14.7 cont'd: Maxima and minima on bounded regions.

9/28/2018

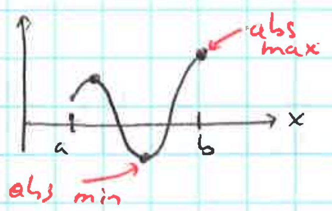
LAST TIME: (a,b) - crit. point of $f(x,y)$ if $\boxed{f_x(a,b)=0,\ f_y(a,b)=0}$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

2nd derivative test:

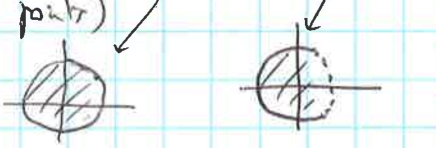
- $D>0,\ f_{xx}>0 \rightarrow$ $f(a,b)$ is a local maximum
- $D>0,\ f_{xx}<0 \rightarrow$ $f(a,b)$ is a local minimum
- $D<0 \rightarrow (a,b)$ - saddle point

Recall: $f(x)$ continuous on $[a,b]$ has an absolute maximum and an abs. minimum on $[a,b]$
- we find them by evaluating ~~[illegible]~~ f on crit. points and on endpoints.



D - closed set in $\mathbb{R}^2$ (i.e. contains boundary points)

bounded set - contained in some disk ("finite in extent")



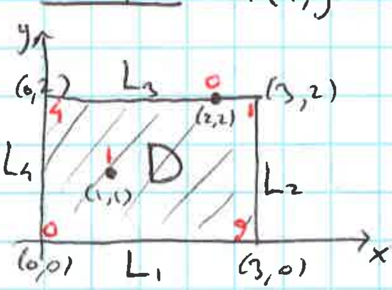
* f, continuous on a closed, bounded set $D \subset \mathbb{R}^2$, attains an absolute maximum value $f(x_1,y_1)$ and abs. min. value $f(x_2,y_2)$ at some points in D

$(x_1,y_1),(x_2,y_2)$ - either a critical pt. of f or a boundary point of D.

To find abs. max, min values of f on a closed, bounded D:

1. find values of f at crit. points in D
2. find extreme values ~~of~~ on the boundary of D.
3. largest of 1, 2 - abs. max.
   smallest of 1, 2 - abs. min.

Example $f(x,y) = x^2 - 2xy + 2y$, find abs. min & max values on rectangle: $D = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$



Sol: 1. $f_x = 2x-2y$, $f_y = -2x+2$

crit. points: $2x-2y=0 \rightarrow y=1$, $-2x+2=0 \rightarrow x=1$

$\rightarrow$ unique crit. pt $(1,1)$

$f(1,1) = 1^2 - 2\cdot1\cdot1 + 2\cdot1 = 1$ - crit. value

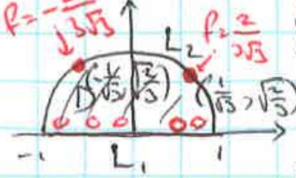
2. boundary: line segments L_1, L_2, L_3, L_4

- on L_1: $f(x,0) = x^2$ (increasing), $0 \le x \le 3$; max: $f(3,0)=9$, min: $f(0,0)=0$
- on L_2: $f(3,y) = 9-4y$ (decreasing), $0 \le y \le 2$; max: $f(3,0)=9$, min: $f(3,2)=1$
- on L_3: $f(x,2) = x^2-4x+4 = (x-2)^2$, $0 \le x \le 3$; min: $f(2,2)=0$, max: $f(0,2)=4$
- on L_4: $f(0,y) = 2y$ (increasing), $0 \le y \le 2$; max: $f(0,2)=4$, min: $f(0,0)=0$

Sol: abs. maximum: $f(3,0) = \boxed{9}$
abs. minimum: $f(0,0) = f(2,2) = \boxed{0}$

Ex: $f(x,y) = xy^2$ on $D = \{(x,y) \mid \begin{smallmatrix} x^2+y^2 \le 1 \\ y \ge 0 \end{smallmatrix}\}$ find abs max & min

$\dfrac{9/28/2018}{2}$



Sol ① $f_x = y^2$ $\quad f_x = 0 \rightsquigarrow y = 0$ ← no crit points in the interior of D

