Intro to Gromov-Witten invariants

Wern

ref: Fulton - Pandharipande 1997
Costin "Kontsevich moduli space of formal maps"

Enumerative geometry $\longleftrightarrow$ quantum cohomology

solutions to enum geom. $\longleftrightarrow$ coeff in multipl table of q.c. ring - associative!

Idea

$\Rightarrow$ relations between coefficients

solve problem in general

Apply idea to find N_d = # rational plane curves $\subset \mathbb{P}^2$ deg $= d$ passing through $3d-1$ general points

Outline Moduli space & compactifications

$\downarrow$

Kontsevich sp of stable maps

$\downarrow$

Gromov-Witten thm

$\downarrow$

Quantum coho

$\downarrow$

compute $N_d \leftarrow$ q.c.

$\nwarrow$ Int. theory on Kontsevich space.

Talk 2 } Compactifications of moduli spaces

$M_g = \{$proj, nonsing curves of genus $g\}$

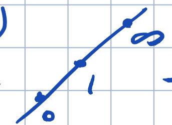
$\overline{M}_g = \{$proj, conn., nodal curves of genus $g\}$ stable

$M_{g,n} = \{$proj, nonsing curves of genus g with n distinct points$\}$

$\overline{M}_{g,n} = \{$ proj, conn. nodal curves, genus $= g$, w/ n distinct nonsing marked pts + stability condition $\Leftrightarrow$ finite autom. groups $\}$

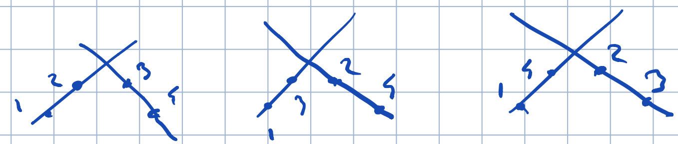
$g=0$: $\overline{M}_{0,n} = \{$trees of proj lines meeting transversely each w/ distinct nonsing marked pts

stable $\Leftrightarrow$ each comp has ≥ 3 special pts (nodes, marked pts) $\}$

Ex: (1) $n=3$ $M_{0,3} = \overline{M}_{0,3} = \{$  $\}$

(2) $n=4$ $M_{0,4} \cong \mathbb{P}^1 \setminus \{0,1,\infty\}$

$\overline{M}_{0,4} \cong \mathbb{P}^1$ Bdry

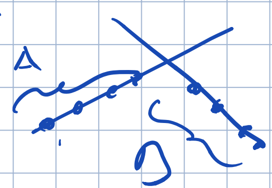


$A \sqcup B = \{1, \dots, n\}$

≥ 2 pts

get divisor $D(A|B) \subset \overline{M}_{0,n}$

$=$

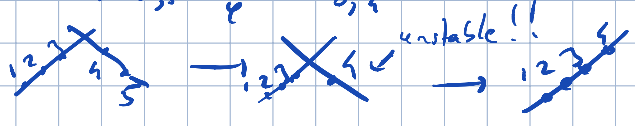
$\{$  $\}$

Forgetful Morphisms $\overline{M}_{g,n} \xrightarrow{\varphi} \overline{M}_{g,m}$

$n \geq m$

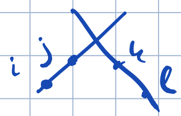
forgets $n-m$ points

Ex: $\overline{M}_{0,5} \xrightarrow{\varphi} \overline{M}_{0,4}$



$\{i,j,k,\ell\} \subseteq \{1,\dots,n\}$

$$\overline{M}_{0,n} \xrightarrow{\varphi} \overline{M}_{0,\{i,j,k,\ell\}} \cong \mathbb{P}^1$$

 $= P(i,j,k,\ell)$

In $\mathbb{P}^1$: $\quad P(i,j \mid k,\ell) = P(i,k \mid j,\ell) = P(i,\ell \mid j,k)$

linearly equivalent divisors (difference corresponds to zeros/poles of a ~~holom.~~ merom. function)

Pullback: $\displaystyle\sum_{A \sqcup B = [n]} D(A \mid B) =$ —— " ——

Kontsevich space of stable maps

(Chow) $\{1\text{-cycles}/\text{rat. equiv}\}$ =

X smooth proj. variety, $\beta \in A_1 X$

$$M_{g,n}(X,\beta) = \left. \left\{ \begin{array}{l} (C, p_1, \dots, p_n, \mu) \\ C \text{ proj non sing curve } g, \\ p_1 \dots p_n \in C \\ \mu : C \to X, \ \mu_*([C]) = \beta \end{array} \right\} \middle/ \sim \right.$$

