Talk on Floer Homology

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1 The Arnold Conjecture

Let (M, ω) be a $2n$-dimensional compact symplectic manifold. Then a **time-dependent Hamiltonian** on (M, ω) is a smooth map $H\colon M \times \mathbb{R} \to \mathbb{R}$, which we also view as a family $\{H_t\}_{t \in \mathbb{R}}$ of Hamiltonians parametrized by $\mathbb{R}$. For each $t \in \mathbb{R}$, we can associate to H_t its Hamiltonian vector field V_{H_t}, which we will abbreviate as V_t. Recall that this is the vector field characterized by

$$\iota_{V_t} \omega = dH_t.$$

The collection of these gives rise to a time-dependent vector field $V\colon M \times \mathbb{R} \to TM$. Since M is assumed to be compact, we have a globally defined time-dependent flow $\theta\colon M \times \mathbb{R} \times \mathbb{R} \to M$. This flow in turn gives rise to two families of objects:

- For each $p \in M$, a smooth curve $x^{(p)}\colon \mathbb{R} \to M$ given by $x^{(p)}(t) = \theta(p, 0, t)$, which satisfies

  $$\dot{x}^{(p)}(t) = V_t \circ x^{(p)}(t). \tag{H}$$

  This is the unique maximal integral curve of V with initial condition $x^{(p)}(0) = p$, and the equation (H) is called **Hamilton's equation**. It is a consequence of the theory of time-dependent flows that the family $\{x^{(p)}\}_{p \in M}$ is a complete set of solutions to (H).
- For each $t \in \mathbb{R}$, a diffeomorphism $x_t\colon M \to M$ given by $x_t(p) = \theta(p, 0, t)$. Observe that

  $$\frac{d}{dt} x_t = V_t \circ x_t, \qquad x_0 = \mathrm{Id}_M\,.$$

  We call the collection $\{x_t\}_{t \in \mathbb{R}}$ the **Hamiltonian diffeotopy generated by H**. We also know that $x_t \in \mathrm{Symp}(M, \omega)$ for all $t \in \mathbb{R}$ by definition, so that each x_t is a symplectomorphism.

Now suppose that the Hamiltonian H satisfies $H_t = H_{t+1}$ for all $t \in \mathbb{R}$, which is to say it is **1-periodic**. We will call a solution $x^{(p)}$ of (H) **1-periodic** if it satisfies $x^{(p)}(t+1) = x^{(p)}(t)$ for all $t \in \mathbb{R}$, meaning that it corresponds uniquely to a smooth curve $x^{(p)}\colon \mathbb{R}/\mathbb{Z} \to M$. Let $\mathcal{P}(H)$ be the set of all 1-periodic solutions of (H).

1.1 Proposition. *The set $\mathcal{P}(H)$ is in bijection with the set $\mathrm{Fix}(x_1)$ of fixed points of $x_1\colon M \to M$.*

Proof. First suppose that $x^{(p)} \in \mathcal{P}(H)$. Then by definition we have $x_1(p) = x^{(p)}(1) = x^{(p)}(0) = p$, so that $p \in \mathrm{Fix}(x_1)$. Conversely, if $p \in \mathrm{Fix}(x_1)$, then define a smooth curve $y^{(p)}\colon \mathbb{R} \to M$ by $y^{(p)}(t) = x^{(p)}(t+1)$ for all $t \in \mathbb{R}$. Then

$$\dot{y}^{(p)}(t) = \dot{x}^{(p)}(t+1) = V_{t+1}(x^{(p)}(t+1)) = V_t(y^{(p)}(t)),$$

so that $y^{(p)}$ is a maximal integral curve of V with initial condition $y^{(p)}(0) = x^{(p)}(1) = x^{(p)}(0) = p$, and so by uniqueness we must have $x^{(p)}(t + 1) = y^{(p)}(t) = x^{(p)}(t)$ for all $t \in \mathbb{R}$, so that $x^{(p)} \in \mathcal{P}(H)$. This completes the proof. $\square$

We will call a 1-periodic solution of (H) **nondegenerate** if

$$\det(I_{2n} - dx_1(p)) \neq 0.$$

We can now formulate the famous conjecture which launched the development of Floer homology.

1.2 Theorem (Arnold Conjecture). *Let (M, ω) be a $2n$-dimensional compact symplectic manifold and $H: M \times \mathbb{R} \to \mathbb{R}$ be a time-dependent 1-periodic Hamiltonian. Suppose that every $x^{(p)} \in \mathcal{P}(H)$ is nondegenerate. Then*

$$|\mathcal{P}(H)| \geq \sum_{i=0}^{2n} \dim H_i(M; \mathbb{Q}).$$

An important observation about the previous discussion is that if the Hamiltonian H is time-independent, so that it is just given by a smooth function $H: M \to \mathbb{R}$, then all the critical points of H are constant solutions of (H), and the assumption that all such solutions are nondegenerate can be shown to imply that H is a Morse function on M. The Arnold conjecture then follows immediately from the standard Morse inequalities.

