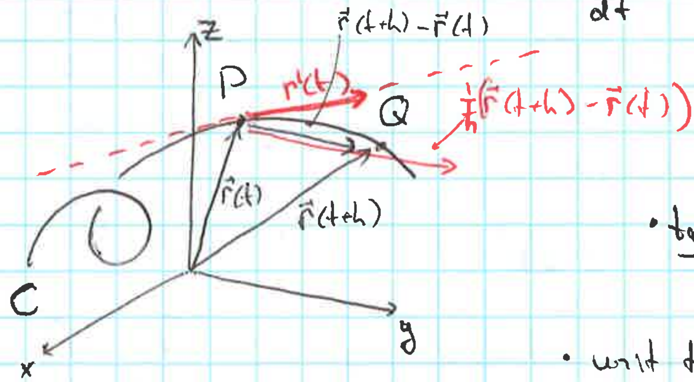
13.2 Derivatives, Integrals

$\vec r(t)$ vector function $\rightsquigarrow$ $\dfrac{d\vec r}{dt} = \vec r'(t) = \lim\limits_{h\to 0} \dfrac{\vec r(t+h)-\vec r(t)}{h}$



$\vec r'(t)$ – tangent vector to curve C at point P (if $\vec r'(t) \neq 0$ and exists)

• tangent line to C at P – line through P, parallel to $\vec r'(t)$

• unit tangent vector: $\vec T(t) = \dfrac{\vec r'(t)}{|\vec r'(t)|}$

* If $\vec r(t) = f(t)\,\vec i + g(t)\,\vec j + h(t)\,\vec k$

then $\vec r'(t) = f'(t)\,\vec i + g'(t)\,\vec j + h'(t)\,\vec k$

Ex $\vec r(t) = (t+t^3)\,\vec i + \cos 2t\,\vec j + te^{-t}\,\vec k$ (a) find $\vec r'(t)$

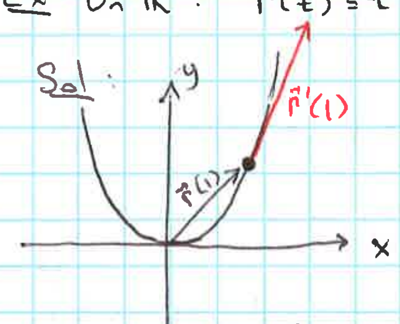
(b) find the unit tangent vector at the point with $t=0$

Sol (a) $\vec r'(t) = (1+3t^2)\,\vec i - 2\sin 2t\,\vec j + (1-t)e^{-t}\,\vec k$

(b) $\vec r(0) = \vec j$, $\vec r'(0) = \vec i + \vec k$, $\vec T(0) = \dfrac{\vec i + \vec k}{|\vec i + \vec k|} = \dfrac{1}{\sqrt 2}\vec i + \dfrac{1}{\sqrt 2}\vec k$

Ex On $\mathbb{R}^2$: $\vec r(t) = t\,\vec i + t^2\,\vec j$. Find $\vec r'(t)$; sketch $\vec r(1)$ and $\vec r'(1)$.

Sol:



$\vec r'(t) = \vec i + 2t\,\vec j$.

So: $\vec r(1) = \vec i + \vec j$ ← draw with tail at O

$\vec r'(1) = \vec i + 2\vec j$ ← draw with tail at (1,1)

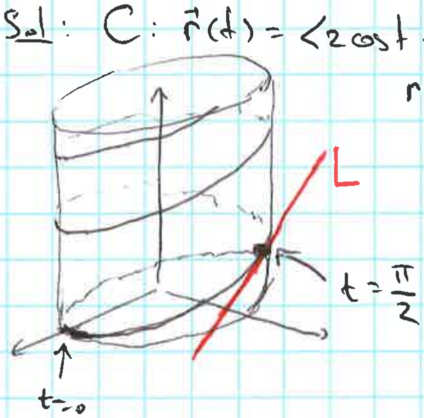
Ex helix C: $x = 2\cos t$, $y = \sin t$, $z = t$. Find eq. of the tangent line to C at point $(0, 1, \frac{\pi}{2})$.

↑ corresp to $t = \frac{\pi}{2}$

Sol: C: $\vec r(t) = \langle 2\cos t, \sin t, t\rangle$

$\vec r'(t) = \langle -2\sin t, \cos t, 1\rangle$

So: $\vec r'(\frac{\pi}{2}) = \langle -2, 0, 1\rangle$



So: L passes through $(0,1,\frac{\pi}{2})$ and is parall. to $\langle -2, 0, 1\rangle$

$\rightsquigarrow$ L: $x = -2t$, $y = 1$, $z = \frac{\pi}{2} + t$

Second derivative: $\vec{r}''(t) = (\vec{r}'(t))'$.

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Ex: $\vec{r}(t) = \langle 2\cos t, \sin t, t \rangle \rightsquigarrow \vec{r}'(t) = \langle -2\sin t, \cos t, 1 \rangle \rightsquigarrow \boxed{\vec{r}''(t) = \langle -2\cos t, -\sin t, 0 \rangle}$

<u>Differentiation rules:</u>

$(\vec{u}(t) + \vec{v}(t))' = \vec{u}'(t) + \vec{v}'(t)$

$(c\,\vec{u}(t))' = c\,\vec{u}'(t)$

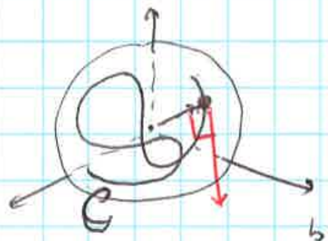
$(f(t)\,\vec{u}(t))' = f'(t)\,\vec{u}(t) + f(t)\,\vec{u}'(t)$

$(\vec{u}(t) \cdot \vec{v}(t))' = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$

$(\vec{u}(t) \times \vec{v}(t))' = \vec{u}(t)' \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$

and $(\vec{u}(f(t)))' = f'(t)\,\vec{u}'(f(t))$ - <u>chain rule</u>.

<u>Ex</u>: if $|\vec{r}(t)| = c$ - constant, then $\vec{r}'(t)$ is orthogonal to $r(t)$ for all t.



Indeed: $\underbrace{\frac{d}{dt}\big(\underbrace{\vec{r}(t) \cdot \vec{r}(t)}_{c^2}\big)}_{=\,0} = \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 2\,\vec{r}'(t) \cdot \vec{r}(t)$.

<u>Integrals</u>

$$\int_a^b \underbrace{\vec{r}(t)}_{\langle f, g, h \rangle} dt = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}$$

* Fund. Thm. of Calculus: (in vector setting)

$$\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big|_a^b = \vec{R}(b) - \vec{R}(a)$$

anti-derivative: $\vec{R}'(t) = \vec{r}(t)$

we denote $\vec{R}(t) = \int \vec{r}(t)\,dt$ (indefinite integral)

<u>Ex</u>: $\vec{r}(t) = 2\cos t\,\vec{i} + \sin t\,\vec{j} + 2t\,\vec{k}$

$\int \vec{r}(t)\,dt = 2\sin t\,\vec{i} - \cos t\,\vec{j} + t^2\,\vec{k} + \underbrace{\vec{C}}_{\text{vector constant of integration}}$

$$\int_0^{\pi/2} \vec{r}(t)\,dt = \Big|2\sin t\,\vec{i} - \cos t\,\vec{j} + t^2\,\vec{k}\Big|_0^{\pi/2} = 2\vec{i} + \vec{j} + \frac{\pi^2}{4}\vec{k}$$