Bundles and connections

Ref: • Madsen, Tornehave
• Milnor, Stasheff

def A fiber bundle is a triple of top spaces E, M, F and a continuous surjective map $\pi: E \to M$ such that:

$\forall p \in M$ $\exists$ open nbhd $\underset{p}{U} \subset M$ and a homeo $\varphi: \pi^{-1}(U) \to U \times F$ ← trivializing neighborhood

such that

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

commutes.

if $\{U_\alpha\}$ - a covering of M, $\{(U_\alpha, \varphi_\alpha)\}$ - local trivialization of the fiber bundle

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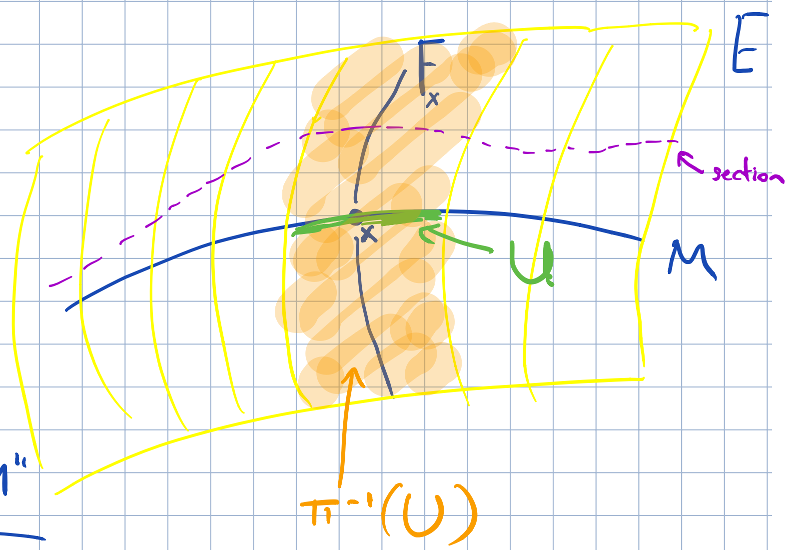
E - total space
M - base
F - typical fiber
$F_x = \pi^{-1}(x)$ - fiber over x
$\underset{\text{homeo}}{\approx} F$

So: a fiber bundle = "family of top spaces $F_x \approx F$ indexed by $x \in M$"



If E, M, F are smooth mfds, π - smooth map, φ's are diffeomorphisms, then we have a <u>smooth</u> fiber bundle.

Section: $s: M \to E$ s.t. $\pi \circ s = id_M$.

Ex: trivial bundle $E = M \times F$
$$\begin{array}{c} M \times F \\ \downarrow \pi = \mathrm{proj}_1 \\ M \end{array}$$

Ex: an n-fold covering space of M is a fiber bundle (with fiber a set with n elements)

Transition functions

Given a local triv. $\{(U_\alpha, \varphi_\alpha)\}$, on each overlap $U_\alpha \cap U_\beta$ one has

$$\varphi_\alpha \varphi_\beta^{-1}: (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$$
$$(x, \xi) \longmapsto (x, t_{\alpha\beta}(x)\xi)$$

where $t_{\alpha\beta}: U_\alpha \cap U_\beta \to G$ - transition function,
G - structure group acting on F by diffeomorphisms

One has:
① $t_{\alpha\alpha}(x) = 1$
② $t_{\alpha\beta}(x) = t_{\beta\alpha}(x)^{-1}$
③ $t_{\alpha\beta}(x) t_{\beta\gamma}(x) t_{\gamma\alpha}(x) = 1$ for $x \in U_\alpha \cap U_\beta \cap U_\gamma$

– and vice versa: given an atlas $\{U_\alpha\}$ on M and transition functions satisfying ①, ②, ③, one can glue a fiber bundle

$$E = \left(\coprod_\alpha U_\alpha \times F \right) \Big/ \text{ for each } x \in U_\alpha \cap U_\beta,\ \underset{\displaystyle U_\alpha \times F}{(x,\xi)_\alpha} \sim \underset{\displaystyle U_\beta \times F}{(x, t_{\beta\alpha}\xi)_\beta}$$

