Moment Maps and Symplectic Reduction

Karim J. Boustany

1 The Hamiltonian Framework

Throughout this section we fix a symplectic manifold (M, ω). If $f \in \mathcal{C}^\infty(M)$, the **Hamiltonian vector field of f**, denoted by X_f, is defined by

$$\iota_{X_f}\omega = df.$$

In local Darboux coordinates (x^i, y^i), this yields the following coordinate formula:

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y^i}\frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial}{\partial y^i} \right).$$

1.1 Proposition. *If $f \in \mathcal{C}^\infty(M)$, then f is constant along each integral curve of X_f.*

Proof. Let $\theta \colon \mathcal{D} \to M$ be the flow of X_f and $p \in M$. Then

$$(f \circ \theta^{(p)})'(t) = df_p(\dot{\theta}^{(p)}(t)) = df_p((X_f)_p) = \omega_p(X_f, X_f) = 0. \qquad \square$$

If $f, g \in \mathcal{C}^\infty(M)$, we define their **Poisson bracket**, denoted by $\{f, g\}$, as $\{f, g\} = \omega(X_f, X_g)$. The Poisson bracket then makes $\mathcal{C}^\infty(M)$ into a Lie algebra. Now $X \in \mathfrak{X}(M)$ is said to be

- **symplectic** if ω is invariant under the flow of X, and we then write $X \in \mathfrak{X}_S(M)$,
- **Hamiltonian** if there is some $f \in \mathcal{C}^\infty(M)$ such that $X = X_f$, and we then write $X \in \mathfrak{X}_H(M)$.

Both $\mathfrak{X}_S(M)$ and $\mathfrak{X}_H(M)$ turn out to be Lie algebras under the usual Lie bracket of vector fields. We also recall Cartan's magic formula:

$$\mathcal{L}_X\,\omega = \iota_X d\omega + d(\iota_X\omega) = d(\iota_X\omega).$$

We can then easily deduce the following two facts:

i. $X \in \mathfrak{X}_S(M)$ if and only if $\iota_X\omega$ is closed.
ii. $X \in \mathfrak{X}_H(M)$ if and only if $\iota_X\omega$ is exact.

It follows that $\mathfrak{X}_H(M) \subset \mathfrak{X}_S(M)$, and we have the following short exact sequence of Lie algebras:

$$0 \longrightarrow \mathfrak{X}_H(M) \longrightarrow \mathfrak{X}_S(M) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(M) \longrightarrow 0,$$

where $\mathrm{H}^1_{\mathrm{dR}}(M)$ is viewed as a trivial Lie algebra and the map $\mathfrak{X}_S(M) \to \mathrm{H}^1_{\mathrm{dR}}(M)$ is $X \mapsto [\iota_X\omega]$.

This map can be seen to be a Lie algebra homomorphism by using the Cartan equations.

1.2 Proposition. *If* $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$*, then* $\mathfrak{X}_S(M) = \mathfrak{X}_H(M)$.

Proof. This is immediate in view of the above short exact sequence. □

For any $H \in \mathcal{C}^\infty(M)$, we will use the following terminology:

- The triple (M, ω, H) is called a **Hamiltonian system**.
- The function H is called the **Hamiltonian** of the system.
- The flow of X_H is called the **Hamiltonian flow** of the system.
- The integral curves of X_H are called the **trajectories** or **orbits** of the system.

In Darboux coordinates, the coordinate formula for X_H implies that a trajectory $\gamma(t) = (x^i(t), y^i(t))$ satisfies the following equations:

$$\begin{aligned} \dot{x}^i(t) &= \frac{\partial H}{\partial y^i}(x(t), y(t)), \\ \dot{y}^i(t) &= -\frac{\partial H}{\partial x^i}(x(t), y(t)). \end{aligned}$$

These are known classically as **Hamilton's equations**.

A function $f \in \mathcal{C}^\infty(M)$ is called a **conserved quantity** of the system if it is constant on its trajectories. A vector field $V \in \mathfrak{X}(M)$ is called an **infinitesimal symmetry** of the system if both ω and H are invariant under the flow of V. We have the following proposition.

1.3 Proposition.

(a) $f \in \mathcal{C}^\infty(M)$ *is a conserved quantity if and only if* $\{f, H\} = 0$.

