Symplectic Geometry Talk for the Intermediate Geometry and Topology Seminar

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Symplectic Linear Algebra

Let V be an m-dimensional real vector space and let $\omega : V \times V \to \mathbb{R}$ be an skew-symmetric bilinear map. We can also view ω as a map from V to V^* by

$$v \mapsto \omega(v, -)[w \mapsto \omega(v, w)]$$

Example 1. Let $V = \mathbb{R}^{2n}$ let $(x_1, ..., x_n, y_1, ..., y_n)$ be a basis for V. Where $x_i = (0, ..., 1, ...0)$ has a 1 the i^{th} position and $y_i = (0, ..., 1, ...0)$ has a 1 the $n + i^{\text{th}}$ position. Let $(dx_1, ..., dx_n, dy_1, ..., dy_n)$ be the corresponding dual basis. The map

$$\omega_1 = \sum_{i=1}^{n} dx^i \wedge dy^i$$

is skew-symmetric and bilinear. We will call $(\mathbb{R}^{2n}, \omega_1)$ the standard symplectic space.

Example 2. If $dim(V) = 2n$, let $(a_1, b_1, ..., a_n, b_n)$ be a basis for V with corresponding dual basis $(\alpha^1, \beta^1, ..., \alpha^n, \beta^n)$ for V^*. Then define ω by

$$\omega = \sum_{i=1}^{n} \alpha^i \wedge \beta^i$$

and note that the following conditions are satisfied: $\omega(a_i, a_j) = 0 = \omega(b_i, b_j)$ for all i, j from 1 to n and furthermore $\omega(a_i, b_j) = \delta_{i,j} = -\omega(b_j, a_i)$.

Associated to skew symmetric bilinear map we have the following subspace $U = \{u \in V | \ \omega(u, v) = 0, \ \forall v \in V\}$. Note that in the previous examples $U = \{0\}$.

Proposition 1. Let V be a vector space with an skew-symmetric bilinear map to $\mathbb{R}$ called ω. Let $u_1, ..., u_k$ be a basis for U. This can be extended to a basis of the whole space V by $u_1, ..., u_k, e_1, f_1, ..., e_n, f_n$ such that $\omega(e_i, e_j) = 0 = \omega(f_i, f_j)$ for all i, j from 1 to n and $\omega(e_i, f_j) = \delta_{i,j} = -\omega(f_j, e_i)$.

Proposition 2. The linear map $\omega : V \to V^*$ is an isomorphism if and only if $U = \{0\}$.

Suppose v is in the kernel of this map, then it must be in U so the injectivity part of the statement is clear as is the forward direction. For surjectivity we will need the following proposition showing there is a standard form for such skew symmetric bilinear maps.

We say that ω is symplectic if ω is nondegenerate, that is if $\omega : V \to V^*$ is an isomorphism, equivalently if $U = \{0\}$. In this case the dimension of V is always even.

Let $U = \{0\}$. Then $\omega(f_i, -)$ is the dual vector associated to e_i and $\omega(e_i, -)$ is the dual vector associated to f_i. To show the surjectivity of $\omega : V \to V^*$, let $\phi \in V^*$. It is simple to verify that $\phi(-) = \omega(v, -)$ where

$$v = \sum_{i=1}^{n} \phi(e_i) f_i + \phi(f_i) e_i.$$

If $S \subset V$ define the symplectic complement of S denoted S^ω by

$$S^\omega = \{v \in V |\ \omega(v, w) = 0,\ \forall w \in S\}.$$

Let S be a subspace of V. Applying the Rank-Nullity theorem to the map $\omega : V \to S*$ by sending v to $\omega(v, -)|_S$ we obtain the following result.

Proposition 3. $dim(V) = dim(S) + dim(S^\omega)$.

We will say that if S is a subspace of a symplectic vector space then S is symplectic if $S \cap S^\omega = \{0\}$, that is S is non-degenerate. S is isotropic if $S \subset S^\omega$, e.g. any collection of purely a_i or purely b_i, that is $\omega|S = 0$. Lagrangian if $S = S^\omega$, i.e. Span$(a_1, ..., a_n)$ or Span$(b_1, ..., b_n)$. Note that for a Lagrangian subspace S, $dim(S) = \frac{1}{2} dim(V)$. Co-isotropic if S^ω is isotropic, e.g. Span$(a_1, ..., a_n, b_1, b_2, b_3)$.

Proposition 4. If $dim(V) = 2n$ then ω is symplectic if and only if the n-fold wedge product ω^n is non-zero. The top degree form $\frac{1}{n!}\omega^n$ can serve as a volume form and it is called the Liouville form on V.

Let $(V, \omega$ and (W, η) be symplectic vector spaces and $L : V \to W$ be a linear isomorphism. Then L is called a symplectomorphism if $L^*\eta = \omega$. By proposition 1 every symplectic vector space is symplectomorphic to $(\mathbb{R}^{2n}, \omega_1)$ the standard symplectic vector space of example 1.