This suggests that a plausible approach to the Arnold conjecture in the case of time-dependent Hamiltonians would be to look for some generalization of Morse theory to the infinite-dimensional setting. This is exactly the objective which Floer homology accomplishes. In order to understand exactly how it generalizes Morse homology, we now proceed with a quick review of the latter theory.

2 Morse-Smale Theory

Let (M, g) be a compact orientable Riemannian manifold and $f: M \to \mathbb{R}$ be a smooth function. We write $\mathrm{Crit}(f)$ for the set of critical points of f. For every $p \in M$, there exists a symmetric covariant 2-tensor on M, called the **Hessian of f** and denoted by $H(f)$. In any smooth chart, the Hessian is represented by the matrix of second partial derivatives of f. We say f is a **Morse function** if $H(f)_p$ is nondegenerate for all $p \in \mathrm{Crit}(f)$, and we then define the **index of p** to be the number of negative eigenvalues of $H(f)_p$, hereafter denoted by $\mathrm{Ind}_f(p)$.

Now let $\theta: M \times \mathbb{R} \to M$ be the flow of $-\nabla f$, where ∇f is the **gradient vector field of f**, uniquely defined by $\iota_{\nabla f} g = df$. For every $p \in M$, the **gradient flow line at p** is the unique maximal integral curve $\theta^{(p)}: M \to \mathbb{R}$ of ∇f for which $\theta^{(p)}(0) = p$.

2.1 Proposition. *All gradient flow lines originate from and terminate at critical points.*

In view of the previous proposition, for every $p \in \mathrm{Crit}(f)$, we define the following two sets:

$$W^s(p) = \{q \in M \mid \lim_{t \to \infty} \theta^{(q)}(t) = p\}, \qquad W^u(p) = \{q \in M \mid \lim_{t \to -\infty} \theta^{(q)}(t) = p\}.$$

We call the above two sets the **stable** and **unstable manifolds of p**, respectively. These are in fact submanifolds of M, and one can also show that we have

$$\dim W^u(p) = \mathrm{Ind}_f(p).$$

We will say the pair (f, g) satisfies the **Morse-Smale condition** if for all $p, q \in \mathrm{Crit}(f)$, we have

$$W^s(p) \pitchfork W^u(q),$$

which is to say that these intersect transversally. In this case, the set

$$\mathcal{W}(q, p) = W^s(p) \cap W^u(q)$$

is a smooth manifold whose elements are those $x \in M$ whose gradient flow lines connect q to p, and we call these gradient flow lines **connecting orbits from q to p**. By transversality, we also have that

$$\dim \mathcal{W}(q, p) = \mathrm{Ind}_f(q) - \mathrm{Ind}_f(p).$$

Now $\mathbb{R}$ acts freely and properly on $\mathcal{W}(q, p)$ by $t \cdot x \mapsto \theta^{(x)}(t)$ for $t \in \mathbb{R}$. The quotient

$$\mathcal{M}(q, p) = \mathcal{W}(q, p)/\mathbb{R}$$

is a smooth manifold, called the **moduli space of flow lines from q to p**. We then see that we have

$$\dim \mathcal{M}(q, p) = \mathrm{Ind}_f(q) - \mathrm{Ind}_f(p) - 1.$$

Therefore the Morse-Smale condition implies that $\mathrm{Ind}_f(p) < \mathrm{Ind}_f(q)$ whenever there is a connecting orbit from q to p, so the index of critical points must strictly decrease along flow lines.

One can show that $\mathcal{M}(q, p)$ is a finite set whenever $\dim \mathcal{M}(q, p) = 0$, whose elements represent connecting orbits from q to p up to time translation. After some appropriate choice of orientations for the various unstable manifolds, we can assign to $\gamma \in \mathcal{M}(q, p)$ a number $\varepsilon(\gamma) \in \{\pm 1\}$. We then define

$$n(q, p) = \sum_{\gamma \in \mathcal{M}(q,p)} \varepsilon(\gamma).$$

It can then be shown that $n(q, p)$ is independent of our choice of orientations up to a sign.

For each $0 \leq k \leq n$, we let $\mathrm{Crit}_k(f)$ be the set of critical points of f of index k. Then we define

$$C_k(f) = \mathbb{Z}\langle \mathrm{Crit}_k(f) \rangle, \qquad C_*(f) = \bigoplus_{k=0}^{n} C_k(f),$$

and a homomorphism $\partial\colon C_k(f) \to C_{k-1}(f)$ by

$$\partial(q) = \sum_{p \in \mathrm{Crit}_k(f)} n(q, p)p.$$

It can be shown that $\partial^2 = 0$. We call $(C_*(f), \partial)$ the **Morse-Smale complex of f**, and its homology $H_*^{\mathrm{Mor}}(M; \mathbb{Z})$ is called the **Morse homology of M**. We then have the following result.

