14.6 Directional derivatives, gradient

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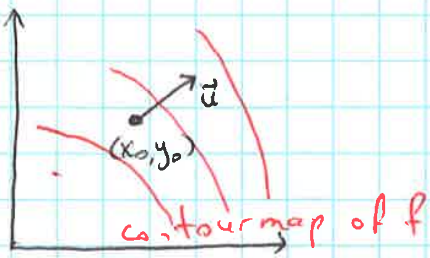
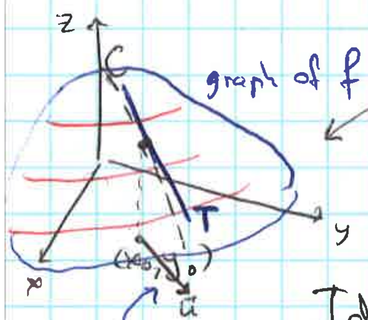
$f(x,y)$, $\vec{u} = \langle a, b \rangle$ - unit vector

$$D_{\vec{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$

- directional derivative of f at (x_0,y_0), in the direction of $\vec{u}$.



- rate of change of f in the direction of $\vec{u}$.
= slope of tangent line T to the curve C



Note for $\vec{u} = \langle 1, 0 \rangle$, $D_{\vec{u}} f = f_x$

for $\vec{u} = \langle 0, 1 \rangle$, $D_{\vec{u}} f = f_y$

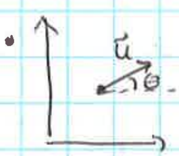
for $\vec{u} = \langle a, b \rangle$,

- $$\boxed{D_{\vec{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b}$$

Idea: set $g(h) = f(x_0 + ha, y_0 + hb)$ - value of f on the line

Then $D_{\vec{u}} f(x_0,y_0) = g'(h=0) \underset{\text{Chain rule}}{=} \frac{\partial f}{\partial x}\frac{dx}{dh} + \frac{\partial f}{\partial y}\frac{dy}{dh}\Big|_{h=0} = f_x(x,y)a + f_y(x,y)b\Big|_{h=0}$

$= f_x(x_0,y_0)a + f_y(x_0,y_0)b$ ☐

- 

unit vector: $\vec{u} = \langle \cos\theta, \sin\theta \rangle$ (θ: angle with x-axis)

So: $$\boxed{D_{\vec{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta}$$

Ex: $f(x,y) = x^3 - 3xy + 4y^2$ $\quad \vec{u}$ unit vector given by $\theta = \pi/6$ $\quad$ Find $D_{\vec{u}} f(1,2)$

Sol: $\vec{u} = \langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6} \rangle = \langle \frac{\sqrt{3}}{2}, \frac{1}{2} \rangle$

$D_{\vec{u}} f = f_x \cdot \frac{\sqrt{3}}{2} + f_y \cdot \frac{1}{2} = (3x^2 - 3y)\frac{\sqrt{3}}{2} + (-3x + 8y)\cdot\frac{1}{2}$

$D_{\vec{u}} f(1,2) = (3 - 3\cdot 2)\frac{\sqrt{3}}{2} + (-3 + 16)\cdot\frac{1}{2} = -\frac{3}{2}\sqrt{3} + \frac{13}{2}$.

Gradient

$D_{\vec{u}} f = f_x(x,y)\,a + f_y(x,y)\,b = \underbrace{\langle f_x(x,y), f_y(x,y) \rangle}_{\text{grad } f \text{ or } \nabla f} \cdot \underbrace{\langle a, b \rangle}_{\vec{u}}$

- gradient of f at (x,y)

$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$ - vector function of x,y

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Ex: $f(x,y) = \sin x + e^{xy}$ then $\nabla f(x,y) = \langle f_x, f_y\rangle = \langle \cos x + y e^{xy}, x e^{xy}\rangle$

$\nabla f(0,1) = \langle 2, 0\rangle$

we have: $\boxed{D_{\vec u} f = \underbrace{\nabla f \cdot \vec u}_{\text{scalar proj. of } \nabla f \text{ onto } \vec u}}$

Ex: $f(x,y) = x^2y^3 - 4y$. Find directional derivative at $(2,-1)$ in the direction of $\vec v = 2\vec i + 5\vec j$.

Sol ① $\nabla f = \langle f_x, f_y\rangle = \langle 2xy^3, 3x^2y^2 - 4\rangle$; $\nabla f(2,-1) = \langle -4, 8\rangle$

② $\vec u = \frac{\vec v}{|\vec v|} = \frac{1}{\sqrt{29}}\langle 2,5\rangle$ - unit vector in dir. of $\vec v$. So:

$D_{\vec u} f = \langle -4, 8\rangle \cdot \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\rangle = \frac{32}{\sqrt{29}}$

3D example $f(\underbrace{x,y,z}_{\text{3 variables!}}) = x \sin yz$

a) find gradient $\nabla f = \langle f_x, f_y, f_z\rangle$ ~~at (1,3,0)~~

b) find directional derivative of f at $(1,3,0)$ in the direction of $\vec v = \langle 1,2,-1\rangle$

Sol: a) $\nabla f = \langle f_x, f_y, f_z\rangle = \langle \sin yz, zx\cos yz, yx\cos yz\rangle$

b) $\nabla f(1,3,0) = \langle 0, 0, 3\rangle$; $\vec u = \frac{\vec v}{|\vec v|} = \frac{1}{\sqrt 6}\langle 1,2,-1\rangle$ unit vector along $\vec v$.

So: $D_{\vec u} f(1,3,0) = \nabla f \cdot \vec u = \frac{1}{\sqrt 6}\langle 0,0,3\rangle\langle 1,2,-1\rangle = -\frac{3}{\sqrt 6}$

In which direction does $f(x,y)$ (or $f(x,y,z)$) change fastest? what is maximum rate of change?

Answer: • maximum value of dir. derivative $D_{\vec u} f(x,y)$ is $|\nabla f(x,y)|$

• it occurs when $\vec u$ has same direction as $\nabla f(x,y)$!

Idea: $D_{\vec u} f(x,y) = \nabla f(x,y) \cdot \vec u = \cancel{|\nabla f(x,y)| \cos}\ |\nabla f| \underbrace{|\vec u|}_{1} \cos \underbrace{\theta}_{\text{angle between } \nabla f \text{ and } \vec u}$ ← maximized at $\theta = 0$

max value: $|\nabla f|$ ☒

Ex: $f = x^2 + y^2$ find max. rate of change and direction of max rate of change at $(x_0,y_0) = (3,4)$

Sol $\nabla f = \langle 2x, 2y\rangle$ $\nabla f(3,4) = \langle 6, 8\rangle$ $|\nabla f| = \sqrt{6^2+8^2} = 10$ - max rate of change

$\vec u = \frac{\langle 6,8\rangle}{10} = \langle \frac35, \frac45\rangle$ - direction of fastest change