Symplectic Manifolds

Let M be a smooth manifold. Let ω be a closed 2-form on M such that ω_p is symplectic on T_pM for all $p \in M$. We call (M, ω) a symplectic manifold and ω the associated symplectic form on M. From the previous section we know M must have even dimension and must be orientable.

Example 3. Let $M = \mathbb{R}^{2n}$ with linear coordinates $(x_1, ..., x_n, y_1, ..., y_n)$. The form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$$

is symplectic and the set $\{(\frac{\partial}{\partial x_1})_p, ..., (\frac{\partial}{\partial x_n})_p, (\frac{\partial}{\partial y_1})_p, ..., (\frac{\partial}{\partial y_n})_p\}$ is a symplectic basis of T_pM

Example 4. Let $M = \mathbb{C}^n$ with coordinates $z_1, ..., z_n$ then

$$\omega = \frac{i}{2} \sum_{i=1}^{n} dz_k \wedge d\bar{z}_k$$

is symplectic.

Example 5. Let $M = S^2 \subset \mathbb{R}^3$. Let p be a point on S^2. Then $\omega_p(u, v) = \langle p, u \times v \rangle$ for $u, v \in T_pS^2$ is symplectic.

Let (M, ω) be a symplectic manifold and $F : N \to M$ be a smooth immersion. Then we say F is symplectic, isotropic, co-isotropic or Lagrangian if the subspace $dF_p(T_pN) \subset T_{F(p)}M$ has the corresponding property for each $p \in N$.

Let (M_1, ω_1) and (M_2, ω_2) be $2n$-dimensional manifolds and $\varphi : M_1 \to M_2$ be a diffeomorphism. We say φ is a symplectomorphism if $\varphi^*\omega_2 = \omega_1$.

Symplectic Structure on the Cotangent Bundle

Theorem 1. If M is a smooth manifold, then there exists a canonical 1-form τ on the cotangent bundle T^*M such that $-d\tau$ is a symplectic form on T^*M. Furthermore this 1-form has the following property $\forall \sigma \in \Omega^1(M)$ $\sigma^*\tau = \sigma$.

If $\pi : T^*M \to M$ is the usual projection then $d\pi : T(T^*M) \to TM$ and for any $(p, \phi) \in T^*M$ we have a pointwise pullback $d\pi^*_{(p,\phi)} : T^*M \to T^*(T^*M)$.

Define $\tau : T^*M \to T^*(T^*M)$ by

$$\tau_{(p,\phi)} = d\pi^*_{(p,\phi)}(\phi)$$

so that for $v \in T_{(p,\phi)}(T^*M)$ we have

$$\tau_{(p,\phi)}(v) = \phi(d\pi_{(p,\phi)}(v)).$$

In local coordinates $\phi = \xi_i dx_i \in T^*M$, $v = (\sum \alpha_i \frac{\partial}{\partial x_i} + \sum \beta_i d\frac{\partial}{\partial \xi_i}) \in T(T^*M)$ and $d\pi(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}$ and $d\pi(\frac{\partial}{\partial \xi_i}) = 0$ for all i.

Thus $\tau_{(p,\phi)}(v) = \xi_i dx_i(d\pi(\sum \alpha_i \frac{\partial}{\partial x_i} + \sum \beta_i d\frac{\partial}{\partial \xi_i})) = \alpha_i \xi_i$. Note that $\alpha_i \xi_i = \xi_i dx_i(\sum \alpha_i \frac{\partial}{\partial x_i} + \sum \beta_i d\frac{\partial}{\partial \xi_i})$ and so locally $\tau_{(p,\xi_i dx_i)} = \xi_i dx_i$. Furthermore,

$$-d\tau_{(p,\xi_i dx_i)} = \sum_{i=1}^{n} dx_i \wedge d\xi_i.$$

The result that $\forall \sigma \in \Omega^1(M)$ $\sigma^*\tau = \sigma$ is left as an exercise. Hint: σ is a section of π.

Construction and Application of Lagrangian Submanifolds

Proposition 5. Let M be a smooth n-manifold and $\sigma \in \Omega^1(M)$. Then $\sigma(M)$ is a Lagrangian sub-manifold of T^*M if and only if σ is closed.

Proof. Since $dim(M) = n = \frac{1}{2}dimT^*M$ $\sigma(M)$ is Lagrangian if and only if $\sigma(M)$ is isotropic, meaning $\sigma^*\omega = \omega|_{\sigma(M)} = 0$. But

$$\sigma^*(\omega) = -\sigma^* d\tau = -d(\sigma^*\tau) = -d\sigma$$

so $\sigma(M)$ is Lagrangian if and only if σ is closed. □

Example 6. Let σ_0 be the section sending every point to the zero vector. Clearly, $\sigma_0^*\omega = 0$ and so $\sigma_0(M)$ must be Lagrangian. This is called the zero section.