2.2 Theorem (Morse Homology). $H_*^{\mathrm{Mor}}(M; \mathbb{Z}) \cong H_*(M; \mathbb{Z})$.

Among other things, this result shows that the Morse homology is independent of the choice of Morse function. From it we can also deduce the Morse inequalities, which we can in turn use to deduce the Arnold conjecture in the time-independent case, as mentioned at the end of the previous section.

Let us summarize this section, so as to make the analogy with the infinite-dimensional case clearer:

i. We started with a Morse function and analyzed its critical points.
ii. We studied gradient flow lines on the manifold and saw that these start and end at critical points.
iii. We imposed the Morse-Smale condition so as to get a nice structure on the moduli space of connecting orbits between critical points of decreasing index.
iv. We assigned meaningful signs to the connecting orbits which reflect their compatibility with the orientation of the manifold.
v. We constructed a chain complex from the set of critical points graded by index.
vi. The homology of this complex turned out to be isomorphic to the singular homology, and gave meaningful topological information.

This procedure will be repeated to the letter in the next section.

3 Symplectic Floer Homology

Let (M, ω) be a compact connected symplectic manifold. For technical reasons we will ask that $\pi_2(M) = 0$ (although strictly speaking it is not necessary). Let $H : M \times \mathbb{R} \to \mathbb{R}$ be a 1-periodic time-dependent Hamiltonian which is "**nice enough**", a requirement which among other implications assures us that all 1-periodic solutions of (H) are nondegenerate.

An **almost complex structure on M** is a bundle endomorphism $J \in \mathrm{End}(TM)$ with the property that $J^2 = -\mathrm{Id}_{TM}$. We say J is **compatible with ω** if $\omega(JX, JY) = \omega(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$ and if the covariant 2-tensor

$$g(-,-) = \omega(-, J-)$$

defines a metric on M. The space of compatible structures on M is denoted by $\mathcal{J}(M, \omega)$.

3.1 Theorem. *The space $\mathcal{J}(M, \omega)$ is nonempty and contractible.*

Let $\mathcal{L}M$ be the space of smooth contractible loops in M, which are smooth maps $x : \mathbb{R}/\mathbb{Z} \to M$ for which there is a smooth map $\tilde{x} : \mathbb{D}^2 \to M$ such that $\tilde{x}|_{\mathbb{S}^1} = x$. This turns out to have the structure of a **Banach manifold**, which among other things means that we can carry out many of the constructions on it that we can for finite-dimensional smooth manifolds. We now define the **action functional on $\mathcal{L}M$** as the map $a_H : \mathcal{L}M \to \mathbb{R}/\mathbb{Z}$ given by

$$a_H(x) = -\int_{\mathbb{D}^2} \tilde{x}^*\omega - \int_0^1 H_t(x(t))\, dt,$$

for some smooth extension $\tilde{x} : \mathbb{D}^2 \to M$ of x. One can then show that the critical points of a_H are precisely the elements of $\mathcal{P}(H)$. Therefore the action functional will be our generalization of a Morse function to infinite dimensions.

Now let $\{J_t\}_{t\in\mathbb{R}}$ be a 1-periodic family of almost complex structures on M compatible with ω, and let g_t be the associated Riemannian metric for each $t \in \mathbb{R}$. For $x \in \mathcal{L}M$, the tangent space $T_x\mathcal{L}M$ consists of vector fields along x, and therefore for $V, W \in T_x\mathcal{L}M$, we can define a metric by

$$\langle V, W \rangle = \int_0^1 g_t(V_{x(t)}, W_{x(t)})\, dt.$$

The gradient of the action functional a_H with respect to this metric is

$$\nabla a_H(x)(t) = J_t(x(t))\dot{x}(t) - \nabla H_t(x(t)),$$

where ∇H_t is the gradient of H_t with respect to g_t. The flow lines of this gradient are exactly the smooth curves $\gamma\colon \mathbb{R} \to \mathcal{L}M$ satisfying

$$\frac{\partial \gamma}{\partial s} + J_t(\gamma)\frac{\partial \gamma}{\partial t} - \nabla H_t(\gamma) = 0, \tag{F}$$

and $\gamma(s,t) = \gamma(s,t+1)$ for all $t \in \mathbb{R}$. Now in the Morse-Smale theory, we saw that the gradient flow lines originated and terminated at critical points of the Morse function. In the infinite-dimensional context, this remains true for a flow line γ if and only if $E(\gamma) < \infty$, where

$$E(\gamma) = \frac{1}{2}\int_0^1 \int_{-\infty}^{\infty} \left(\left|\frac{\partial \gamma}{\partial s}\right|^2 + \left|\frac{\partial \gamma}{\partial t} - V_t(\gamma)\right|^2 \right) ds\, dt.$$