Def a vector bundle over $\Bbbk$ $(=\mathbb{R},\mathbb{C})$ is a fiber bundle with $F = V$ - $\Bbbk$ - vector space and transition functions $t_{\alpha\beta}$ - linear transformations of V (i.e. $G = GL(V)$)

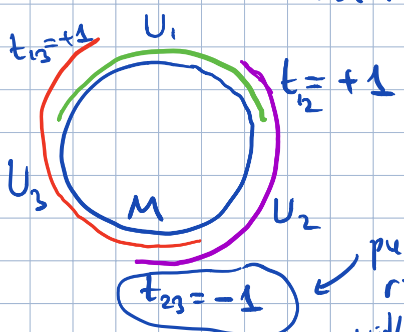
<put another way, F is linear, F_x's are linear and diffeos φ induce linear isos between F_x and F>

$\dim V$ = "rank" of the vector bundle

rank 1 vector bundles are also called "line bundles"

Ex: (Möbius band) $M = S^1$, $V = \mathbb{R}$

define E via transition functions:



purple and red ribbons are identified with a twist on the overlap

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A morphism of fiber bundles $\begin{array}{c} E \\ \downarrow \pi \\ M \end{array}$, $\begin{array}{c} E' \\ \downarrow \pi' \\ M' \end{array}$ is a pair of maps $\begin{array}{c} E \xrightarrow{\Phi} E', \\ M \xrightarrow{f} M' \end{array}$

such that the diagram $\begin{array}{ccc} E & \longrightarrow & E' \\ \downarrow \pi & & \downarrow \pi' \\ M & \longrightarrow & M' \end{array}$ commutes <in particular, fibers are sent to fibers>

Exercise: show that Möbius band (above) is not isomorphic to the trivial bundle $\begin{array}{c} S^1 \times \mathbb{R} \\ \downarrow \\ S^1 \end{array}$

Ex: the tangent bundle TM <sections = vector fields>

$$TM = \left\{ (x,u) \,\middle|\, \begin{array}{l} x \in M \\ u \text{ - tangent vector to } M \text{ at } x \end{array} \right\}$$

Exercise: given an atlas of loc. coordinate charts on M, describe the transition functions of TM

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Rem. fiber bundles $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ $\begin{array}{c} E' \\ \downarrow \\ M \end{array}$ are isomorphic, with iso. covering identity $M \xrightarrow{id} M$

$\iff$ one has a collection of maps $h_\alpha : U_\alpha \to G$

s.t. $t'_{\alpha\beta}(x) = h_\alpha(x)\, t_{\alpha\beta}(x)\, h_\beta^{-1}(x)$

$\forall x \in U_\alpha \cap U_\beta$

- In particular, $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ is iso. to the trivial bundle iff $t_{\alpha\beta}(x) = h_\alpha(x) h_\beta^{-1}(x)$ for some functions $\{h_\alpha : U_\alpha \to G\}$

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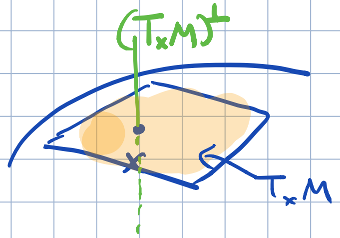
Ex: $M \subset \mathbb{R}^n$ embedded submanifold, $\dim M = k$

normal bundle $\begin{matrix} NM \\ \downarrow \\ M \end{matrix}$ has fiber $(T_xM)^\perp$ over $x \in M$

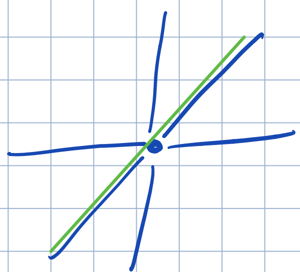


$\mathrm{rank}(NM) = n-k$

Ex: $\mathbb{R}P^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} - \{0\}\} \big/ (x_0,\dots,x_n) \sim (\lambda x_0, \dots, \lambda x_n) \;\; \forall \lambda \neq 0$ $= \left\{ \begin{matrix} \text{lines in} \\ \mathbb{R}^{n+1} \end{matrix} \right\}$

(real) projective space

tautological line bundle $\begin{matrix} \tau \\ \downarrow \\ \mathbb{R}P^n \end{matrix}$: fiber over a line $(x_0 : x_1 : \dots : x_n)$ is the line itself $\{(\mu x_0, \dots, \mu x_n) \mid \mu \in \mathbb{R}\}$



Similarly, one has a tautological complex line bundle over $\mathbb{C}P^n$.

