial z}{\partial x} = 0 \quad \rightsquigarrow \dfrac{\partial z}{\partial x} = -\dfrac{2x}{3z^2}$

$\dfrac{\partial}{\partial y}(*)$: $\quad 0 + 5y^4 + 3z^2\dfrac{\partial z}{\partial y} = 0 \quad \rightsquigarrow \dfrac{\partial z}{\partial y} = -\dfrac{5y^4}{3z^2}$