If this is the case we say γ has **finite energy**, and there exists $x^+, x^- \in \mathcal{P}(H)$ such that

$$\lim_{s \to \pm\infty} \gamma(s,t) = x^{\pm}(t).$$

Let $\mathcal{W}(x^-, x^+)$ be the space of all such γ. Then because H was chosen to be "nice enough", these all turn out to be finite-dimensional manifolds and there is a function $\eta_H : \mathcal{P}(H) \to \mathbb{R}$ such that

$$\dim \mathcal{W}(x^-, x^+) = \eta_H(x^-) - \eta_H(x^+).$$

Intuitively, the map η_H is playing the role of the index of critical points in the Morse-Smale theory. The number $\eta_H(x)$ is called the **Conley-Zehnder index of x**. (Its actual definition is quite complicated, and our description of it here is really only a special case which applies when $\pi_2(M) = 0$.)

Let $x^{\pm} \in \mathcal{P}(H)$ be such that $\eta_H(x^-) - \eta_H(x^+) = 1$. Then $\mathbb{R}$ acts freely and properly on $\mathcal{W}(x^-, x^+)$ by $a \cdot u(s,t) = u(s+a,t)$ for $a \in \mathbb{R}$. The quotient manifold

$$\mathcal{M}(x^-, x^+) = \mathcal{W}(x^-, x^+)/\mathbb{R}$$

is zero-dimensional, and called the **moduli space of connecting orbits** from x^- to x^+.

3.2 Theorem. *$\mathcal{M}(x^-, x^+)$ is finite for any $x^{\pm} \in \mathcal{P}(H)$ with $\eta_H(x^-) - \eta_H(x^+) = 1$.*

It can then be shown that all such moduli spaces are orientable, and after choosing a system of appropriate orientations for each, which is done by a rather involved procedure, we can assign to every $\gamma \in \mathcal{M}(x^-, x^+)$ a well-defined number $\varepsilon(\gamma) \in \{\pm 1\}$.

Let $\mathbb{F}$ be a principal ideal domain and $CF_k(M)$ be the free $\mathbb{F}$-module generated by critical points $x \in \mathcal{P}(H)$ with $\eta_H(x) = k$. Define a homomorphism $\partial\colon CF_k(M, \omega, \mathbb{F}) \to CF_{k-1}(M, \omega, \mathbb{F})$ by

$$\partial(y) = \sum_{\eta_H(x) = k-1} \sum_{u \in \mathcal{M}(y,x)} \varepsilon(u) x.$$

Then we have the following easy to state, but rather difficult to prove, result.

3.3 Theorem (Floer). $\partial^2 = 0$.

We call $(CF_*(M,\omega;\mathbb{F}),\partial)$ the **Floer complex of M** and its homology $HF_*(M,\omega;\mathbb{F})$ is called the **Floer homology of M**. This can all be shown to be independent of the choices of time-dependent Hamiltonian H and complex structure J. In particular, we can choose them to be time-independent. In this case an element $x \in \mathcal{P}(H)$ is just a critical point of H, and furthermore H can be chosen in a special enough way that the moduli spaces $\mathcal{M}(x^-,x^+)$ coincide with those defined in the Morse-Smale theory. We can therefore reduce the computation of Floer homology to that of Morse homology, and by the main result obtained for that theory, we have the following.

3.4 Theorem (Floer). $HF_*(M,\omega;\mathbb{F}) \cong H_*(M;\mathbb{F})$.

This allows us to prove the Arnold conjecture in a rather general setting.

3.5 Corollary (Arnold Conjecture, Special Case). *Let (M,ω) be a $2n$-dimensional compact connected symplectic manifold with $\pi_2(M)=0$ and $H\colon M\times\mathbb{R}\to\mathbb{R}$ be a time-dependent 1-periodic Hamiltonian. Suppose that every $x^{(p)} \in \mathcal{P}(H)$ is nondegenerate. Then*

$$|\mathcal{P}(H)| \geq \sum_{i=0}^{2n} \dim H_i(M;\mathbb{Q}).$$

Proof. We have $|\mathcal{P}(H)| = \operatorname{rank}(CF_*(M,\omega;F)) \geq \operatorname{rank}(H_*(M;\mathbb{Q})) = \sum_{i=0}^{2n} \dim H_i(M;\mathbb{Q})$. □