(b) $X \in \mathfrak{X}(M)$ *is an infinitesimal symmetry if and only if* X *is symplectic and* $XH = 0$.

Proof. These are straightforward to prove and are left as an exercise. □

1.4 Theorem (Noether's Theorem).

(a) *If* $f \in \mathcal{C}^\infty(M)$ *is a conserved quantity, then* X_f *is an infinitesimal symmetry.*

(b) *If* $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$*, then each infinitesimal symmetry is the Hamiltonian vector field of a conserved quantity, which is unique up to a locally constant function.*

Proof. To prove (a), note that if X_f is Hamiltonian, and hence symplectic. Also, since f is a conserved quantity, we have $0 = \{f, H\} = \{H, f\} = X_f H$. Therefore X_f is an infinitesimal symmetry.

To prove (b), assume that X is an infinitesimal symmetry. Then X is symplectic and therefore Hamiltonian since $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$. So there is some $f \in \mathcal{C}^\infty(M)$ such that $X = X_f$. Then we also have $\{f, H\} = -\{H, f\} = -X_f H = 0$, by our characterization of infinitesimal symmetries. It g is any other function which satisfies $X_g = X$, then $d(g - f) = \iota_{X_g - X_f}\omega = 0$, so that $g - f$ must be a locally constant function on M. □

Observe that since $\{H, H\} = 0$, the Hamiltonian H is itself a conserved quantity of the system. The corresponding infinitesimal symmetry is just X_H. Since H is usually interpreted as an energy function of a physical system, the fact that it is conserved is classically called **conservation of energy**.

2 Lie Theory

If G is a Lie group, the **Lie algebra of G**, denoted by $\mathfrak{g}$, is the Lie algebra of all left-invariant vector fields on G, and this is canonically isomorphic to T_eG via the isomorphism $X \mapsto X_e$. If $X \in \mathfrak{g}$, the maximal integral curve of X starting at the identity (which is necessarily defined for all time since left-invariant vector fields are complete) is called the **one-parameter subgroup generated by X**.

The **exponential map of G** is the map $\exp\colon \mathfrak{g} \to G$ defined by $\exp X = \gamma(1)$, where γ is the integral curve of X starting at the identity. It can then be shown that the curve $\gamma\colon \mathbb{R} \to G$ given by $\gamma(t) = \exp tX$ is exactly the one-parameter subgroup generated by X.

If M is a smooth manifold, a **left G-action on M** is a homomorphism $\varphi\colon G \to \mathrm{Diffeo}(M)$, denoted as $g \mapsto \varphi_g$. We will then say the action φ is **smooth** if the map $G \times M \to M$ given by $(g, p) \mapsto \varphi_g(p)$ is smooth. If $X \in \mathfrak{g}$ generates the one-parameter subgroup $\exp tX$, we can define a smooth vector field $X^\sharp$ on M by

$$X^\sharp_p = \left.\frac{d}{dt}\right|_{t=0} \varphi_{\exp tX}(p).$$

For each $g \in G$, we have the conjugation mapping $C_g : G \to G$ defined as $C_g(h) = ghg^{-1}$. We then define the **adjoint representation of G** to be the homomorphism $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$ given by $g \mapsto \mathrm{Ad}_g$, where

$$\mathrm{Ad}_g = (dC_g)_e\colon \mathfrak{g} \to \mathfrak{g}.$$

We then define the **coadjoint representation of G** to be the homomorphism $\mathrm{Ad}^*\colon G \to GL(\mathfrak{g}^*)$ given by $g \mapsto \mathrm{Ad}^*_g$, where

$$\mathrm{Ad}^*_g = (\mathrm{Ad}_{g^{-1}})^*\colon \mathfrak{g}^* \to \mathfrak{g}^*.$$

3 Moment Maps

Throughout this section we fix a symplectic manifold (M, ω) and a connected Lie group G. We say a left G-action $\varphi\colon G \to \mathrm{Diffeo}(M)$ is **symplectic** if $\varphi(G) \subset \mathrm{Symp}(M)$. This means that the diffeomorphism $\varphi_g : M \to M$ is a symplectomorphism for each $g \in G$.