- More generally, over the Grassmanian $Gr(k,n)$ of k-dimensional subspaces in $\mathbb{R}^n / \mathbb{C}^n$, one has the taut. vector bundle of rank k

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There are lots of ways to cook up new vector bundles from old ones:

- Whitney sum $E_1 \oplus E_2$ - $(E_1 \oplus E_2)_x = (E_1)_x \oplus (E_2)_x$
- tensor product $E_1 \otimes E_2$ $(E_1 \otimes E_2)_x = (E_1)_x \otimes (E_2)_x$ ← especially interesting for line bundles
- dual E^* $(E^*)_x = (E_x)^*$ with inverse-transpose transition functions

  (ex: $(TM)^* = T^*M$ - cotangent bundle)

- box tensor product $\begin{matrix} E_1 \\ \downarrow \\ M_1 \end{matrix} \boxtimes \begin{matrix} E_2 \\ \downarrow \\ M_2 \end{matrix}$ is a bundle $\begin{matrix} E_1 \boxtimes E_2 \\ \downarrow \\ M_1 \times M_2 \end{matrix}$ over the direct product manifold, $(E_1 \boxtimes E_2)_{(x_1,x_2)} = (E_1)_{x_1} \otimes (E_2)_{x_2}$
- quotient by a subbundle. Ex: $\underset{\text{normal bundle}}{NM} \overset{\text{iso}}{\approx} \underbrace{M \times \mathbb{R}^n}_{\text{triv. bundle}} / TM$ ← quotient
- symmetric & exterior powers.

  Ex: $\Lambda^p T^* M$ - the bundle of p-forms on M.

  Ex: metric $\in \Gamma(\mathrm{Sym}^2 T^* M)$

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pullback of a fiber bundle: given a bundle $\begin{matrix} E \leftarrow F \\ \downarrow \\ M \end{matrix}$ and a map $f: M' \to M$,

one can form the bundle $\begin{matrix} f^*E \\ \downarrow \\ M' \end{matrix}$ where $f^*E := \{(x', e) \in M' \times E \mid f(x') = \pi(e)\} \subset M' \times E$

$proj_1$ - bundle projection $f^*E \xrightarrow{\pi'} M'$

$proj_2$ gives the map $f^*E \xrightarrow{h} E$

s.t.
$$\begin{array}{ccc} f^*E & \xrightarrow{h} & E \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M' & \xrightarrow{f} & M \end{array}$$
commutes

- fiber of f^*E over $x' \in M'$
  = fiber of E over $f(x')$

Theorem (Thm 15.21 from MT, p. 155)

for $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ a vector bundle and $f_0, f_1 : \underbrace{X}_{\text{compact}} \rightrightarrows M$ two homotopic maps,

then the pullbacks $\begin{matrix} f_0^*E \\ \downarrow \\ X \end{matrix}$, $\begin{matrix} f_1^*E \\ \downarrow \\ X \end{matrix}$ are isomorphic as bundles over X.

Corollary every vector bundle over a contractible base is trivial

Rem: in fact, Thm works for X paracompact - see Dan Freed's lecture

https://web.ma.utexas.edu/users/dafr/M392C-2015/Notes/lecture2.pdf

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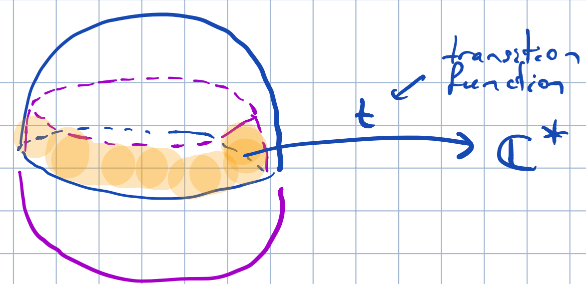
Ex: complex line bundles over $\mathbb{C}P^1$