$(C, p_1, \dots, p_n, \mu) \sim (C', p_1', \dots, p_n', \mu')$

if $\exists$

$$\begin{tikzcd} C \arrow[rr, "\tau", "\sim"'] \arrow[dr, "\mu"'] & & C' \arrow[dl, "\mu'"] \\ & X & \end{tikzcd}$$

$p_i \longmapsto p_i'$

$$\overline{M}_{g,n}(X,\beta) = \{(C, p_1, \dots, p_n, \mu)\}$$

μ stable $\iff$ $\forall E \subseteq C$ irred comp

(1) $E \cong \mathbb{P}^1$ gets mapped to a pt then $E \ni \geq 3$ special pts

(2) E genus 1 mapped to a pt then $E \ni \geq 1$ special pt

Ex: (1) $\overline{M}_{g,n}(pt, 0) = \overline{M}_{g,n}$

(2) $\overline{M}_{0,0}(\mathbb{P}^n, 1) \cong \mathbb{G}(1,n)$
Grassmannian

(3) $\overline{M}_{g,n}(X, 0) \cong \overline{M}_{g,n} \times X$.

properties on X to make $\overline{\mathcal{M}}_{0,n}(X,\beta)$ nice?

(1) Convex X is convex if
$\forall \mu : \mathbb{P}^1 \to X$, $H^1(\mathbb{P}^1, \mu^* TX) = 0$
- makes $\overline{M}_{0,n}(X,\beta)$ "smooth"

(2) Homogeneous G Lie gp acting transitively on X $\Rightarrow$

$\Rightarrow TX$ glob generated $\Rightarrow X$ convex. $\Rightarrow$ have natural basis of Schubert classes for $A^i_{(i)} X$

When X proj nonsing, cvx, have Thms 1-3:

- $\overline{\mathcal{M}}_{0,n}(X,\beta)$ normal proj var, whose sing are quot of nonsing var by finite group

- $\overline{\mathcal{M}}_{0,n}(X,\beta)$ dim
$= \dim X + \int_\beta c_1 TX + n - 3$, if ≥ 0

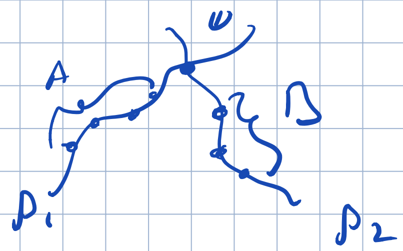
- the open locus of stable maps with no nontriv. aut. $\overline{M}^*_{0,n}(X,\beta)$ is smooth

- Bdry of $\overline{M}_{0,n}(X,\beta)$ is a divisor w/ normal..

Boundary of $\overline{M}_{0,n}(X,\beta)$

$n \geq 4$ $A \amalg B = [n]$, $\beta_1 + \beta_2 = \beta \in A_1 X$

have divisor $D(A, B; \beta_1, \beta_2)$
$$\overline{M}_{0,A\cup\{0\}}(X,\beta_1) \times_X \overline{M}_{0,B\cup\{0\}}(X,\beta_2)$$



$\{i,j,k,l\} \subseteq [n]$

$$D(i,j\,|\,k,l) = \sum_{\substack{A \sqcup B = [n] \\ i,j \in A,\ k,l \in B \\ \beta_1+\beta_2=\beta}} D(A,B\,;\,\beta_1,\beta_2)$$

Forgetful mor $\quad \overline{M}_{0,n}(X,\beta) \longrightarrow \overline{M}_{0,\{i,j,k,l\}} \cong \mathbb{P}^1$

$$P(i,j\,|\,k,l) = \underline{\qquad\qquad}$$

$$D(i,j\,|\,k,l) = \underline{\qquad\qquad}$$ "Keel relations"