A **moment map for G** is a smooth map $\mu\colon M \to \mathfrak{g}^*$ satisfying the following properties:

i. If $\mu^*(X)\colon M \to \mathbb{R}$ is the map $\mu^*(X)(p) = \mu(p)(X)$ with $X \in \mathfrak{g}$, then

$$d\mu^*(X) = \iota_{X^\sharp}\omega.$$

ii. μ is equivariant with respect to the coadjoint action on $\mathfrak{g}^*$, so that for all $g \in G$, we have

$$\mu \circ \varphi_g = \mathrm{Ad}^*_g \circ \mu.$$

If such a map exists, we say φ is a **Hamiltonian group action** and call the quadruple (M, ω, G, μ) a **Hamiltonian G-space**. From the moment map, we can construct the corresponding **comoment map**, which is the linear map $\mu^*\colon \mathfrak{g} \to C^\infty(M)$ given by

$$\mu^* : X \mapsto \mu^*(X).$$

It can then be shown that the defining properties of μ imply the following two properties for μ^*:

i. $\mu^*(X)$ is a Hamiltonian function for $X^\sharp$.

ii. μ^* is a Lie algebra homomorphism:

$$\mu^*[X,Y] = \{\mu^*(X), \mu^*(Y)\},$$

In fact, the two pairs of properties are equivalent, which is to say that moment maps and comoment maps encode the same data. (This relies on our assumption that G is connected.)

Any Hamiltonian group action is symplectic by definition, but we also have the following useful criterion for determining when the converse is true when M is compact and connected. Suppose that every symplectic vector field on M was Hamiltonian, and fix a basis $\{X_1, \dots, X_k\}$ of $\mathfrak{g}$. Then for each such X_i, the vector field $X_i^\sharp$ is symplectic, and by assumption we can find a smooth function $\mu^*(X_i)$ on M for which

$$\iota_{X_i^\sharp}\omega = d\mu^*(X_i).$$

The $\mu^*(X_i)$ are only unique up to a constant, but we can fix said constant by requiring that

$$\int_M \mu^*(X_i)\omega^n = 0.$$

Extending by linearity, we obtain a linear map $\mu^* \colon \mathfrak{g} \to \mathcal{C}^\infty(M)$ with

$$\iota_{X^\sharp}\omega = d\mu^*(X).$$

The integration condition above actually shows that μ^* is a Lie algebra homomorphism.

3.1 Theorem. *If $\mathrm{H}^1_{\mathrm{dR}}(M) = 0$, then every symplectic G-action on M is Hamiltonian. This is true in particular when M is simply connected.*

3.2 Theorem (Generalized Noether's Theorem). *A function $f \in \mathcal{C}^\infty(M)$ is G-invariant if and only if the moment map μ is constant on the integral curves of X_f.*

Proof. Observe that μ being constant on the integral curves of X_f is equivalent to $\mu^*(X)$ being constant on the integral curves of X_f for each $X \in \mathfrak{g}$. But then we have

$$\begin{aligned}
\mathcal{L}_{X_f}\, \mu^*(X) &= \iota_{X_f} d\mu^*(X) \\
&= \iota_{X_f}\iota_{X^\sharp}\omega \\
&= -\iota_{X^\sharp}\iota_{X_f}\omega \\
&= -\iota_{X^\sharp} df \\
&= -\mathcal{L}_{X^\sharp}\, f.
\end{aligned}$$

Since G is connected, the vanishing of the final expression is equivalent to the G-invariance of f. The theorem now follows immediately from this equality. □

Observe that a Hamiltonian system (M, ω, H) is a trivial example of a Hamiltonian $\mathbb{R}$-space when X_H is complete. The $\mathbb{R}$-action in this case is just action by the flow of X_H, and the moment map is just H itself. An $\mathbb{R}$-invariant function is then the same as a conserved quantity, and H being constant on the integral curves of X_f is the same as X_f being an infinitesimal symmetry. So the previous theorem does indeed generalize Noether's theorem as presented previously.

3.3 Example. Consider a particle moving in $\mathbb{R}^3$, and let (x, ξ) be coordinates on $T^*\mathbb{R}^3 = \mathbb{R}^6$, which is given the standard symplectic form. Suppose we are given a Hamiltonian $H : \mathbb{R}^6 \to \mathbb{R}$ which describes the dynamics of the particle. We will now look at two group actions on $\mathbb{R}^6$:

(a) Consider the action of $\mathbb{R}^3$ on $\mathbb{R}^6$ by $(a, (x, \xi)) \mapsto (x + a, \xi)$ for $a \in \mathbb{R}^3$. Then one can show that the corresponding moment map $\mu : \mathbb{R}^6 \to \mathbb{R}^3$ is given by $\mu(x, \xi) = \xi$. Classically, this is called the **linear momentum**. If H is invariant under this action, then Noether's theorem implies that the linear momentum is invariant under translation in the space variable. This is known as **conservation of linear momentum**.

