14.6 cont'd Directional derivatives, gradient ... tangent planes, normal lines

9/24/2018
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Ex

$$T(x,y,z) = \frac{80}{1+x^2+2z^2}$$ temperature at (x,y,z)

At the point $(1,0,1)$, ⓐ in which direction the temperature increases fastest?

ⓑ what is the max. rate of increase?

Sol: $\nabla T = \langle T_x, T_y, T_z \rangle = \left\langle -\frac{80x}{(1+x^2+2z^2)^2}, 0, \frac{-160z}{(1+x^2+2z^2)^2} \right\rangle$

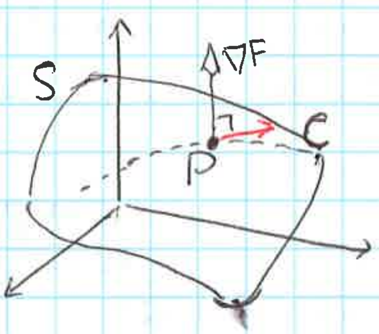
$$\nabla T(1,0,1) = \left\langle -\frac{80}{4^2}, 0, -\frac{160}{4^2} \right\rangle = \langle -5, 0, -10 \rangle$$

ⓐ ~~dir of max increase:~~ T increases fastest at $(1,0,1)$ in dir of $\langle -5,0,-10\rangle$, or unit vector $\frac{1}{\sqrt{5}}\langle -1,0,-2\rangle$

ⓑ max rate: $|\nabla T| = 5\sqrt{5}$ $(\approx 11.18°C/m)$

Tangent planes to level surfaces

$F(x,y,z)$, $S: F(x,y,z) = k$ - level surface $\qquad P(x_0,y_0,z_0)$ point on S



a curve C on S through P is given by $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$

such that: (1) $F(x(t), y(t), z(t)) = k$ $\quad(*)$

(2) $\vec{r}(t_0) = \langle x_0, y_0, z_0\rangle$

some value of parameter

$$\frac{d}{dt}(*): \quad 0 = \frac{dF}{dt} \underset{\text{chain rule}}{=} \underbrace{\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt}}$$

$$\langle F_x, F_y, F_z\rangle \cdot \langle x'(t), y'(t), z'(t)\rangle = \nabla F \cdot \vec{r}\,'(t)$$

In particular, for $t = t_0$:

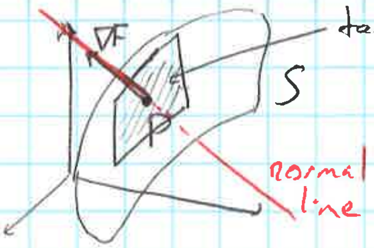
$$\nabla F(x_0,y_0,z_0) \cdot \underbrace{\vec{r}\,'(t_0)}_{\text{tangent vector to } C \text{ at } P} = 0$$

Thus: gradient vector at P, $\nabla F(x_0,y_0,z_0)$ is perpendicular to the tan. vector to any curve C on S passing through P.

So: if ~~$\nabla F \neq 0$~~ $\nabla F(x_0,y_0,z_0) \neq 0$,

then the tangent plane to level surface $F(x,y,z) = k$ at $P(x_0,y_0,z_0)$ is the plane through P with normal vector $\nabla F(x_0,y_0,z_0)$.

I.e. tangent plane: $\boxed{F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0}$



normal line to S at P

– line through P and perpendicular to tangent plane (i.e. its direction is $\nabla F(x_0,y_0,z_0)$):

$$\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}$$

Ex: S: $z=f(x,y)$ – graph of f or: $F(x,y,z)=f(x,y)-z=0$

$$\nabla F = \langle f_x(x_0,y_0), f_y(x_0,y_0), -1\rangle$$

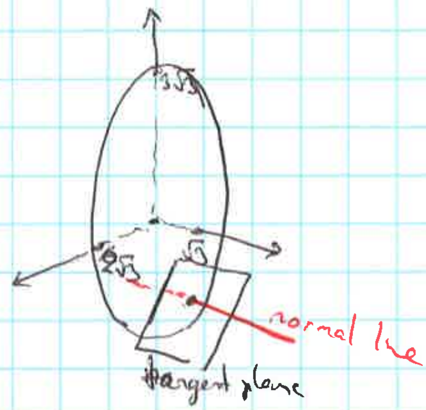
tangent plane: $\boxed{f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0}$ (*) f_x, f_y – "slopes" of the plane