Gromov-Witten invariants

$1 \le i \le n$ have evaluation maps $\quad \rho_i : \overline{M}_{0,n}(X,\beta) \longrightarrow X$

$$[C, p_1, \dots, p_n, \mu] \longmapsto \mu(p_i)$$

Given $\gamma_1, \dots, \gamma_n \in A^*X$

$$\int_{\overline{M}_{0,n}(X,\beta)} \rho_1^*(\gamma_1) \cup \dots \cup \rho_n^*(\gamma_n)$$

$$=:$$

$$I_\beta(\gamma_1 \dots \gamma_n)$$ Gromov-Witten invariants

Properties of $I_\beta(\gamma_1 \dots \gamma_n)$

(a) symmetric in γ_i

(b) if γ_i homogeneous, then $I_\beta(\gamma_1, \dots, \gamma_n)$ possibly nonzero $\iff \sum_i \operatorname{codim} \gamma_i = \dim \overline{M}$

(c) link between GW and enumerative geometry

Lemma 14 $\Gamma_1, \dots, \Gamma_n$ pure dim ? subvar of X corresp $\gamma_1 \dots \gamma_n \in A^* X$

$\sum \operatorname{codim} \Gamma_i = \dim \bar{M}$ $\quad G \ni g_1, \dots, g_n$ general elts

Then (1) The scheme-theoretic intersection

$\rho_1^{-1}(g_1 \Gamma_1) \cap \dots \cap \rho_n^{-1}(g_n \Gamma_n)$

is a fin # of reduced pts supp in $/\!/ M_{0,n}(X, \beta)$

(2) $I_\beta(\gamma_1 \dots \gamma_n)$

Consequences (1) $I_\beta(\gamma_1 \dots \gamma_n) = \# \left\{ \begin{array}{l} \text{ptd maps } \mu : \mathbb{P}^1 \to X \text{ rep} \\ \text{class } \beta \in A_1 X \text{ st } \mu(p_i) \in g_i P_i \end{array} \right\}$

(2) $X = \mathbb{P}^2$ $\quad \beta = d\,[\text{line}]$ $\quad \Gamma_i = \{p\}$

$n = 3d - 1$ $\quad 2n = 2 + \underbrace{\int_{d[\text{line}]} c_1(T\mathbb{P}^2)}_{3d} + n - 3$

$I_d(\underbrace{[p] \dots [p]}_{3d-1}) = N_d$ $\quad n = 3d - 1$

Talk III $\Big\}$ $A_i X = [i\text{-cycles}]$

$A^i X = A_{n-i} X = [(n-i)\text{-cycles}]$

$f: X \to Y$ proper

git $f_*: A_* X \to A_* Y$

$$[V] \longmapsto \begin{cases} \deg(V/f(V)) \cdot [f(V)] & \text{if } \dim V = \dim f(V) \\ 0 & \text{otherwise} \end{cases}$$

Properties of $I_\beta(\gamma_1, \dots, \gamma_n)$

(I) $\beta = 0$

$$\overline{M}_{0,n}(X, \beta = 0) = \overline{M}_{0,n} \times X \ni ([C, p_1, \dots, p_n], \mu(C))$$

$$[C, p_1, \dots, p_n, \mu] \in \overline{M}_{0,n}(X,\beta=0), \quad \overline{M}_{0,n}(X,\beta=0) \xrightarrow{\rho_i} X, \quad \overline{M}_{0,n} \times X \xrightarrow{p} X$$

$$[C, p_1, \dots, p_n, \mu] \mapsto \mu(p_i) = \mu(C) \leftarrow ([C, p_1, \dots, p_n], \mu(C))$$

$\rho_1^*(\gamma_1) \cup \dots \cup \rho_n^*(\gamma_n)$

$= p^*(\gamma_1 \cup \dots \cup \gamma_n)$

$$I_\beta(\gamma_1 \dots \gamma_n) = \int_{\overline{M}_{0,n} \times X} p^*(\gamma_1 \cup \dots \cup \gamma_n) = \int_{p_*(\overline{M}_{0,n} \times X)} \gamma_1 \cup \dots \cup \gamma_n$$

$0 \leq n \leq 2$, $\overline{M}_{0,n} = \emptyset$

$n \geq 3$ $\overline{M}_{0,n}$ positive-dim

$\Rightarrow p$ has positive-dim fibers

$\Rightarrow p_*(\overline{M}_{0,n} \times X) = 0$

$n = 3$, $\overline{M}_{0,n} = \{pt\}$

$$I_\beta(\gamma_1 \gamma_2 \gamma_3) = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3$$