$f_y = 2xy$ $\quad f_y = 0 \rightsquigarrow x$ any $\qquad f(x,0) = 0$

② on $\underset{y=0,\ -1\le x\le 1}{L_1}$: $f = 0$

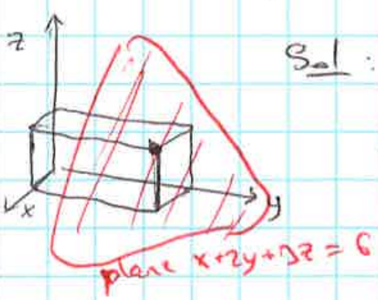
On L_2:

$x^2+y^2 = 1,\ y \ge 0 \rightsquigarrow y = \sqrt{1-x^2}$, $\underset{-1\le x\le 1}{f = \overset{=g(x)}{x(1-x^2)}} \rightsquigarrow$ $\frac{d}{dx} g(x) = 1 - 3x^2$ $\qquad g'(x) = 0$ iff $x = \pm\frac{1}{\sqrt{3}}$ - crit points of g

$g(\frac{1}{\sqrt{3}}) = \frac{2}{3\sqrt{3}}$, $g(-\frac{1}{\sqrt{3}}) = -\frac{2}{3\sqrt{3}}$. So: abs maximum: $f(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}) = \frac{2}{3\sqrt{3}}$

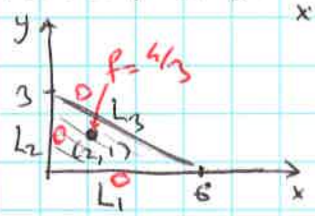
abs minimum: $f(-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}) = -\frac{2}{3\sqrt{3}}$

Ex: find volume of largest rectangular box in 1st octant with faces parall. to xy, xz, yz planes and one vertex ~~at~~ in the plane $x+2y+3z = 6$



Sol: $z = \dfrac{6-x-2y}{3}$, $f(x,y) = x \cdot y \cdot \dfrac{6-x-2y}{3}$,

maximize f in D: $x+2y \le 6$, $x \ge 0,\ y \ge 0$



① $f_x = y\dfrac{6-x-2y}{3} - \dfrac{1}{3}xy = y\dfrac{6-2x-2y}{3}$

$f_y = x\dfrac{6-x-2y}{3} - \dfrac{2}{3}xy = x\dfrac{6-x-4y}{3}$

$f_x = 0, f_y = 0$: $y = 0$ or $x = 0$ ← on bdry of D

or $\begin{cases} 6-2x-2y=0 \\ 6-x-4y=0 \end{cases} \Leftrightarrow \begin{matrix} 2x+2y=6 \\ x+4y=6 \end{matrix} \Leftrightarrow \begin{matrix} x=2 \\ y=1 \end{matrix}$

$f(2,1) = 2 \cdot 1 \cdot \dfrac{6-2-2}{3} = \dfrac{4}{3}$

② on L_1: $\begin{smallmatrix} y=0 \\ 0\le x\le 6 \end{smallmatrix}$, $f = 0$

on L_2: $\begin{smallmatrix} x=0 \\ 0\le y\le 3 \end{smallmatrix}$, $f = 0$

on L_3: $\begin{smallmatrix} 0\le x\le 6 \\ x+2y=6 \end{smallmatrix}$ $\quad y = \dfrac{6-x}{2}$, $f(x,y) = x \cdot \dfrac{6-x}{2} \cdot \dfrac{\overset{=0}{6-x-2\frac{6-x}{2}}}{3} = 0$

So: abs. maximum: $f(2,1) = \boxed{\frac{4}{3}}$. - box of size $\boxed{\underset{x}{2} \times \underset{y}{1} \times \underset{z}{\frac{2}{3}}}$

~~Ex: surface area of a rectangular box is 12. Find the largest volume~~

~~Sol: x, y, z - dimensions. surface area: $2xy + 2xz + 2yz = 12 \rightsquigarrow z = \frac{6-xy}{x+y}$~~

~~Volume $= xyz = xy\frac{6-xy}{x+y} = f(x,y)$ $\quad f_x = y\frac{6-xy}{x+y} + \frac{y(x+y)-(6-xy)}{(x+y)^2}xy$~~