- classified by a transition map
  $t: S^1 \times [-\varepsilon, \varepsilon] \longrightarrow GL(1, \mathbb{C}) = \mathbb{C}^*$

iso classes of bundles $\sim$ homotopy classes of maps t
- classified by the winding number $\in \mathbb{Z}$

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def A principal G-bundle (with G a topological group) is a fiber bundle $\underset{\text{total space}}{\pi: P} \to \underset{\text{base}}{M}$, with a continuous right action $P \times G \to P$ preserving fibers of π,

i.e. for $p \in P_x$, $p \cdot g \in P_x$ $\forall g$, and such that G acts on P_x for each x freely-transitively

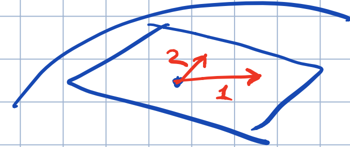
$\langle$ + so that $\forall p \in P_x$, the map $\begin{aligned} G &\to P_x \\ g &\mapsto p\cdot g \end{aligned}$ is a homeo $\rangle$

- G-orbits of P are the fibers of π
- fibers of π are "G-torsors" - copies of G without a marked unit

smooth setting: G - Lie group, all maps in C^∞

Ex: frame bundle of a manifold M: FM

fiber over x = frames (ordered bases) in T_xM.



$$\begin{array}{c} FM \circlearrowleft GL(\underset{\substack{\uparrow \\ \dim M}}{n}, \mathbb{R}) \\ \downarrow \\ M \end{array}$$

Ex: Hopf bundle (Hopf fibration)

$$S^3 = \{(z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1\} \circlearrowleft S^1$$

$(z_0, z_1) \mapsto (e^{i\theta} z_0, e^{i\theta} z_1)$

line through (z_0, z_1)

$$\downarrow$$

$$\mathbb{CP}^1 \overset{\text{diffeo}}{\sim} S^2$$

Sections & trivializations

given a trivialization $\pi^{-1}(U) \xrightarrow{\varphi} U \times G$, one has the associated loc. section $s|_U = x \mapsto \varphi^{-1}(x, \underset{\text{group unit}}{e})$

conversely, given s, one builds an equivariant trivialization $\varphi^{-1}(x, g) = s(x)\cdot g$

(i.e. $\varphi(p\cdot g) = \varphi(p)\cdot g$)

$\exists$ a global section s $\Leftrightarrow$ principal bundle is trivial! (this is special to principal bundles, is not true for general fiber bundles)

transition functions: $\{(U_\alpha, \underset{\text{loc. sections}}{s_\alpha})\}$ - loc. triv., $s_\beta(x) = s_\alpha(x) \cdot t_{\alpha\beta}(x)$, $x \in U_\alpha \cap U_\beta$

$t_{\alpha\beta}: U_\alpha \cap U_\beta \to G$ - transition functions

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* Associated bundle construction

if $\begin{array}{c} P \circlearrowleft G \\ \downarrow \\ M \end{array}$ - principal bundle and G is linearly represented on V - v.sp.,

then $E = P \times_G V = \{(p,v) \in P \times V\} / (p\cdot g, v) \sim (p, g\cdot v) \ \forall g \in G$

- the associated vector bundle

$\pi((p,v)) = \pi(p)$ is well-defined

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more generally: if $\begin{array}{c} E \leftarrow F \\ \downarrow \\ M \end{array}$ - fiber bundle with transition functions $t_{\alpha\beta} : U_\alpha \cap U_\beta \to \underset{\text{str. group}}{G}$ $\quad G \quad \underset{\text{fiber}}{F}$

and F' another top space on which G acts,

one can construct an assoc. bundle $\begin{array}{c} E' \leftarrow \underbrace{F'}_{\text{new fiber}} \\ \downarrow \\ M \end{array}$ with the same transition functions $t_{\alpha\beta}$.