Case

(II) $\gamma_1 = 1 \in A^0 X$

- $\beta \neq 0$

$$I_\beta(\gamma_1 \dots \gamma_n) = \int_{\overline{M}_{0,n}(X,\beta)} \cancel{\rho_1^*(\gamma_1)} \cup \dots \cup \rho_n^*(\gamma_n)$$

$$\varphi : \overline{M}_{0,n}(X,\beta) \to \overline{M}_{0,n-1}(X,\beta) \quad = \int_{\overline{M}_{0,n}(X,\beta)} \varphi^*\left(\rho_2^*(\gamma_2) \cup \dots \cup \rho_n^*(\gamma_n)\right)$$

$$= \int_{\varphi_* \overline{M}_{0,n}(X,\beta)} \rho_2^*(\gamma_2) \cup \dots \cup \rho_n^*(\gamma_n)$$

$$= 0$$

φ has pos-dim. fibers

- $\beta = 0$

$$I_0(\gamma_1 \cdot \gamma_2 \cdot \gamma_3) = \int_X \gamma_2 \cup \gamma_3$$

(III) $\gamma_1 \in A^1 X$, $\beta \in X$

Then $I_\beta(\gamma_1 \dots \gamma_n) = \left(\int_\beta \gamma_1\right) I_\beta(\gamma_2 \dots \gamma_n)$

Lemma 14: $I_\beta(\gamma_1 \dots \gamma_n) = \#\left\{ \begin{array}{l} \text{pointed maps } \mu : C \to X \text{ with} \\ \mu_*([C]) = \beta \text{ s.t. } \mu(p_1) \in \underset{\text{corresp. to } \gamma_1}{\Gamma_1} \text{ etc.} \end{array} \right\}$

(slogan:) There are $\left(\int_\beta \gamma_1\right)$ choices for $\mu(p_1)$

Quantum cohomology: Assume X homogeneous, so we have a basis of Schubert classes

$T_0 = 1 \in A^0 X$, $T_1, \dots, T_p \in A^1 X$, $T_{p+1}, \dots, T_m \in A^* X$ (the rest)

for each $n_i \geq 0$ we define the number $N(n_{p+1}, \dots, n_m; \beta) := I_\beta(T_{p+1}^{n_{p+1}} \cdots T_m^{n_m})$

This # is non-zero $\iff$ codim sums to $\dim \overline{M}_{0,n}(X,\beta)$; in which case it is $\#\{$pointed rational maps meeting n_i general representatives of T_i $\forall\, p+1 \leq i \leq m\}$

$0 \le i,j \le m$ $\qquad g_{ij} := \int_X T_i \cup T_j$

$\{T_i\}$ Schubert basis

$\Rightarrow \forall i \quad \exists! j \quad g_{ij} \neq 0$ in which case $g_{ij} = 1$

define (g^{ij}) to be the inverse of g_{ij}

Now we have:

(44) $$T_i \cup T_j = \sum_{e,f} \int_X (T_i \cup T_j \cup T_e) g^{ef} T_f \overset{(I)}{=} \sum_{e,f} I_0 (T_i T_j T_e) g^{ef} T_f$$

$$g^{ef} = g_{ef} = \int_X T_e \cup T_f$$

T_f-coeff of $T_i \cup T_j$

$$= \int_X T_i \cup T_j \cup T_f = \left(\sum_e \int_X T_i \cup T_j \cup T_e\right)\left(\int_X T_e \cup T_f\right)$$

- usual cohom. ring.

now deform it by deforming β away from 0.

For $\gamma \in A^* X$, define the "potential function"

$$\Phi(\gamma) := \sum_{n \geq 3} \sum_\beta \frac{1}{n!} I_\beta(\gamma^n)$$

enum. info

Lemma 15: $\forall n$ $\exists$ only finitely many effective $\beta \in A_1 X$ s.t. $I_\beta(\gamma^n) \neq 0$

consequence: $\gamma = \sum y_i T_i$

$$\Phi(y_0 \dots y_m) := \Phi(\gamma) = \sum_{n_0 + \dots + n_m} \sum_\beta I_\beta (T_0^{n_0} \dots T_m^{n_m}) \frac{y_0^{n_0}}{n_0!} \cdots \frac{y_m^{n_m}}{n_m!}$$

is a formal power series in $\mathbb{Q}[[y_0, \dots, y_m]] =: \mathbb{Q}[[y]]$

$0 \le i,j,k \le m$, define

(48) $$\Phi_{ijk} := \frac{\partial^3 \Phi}{\partial y_i \partial y_j \partial y_k} = \sum_{n \geq 0} \sum_\beta \frac{1}{n!} I_\beta (\gamma^n \cdot T_i T_j T_k)$$

define "quantum product" $*$ by $\quad T_i * T_j := \sum_{e,f} \Phi_{ije} g^{ef} T_f$