(b) Consider the action of SO(3) on $\mathbb{R}^6$ by $(A, (x, \xi)) \mapsto (Ax, A\xi)$. The corresponding Lie algebra $\mathfrak{so}(3)$ consists of skew-symmetric 3×3 the moment map μ is given by $\mu(x, \xi)(A) = \langle \xi, Ax \rangle$. Under the identification of $\mathfrak{so}(3)$ with $\mathbb{R}^3$ with Lie bracket given by the cross product, the moment map becomes $\mu(x, \xi) = x \times \xi$. Classically, this is called the **angular momentum**. If H is invariant under this action, then Noether's theorem implies that the angular momentum is invariant under rotation in space. This is known as **conservation of angular momentum**.

4 Symplectic Reduction

We let (M, ω, G, μ) be a Hamiltonian G-space. For $\xi \in \mathfrak{g}^*$, let G_ξ be the isotropy subgroup of G under the coadjoint action. Since μ is equivariant under the coadjoint action, the quotient space $M_\xi = \mu^{-1}(\xi)/G_\xi$ is well defined. Denote by $\pi_\xi : \mu^{-1}(\xi) \to M_\xi$ and $\iota_\xi : \mu^{-1}(\xi) \to M$ the canonical projection and inclusion maps, respectively. We then have the following result.

4.1 Theorem (Marsden-Weinstein Theorem). *Suppose that $\xi \in \mathfrak{g}^*$ is a regular value of μ and that G_ξ acts freely and properly on $\mu^{-1}(\xi)$. Then M_ξ is a smooth manifold of dimension* $\dim M - \dim G - \dim G_\xi$*, and there is a symplectic form ω_ξ on M_ξ such that $\iota_\xi^*\omega = \pi_\xi^*\omega_\xi$. The pair (M_ξ, ω_ξ) is called the* ***symplectic reduction of (M, ω) with respect to (G, μ) at the level ξ***.

Note that if G is abelian, then the coadjoint action is trivial and hence $G_\xi = G$ for all $\xi \in \mathfrak{g}^*$. Therefore if G acts freely and properly on $\mu^{-1}(\xi)$, then M_ξ will have dimension $\dim M - 2\dim G$.

4.2 Example. We have observed that a Hamiltonian system (M, ω, H) for which X_H is complete vector field is a Hamiltonian $\mathbb{R}$-space with moment map H. If $p \in \mathbb{R}$ is any regular value of H, then $H^{-1}(e)/\mathbb{R}$ is a symplectic reduction of dimension $\dim M - 2$. It is called the **manifold of solutions of constant energy**.

4.3 Example. If $M = T^*\mathbb{R}^{3k}$, say the phase space of a system of k particles in $\mathbb{R}^3$, and $G = \mathbb{R}^3$ with linear momentum, then for any $c \in \mathbb{R}^3$, the reduction $T^*\mathbb{R}^{3k}/\mathbb{R}^3$ is the process of switching to the center of mass reference frame, which in effect allows the study of the dynamics of the system of particles while ignoring the momentum. In this way the degrees of freedom of the system of reduced.

4.4 Example. If $M = T^*\mathbb{R}^3$ and $G = \mathrm{SO}(3)$ with angular momentum, then for $\xi \in \mathfrak{so}(3) \cong \mathbb{R}^3$ with $\xi \neq 0$, we have $G_\xi = \mathbb{S}^1$, which correspond to rotations about the axis ξ. The reduction $\mu^{-1}(\xi)/\mathbb{S}^1$ is a generalization of a procedure in celestial mechanics called **elimination of nodes**.