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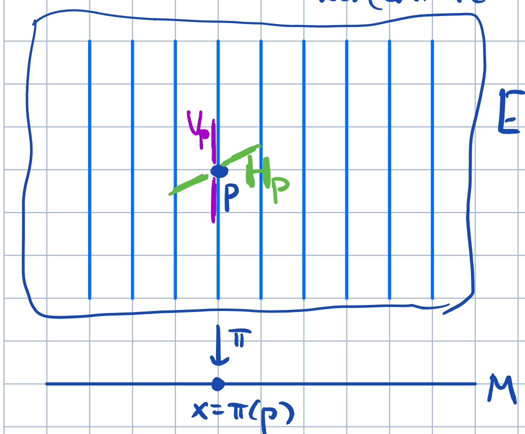
Connections

def An (Ehresmann) connection in a fiber bundle $\begin{array}{c} E \\ \downarrow \pi \\ M \end{array}$ is a subbundle $H \subset TE$

such that $H \oplus \underbrace{V}_{\ker(d\pi : TE \to TM)} = TE$.



I.e. it is a choice of a splitting of $V \hookrightarrow TE \xrightarrow{d\pi} TM$

note: V is canonical (vertical distribution)

whereas H is a choice (of a horizontal distribution)

* projection $\mathcal{A} : TE \to V$ along H

can be seen as a 1-form $\mathcal{A} \in \Omega^1(E, V)$ ← connection 1-form

satisfying: $\mathcal{A}$ is identity on V.

(Sometimes defined as a form with values in TE, satisfying additionally $\mathcal{A}^2 = \mathcal{A}$)

curvature (of the connection)

$F \in \Omega^2(E, V)$ defined by $F(X,Y) = [X_H, Y_H]_V$

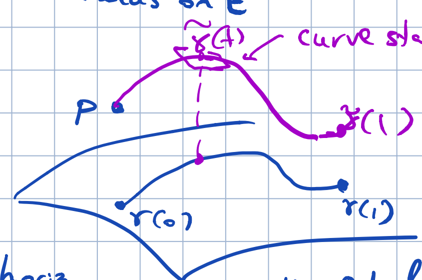
X, Y - two vector fields on E

* parallel transport / holonomy

fix $\gamma : [0,1] \to M$ a curve on the base

$\mathrm{Hol}_\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$

$p \longmapsto \tilde{\gamma}(1)$ where $\tilde{\gamma}(t)$ is the horiz. lift of $\gamma(t)$ starting at p,



i.e. the sol. of

$\tilde{\gamma}(t) \in F_{\gamma(t)}$

$\dot{\tilde{\gamma}}(t) \in H_{\tilde{\gamma}(t)}$ ODE

$\tilde{\gamma}(0) = p$ init. cond.

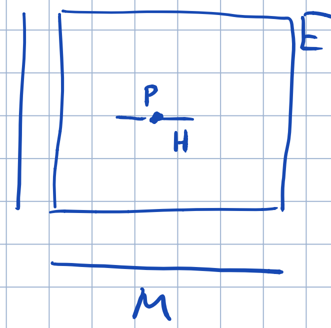
* curvature measures how far is $\tilde{\gamma}(1)$ from p for γ a small closed loop

* connection has zero curvature ("is flat") iff H is <u>Frobenius-integrable</u> (and thus integrates into a foliation of E)

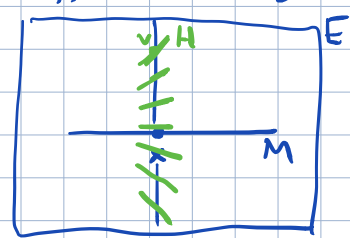
<u>Ex</u>: a trivial bundle $M \times F$ has a "trivial connection"

$$\begin{array}{ccc} M\times F & & \\ \downarrow & \searrow & \\ M & & F \end{array}$$

$H = \ker(d\,\mathrm{proj}_2)$



- For $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ a vector bundle, one considers Ehresmann connections inducing <u>linear</u> holonomy maps (in other words, $H_{(x,v)}$ depends linearly on $v \in E_x$)



Covariant derivative

$$\nabla_X S\big|_x = \frac{d}{dt}\Big|_{t=0} \underbrace{(\mathrm{Hol}_\gamma)^{-1}}_{E_x \to E_{\gamma(t)}} S(\gamma(t))$$

vector field on M — section of E

$\gamma : [0,t] \to M$ a curve starting at x with init. velocity $X(x)$

$$\rightsquigarrow \nabla : \underset{\text{sections of } E}{\Gamma(E)} \to \Omega^1(E)$$

one can extend this differential operator by Leibniz rule

$$\nabla(S\alpha) = \nabla S \cdot \alpha + S \cdot d\alpha$$

section — p-form

to an operator $\underset{=\Gamma(E)}{\Omega^0(E)} \xrightarrow{\nabla} \Omega^1(E) \xrightarrow{\nabla} \Omega^2(E) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^n(E)$