Extend $\mathbb{Q}[[y]]$ linearly to $\mathbb{Q}[[y]]$-mod $A^* X \otimes_{\mathbb{Z}} \mathbb{Q}[[y]]$

- Commutative
- unit is $T_0 = 1$ follows from $I - \underline{IV}$

Associative:

$$(T_i * T_j) * T_k = \sum_{e,f} \sum_{c,d} \Phi_{ije}\, g^{ef}\, \Phi_{fkd}\, g^{ld}\, T_d$$

$$T_i * (T_j * T_k) = \sum_{e,f} \sum_{l,d} \Phi_{jke}\, g^{ef}\, \Phi_{ifl}\, g^{el}\, T_d$$

equality follows from Keel relations

Finally define QH^*X quantum cohomology ring

- $V = \mathrm{Grp}(A^*X)$, $\mathbb{Q}[[V^*]] :=$ completion of the graded poly ring $\bigoplus_{i=0}^{\infty} \mathrm{Sym}(V^*) \otimes \mathbb{Q}$ at unique max. ideal

$*$ defines canon. ring structure on free $\mathbb{Q}[[V^*]]$-module $\mathbb{Q}[[V^*]] \underset{\mathbb{Z}}{\otimes} V$ (vector fields on A^*X?)

(Big) quantum cohom. ring is $QH^*X = (V \underset{\mathbb{Z}}{\otimes} \mathbb{Q}[[V^*]], *)$

Compute N_d using q.c.r.

$$\Phi(y_0, \dots, y_m) = \underset{\beta=0}{\Phi_{classical}(y)} + \underset{\beta \neq 0}{\Phi_{quantum}(y)}$$

$$\Phi_{classical}(y) = \sum_{n_0+\dots+n_m=3} \left(\int_X T_0^{n_0} \cup \dots \cup T_m^{n_m} \right) \frac{y_0^{n_0}}{n_0!} \cdots \frac{y_m^{n_m}}{n_m!}$$

replace $\Phi_{quantum}(y)$ by $\Gamma(y) = \sum_{n_{p+1}+\dots+n_m \geq 0} \sum_{\beta \neq 0} N(n_{p+1}, \dots, n_m, \beta)\, e^{(\int_\beta T_1) y_1} \cdots e^{(\int_\beta T_p) y_p} \frac{y_{p+1}^{n_{p+1}}}{n_{p+1}!} \cdots \frac{y_m^{n_m}}{n_m!}$

Case: $X = \mathbb{P}^2$, $T_0 = 1$, $T_1 = [\text{line}]$, $T_2 = [pt]$

$$T_i * T_j = \Phi_{ij0} T_2 + \Phi_{ij1} T_1 + \Phi_{ij2} T_2$$

$$\begin{cases} T_1 * T_1 = T_2 + \Gamma_{111} T_1 + \Gamma_{112} T_0 \\ T_1 * T_2 = \Gamma_{121} T_1 + \Gamma_{122} T_0 \end{cases}$$

$$T_2 * T_2 = \Gamma_{221} T_1 + \Gamma_{222} T_0$$

$$(T_1 * T_1) * T_2 = T_1 * (T_1 * T_2) \Rightarrow \Gamma_{222} = \Gamma_{112}^2 - \Gamma_{122}\Gamma_{111}$$

when $\beta = d[\text{line}]$, $N(n, \beta) \neq 0$ only when $n = 3d-1$

in which case it is N_d

$$\Gamma(y) = \sum_{d \geq 1} N_d \, e^{d y_1} \frac{y_2^{3d-1}}{(3d-1)!}$$

compute Γ_{ijk} and use (58)

$\Rightarrow$ $d \geq 2$

$$N_d = \sum_{d_1 + d_2 = d} N_{d_1} N_{d_2} \left[d_1^2 d_2^2 \binom{3d-4}{3d_1-2} - d_1^3 d_2 \binom{3d-4}{3d_1-1} \right]$$

(59)