4.5 Theorem. *With the assumptions of the Marsden-Weinstein theorem, suppose that $H \in \mathcal{C}^\infty(M)$ is a G-invariant Hamiltonian. Then the flow θ of X_H leaves $\mu^{-1}(\xi)$ invariant and commutes with the*

G_ξ-action, so it descends to a flow η on M_ξ satisfying $\pi_\xi \circ \theta_t = \eta_t \circ \pi_\xi$. This flow is the Hamiltonian flow on M_ξ of a Hamiltonian H_ξ which satisfies $H_\xi \circ \pi_\xi = H \circ \iota_\xi$. This is called the ***reduced Hamiltonian****.*

5 Existence and Uniqueness

Assume for this section that:

- (M, ω) is a connected symplectic manifold.
- G is a connected Lie group.

A symplectic action $\varphi\colon G \to \mathrm{Symp}(M)$ yields a Lie algebra anti-homomorphism $d\varphi\colon \mathfrak{g} \to \mathfrak{X}_S(M)$ given by $X \mapsto X^\sharp$, since each $X^\sharp$ is a symplectic vector field. Now since G is connected, the action φ is Hamiltonian if and only if there is a Lie algebra homomorphism $\mu^*\colon \mathfrak{g} \to \mathcal{C}^\infty(M)$ such that the following diagram commutes:

$$\begin{tikzcd} & \mathcal{C}^\infty(M) \arrow[d, "\alpha"] \\ \mathfrak{g} \arrow[ur, "\mu^*"] \arrow[r, "d\varphi"'] & \mathfrak{X}_S(M) \end{tikzcd}$$

Of course, μ^* is nothing but the comoment map. We now digress to talk about Lie algebra cohomology.

Let $\mathfrak{g}$ be a Lie algebra and let $C^k = \Lambda^k \mathfrak{g}^*$. Then it can be shown that there is a linear operator $\delta\colon C^k \to C^{k+1}$ such that $\delta^2 = 0$. We thus have a cochain complex whose cohomology, denoted by $\mathrm{H}^*(\mathfrak{g}; \mathbb{R})$, is called the **Lie algebra cohomology** of $\mathfrak{g}$.

5.1 Theorem. *If G is compact, then $\mathrm{H}^k(\mathfrak{g}; \mathbb{R}) = \mathrm{H}^k_{\mathrm{dR}}(G)$.*

The **commutator ideal** of $\mathfrak{g}$ is

$$[\mathfrak{g}, \mathfrak{g}] = \mathrm{Span}\{[X, Y] \mid X, Y \in \mathfrak{g}\}.$$

It can then be shown that $\mathrm{H}^1(\mathfrak{g}; \mathbb{R})$ is the space of those $c \in \mathfrak{g}^*$ for which $c|_{[\mathfrak{g},\mathfrak{g}]} = 0$.

We can now answer the question of uniqueness of moment maps when G is compact. For if μ_1^* and μ_2^* are both comoment maps for the same symplectic action $\varphi\colon G \to \mathrm{Symp}(M)$, then by definition, for every $X \in \mathfrak{g}$ the maps $\mu_1^*(X)$ and $\mu_2^*(X)$ are both Hamiltonian functions for the vector field $X^\sharp$. We conclude that

$$\mu_1^*(X) - \mu_2^*(X) = C^X$$

is a constant function on M, and so the mapping $X \mapsto C^X$ defines an element $C \in \mathfrak{g}^*$. Therefore,

$$\mu_1 = \mu_2 + C,$$

and using the fact that μ_1^* and μ_2^* are Lie algebra homomorphisms, it can be shown that $C([X, Y]) = 0$ for all $X, Y \in \mathfrak{g}$. Thus $C \in \mathrm{H}^1(\mathfrak{g}; \mathbb{R})$, and we have therefore proved the following result.

5.2 Theorem. *If $\mathrm{H}^1(\mathfrak{g}; \mathbb{R}) = 0$, then moment maps for Hamiltonian G-actions are unique.*

5.3 Theorem. *Let G be a connected Lie group with* $\mathrm{H}^1(\mathfrak{g};\mathbb{R}) = \mathrm{H}^2(\mathfrak{g};\mathbb{R}) = 0$. *Then every symplectic G-action is Hamiltonian.*

This is realized in case where G is **semisimple**, that is, compact and $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

5.4 Proposition. *Let G be a compact Lie group. Then G is semisimple if and only if* $\mathrm{H}^1(\mathfrak{g};\mathbb{R}) = \mathrm{H}^2(\mathfrak{g};\mathbb{R}) = 0$.