Ex: $\underbrace{\frac{x^2}{4}+y^2+\frac{z^2}{9}}_{F(x,y,z)} = \underbrace{3}_{k}$ – ellipsoid   Find tangent plane, normal line at $P(-2,1,-3)$

Sol: $\nabla F = \langle \frac{x}{2}, 2y, \frac{2z}{9}\rangle$ $\nabla F(-2,1,-3) = \langle -1, 2, -\frac{2}{3}\rangle$

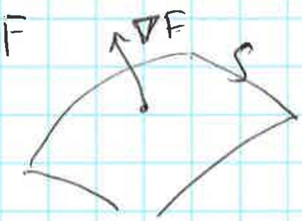
tan. plane: $\boxed{-(x+2)+2(y-1)-\frac{2}{3}(z+3)=0}$

normal line: $\boxed{\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-2/3}}$

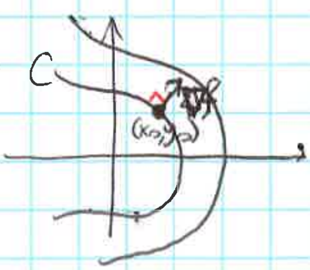


Gradient, Summary:

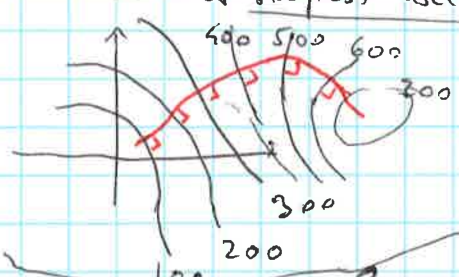
- for $F(x,y,z)$, $\nabla F(x_0,y_0,z_0)$ – direction of max increase of F
  – perp. to level surface S



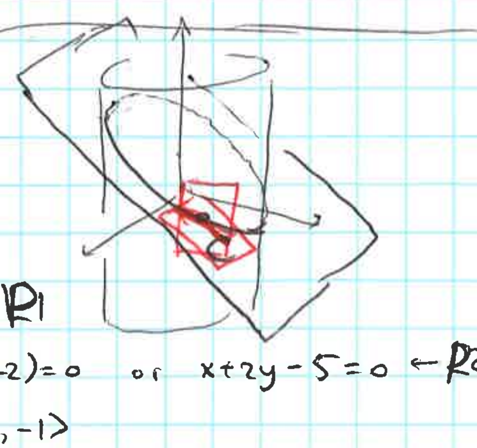
- for $f(x,y)$, $\nabla f(x_0,y_0)$ – dir. of max. increase of f
  – perp to level curves C



Curve of steepest ascent – a curve everywhere perpendicular to contour lines of f optional



Ex: find tangent line L to the intersection curve C of $y+z=3$, $x^2+y^2=5$ at $P(1,2,1)$



Sol: ① find tan plane to $y+z=3$: (*) $0\cdot(x-1)+1(y-2)+(z-1)=0$
or $y+z-3=0$ or $y+z-3=0 \leftarrow P1$

② tan. line to $x^2+y^2-5=0$: $2x_0(x-x_0)+2y_0(y-y_0)=0 \to 2(x-1)+4(y-2)=0$ or $x+2y-5=0 \leftarrow P2$

③ intersection of P1 and P2: $\langle 0,1,1\rangle \times \langle 1,2,0\rangle = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 1 & 2 & 0\end{vmatrix} = \langle -2, 1, -1\rangle$

So, L: $\boxed{-2(x-1)+(y-2)-(z-1)=0}$