* locally, in a trivializing nbhd U_α one has ∇: $S_\alpha(x) \longmapsto (d + A_\alpha) S_\alpha(x)$

$$\begin{tikzcd} \pi^{-1}(U_\alpha) \arrow[r, "\varphi_\alpha"] & U_\alpha \times V \\ U_\alpha \arrow[u, "s"] \arrow[ur, "s_\alpha"'] & \end{tikzcd}$$

local (matrix-valued) connection 1-form $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}\, V)$

or: $\sum_a e_a s^a(x) \longmapsto \sum_a e_a \left(d s^a(x) + \sum_b A^a_b(x) s^b(x) \right)$

basis sections

* $\nabla^2 = F$.

with $F \in \Omega^2(M, \mathrm{End}\, E)$ - curvature

($\rightsquigarrow$ Ehresmann curvature 2-form: $\mathcal{F} \in \Omega^2(E, V)$, $\mathcal{F}_{(x,v)}(X,Y) = F_x(d\pi(X), d\pi(Y)) \circ v$)

<u>Exercise</u>: ① show that locally one has $F = dA + \frac{1}{2}[A, A]$

② show that on an overlap $U_\alpha \cap U_\beta$ one has

$$A_\beta = t_{\alpha\beta}^{-1} A_\alpha t_{\alpha\beta} + t_{\alpha\beta}^{-1} d t_{\alpha\beta}$$

with $t_{\alpha\beta}$ the transition function

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Connection in a principal bundle

- an Ehresmann connection on $\begin{array}{c} P \circlearrowleft G \\ \downarrow \\ M \end{array}$ which is <u>G-equivariant</u>, i.e. $H_{p\cdot g} = d(R_g)_p H_p$

$$R_g : P \to P$$
$$y \mapsto y\cdot g$$

connection 1-form $\mathcal{A} \in \Omega^1(P, \mathfrak{g})$

- satisfies ① equivariance $\quad Ad_g(R_g^* \mathcal{A}) = \mathcal{A}$

② normalization $\quad \iota_{X_\xi} \mathcal{A} = \xi \quad$ with any $\xi \in \mathfrak{g}$ and X_ξ the corresp. fund. v.f. on P

curvature: $\mathcal{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \in \Omega^2(P, \mathfrak{g})$ - G-equivariant, horizontal 2-form;

it corresponds to an element $F \in \Omega^2(\underset{\sim}{M}, \underset{\substack{\uparrow \\ P \times_G \mathfrak{g}}}{ad(P)})$

$P \times_G \mathfrak{g}$ - adjoint bundle

- in a trivializing nbhd U_α with section s_α,

$$F_\alpha := s_\alpha^* \mathcal{F} \in \Omega^2(U_\alpha, \mathfrak{g})$$

on $U_\alpha \cap U_\beta$: $\quad F_\beta = \underbrace{ad(t_{\beta\alpha})}_{\text{correct transition function for the bundle } ad(P)} \circ F_\alpha$.

- Another way to compare F and $\mathcal{F}$:

$$F_x(d\pi(X), d\pi(Y)) = (p, \mathcal{F}_p(X,Y)) \big/ G \quad \text{for } p \in P_x,\ X, Y \in T_pP$$

- In fact, one has a general <u>Lemma:</u> $\Omega^k(P, V)^{\text{horizontal, } G\text{-equivariant}} \simeq \Omega^k(M, P \times_G V)$

V ← vect. space carrying a rep. of G

<u>Cor</u>: the space of connections is an affine space modeled on $\Omega^1(M, ad(P))$.

(proof: a difference of two connection forms $\mathcal{A}_1 - \mathcal{A}_0$ is equivar. and <u>horizontal</u> $\Rightarrow$

$\Rightarrow \mathcal{A}_1 - \mathcal{A}_0$ corresponds to an element of $\Omega^1(M, ad(P))$) (see ② above)