Let S be a k dimensional submanifold of an n dimensional manifold X. The conormal space at $x \in S$ is defined to be

$$N_x^*S = \{\xi \in T_x^*X | \xi(v) = 0, \ \forall v \in T_xS\}.$$

The conormal bundle is

$$N^*S = \{(x,\xi) \in T^*X | x \in S, \xi \in N_x^*S\}$$

Proposition 6. Let $i : N^*S \to T^*X$ be the inclusion, and let τ be the tautological 1-form on T^*X. Then $i^*\tau = 0$.

As a corollary we have that for any submanifold $S \subset X$, the conormal bundle N^*S is a Lagrangian submanifold of T^*X.

The notion of a lagrangian submanifold will also give us a method to discern whether a given diffeomorphism is a symplectomorphism.

Let (M_1, ω_1) and (M_2, ω_2) be two $2n$-dimensional symplectic manifolds. Given a diffeomorphism $\varphi : M_1 \to M_2$ does $\varphi^*\omega_2 = \omega_1$?

Use the projection maps $p_1 : M_1 \times M_2 \to M_1$ and $p_2 : M_1 \times M_2 \to M_2$ to creat a form on $M_1 \times M_2$ given by

$$\omega = p_1^*\omega_1 + p_2^*\omega_2$$

which is closed since differential commutes with pullback and symplectic which you can check by showing $\omega^2 n \neq 0$ using proposition 4. More generally

$$\omega = \lambda_1 p_1^*\omega_1 + \lambda_2 p_2^*\omega_2$$

is a symplectic form for all $\lambda_1, \lambda_2 \in \mathbb{R}^\times$. In particular examine the twisted product

$$\widetilde{\omega} = p_1^*\omega_1 - p_2^*\omega_2$$

We define the graph of a diffeomorphism $\varphi : M_1 \to M_2$ as follows

$$\Gamma_\varphi := \{(p, \varphi(p)) | p \in M_1\}$$

The submanifold Γ_φ is an embedded image of M_1 in $M_1 \times M_2$, the embedding being the map $\gamma : M_1 \to M_1 \times M_2$ by $(p \mapsto (p, \varphi(p)))$

Proposition 7. The diffeomorphism $\varphi : M_1 \to M_2$ is a symplectomorphism if and only if Γ_φ is a Lagrangian submanifold of $M_1 \times M_2$.

Proof. The graph is Lagrangian if and only if $\gamma^*\widetilde{\omega} = 0$ but

$$\gamma^*\widetilde{\omega} = \gamma^* p_1^*\omega_1 + \gamma^* p_2^*\omega_2 = (p_1 \circ \gamma)^*\omega_1 - (p_2 \circ \gamma)^*\omega_2$$

but since $p_1 \circ \gamma$ is the identity on M_1 and $p_2 \circ \gamma$ is exactly φ we have

$$\gamma^*\widetilde{\omega} = \omega_1 - \varphi^*\omega_2.$$

note that φ is a symplectomorphism if and only if $\omega_1 - \varphi^*\omega_2 = 0$. $\square$

Darboux's Theorem and Moser's Trick

Just as any n-dimensional manifold looks locally like $\mathbb{R}^n$ any $2n$-dimensional symplectic manifold looks locally like $(\mathbb{R}^{2n}, \omega_1)$.

Theorem 2. (Darboux) Let (M, ω_0) be a 2n-dimensional symplectic manifold, and let $p \in M$. Then there is a coordinate chart $(U_0, x_1, ..., x_n, y_1, ..., y_n)$ centered at p_0 such that on U_0

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

A chart with this property is called a Darboux chart and $x_1, ..., x_n, y_1, ..., y_n$ are called Darboux coordinates.

Proof. (sketch) Let p be an arbitrary point of M. Let $p_0 = \varphi(p)$ We need a chart (U_0, φ) such that $\varphi^*(\omega_1) = \omega_0$ where ω_1 is the standard form on $\mathbb{R}^{2n}$. This is a local question so replace ω_0 with $(\varphi^{-1})^*\omega_0 = \omega_0(\varphi^{-1}(-), \varphi^{-1}(-))$. So ω_1 and ω_0 are both forms on U_0. By a linear change of coordinates we can require that $\omega_1|_{p_0} = \omega_0|_{p_0}$.

Let $\eta = \omega_1 - \omega_0$ since η is closed the poincare lemma says there is a smooth 1-form α on U_0 such that $d\alpha = -\eta$. For each $t \in \mathbb{R}$ define a closed 2-form ω_t by $\omega_t = \omega_0 + t\eta = (1-t)\omega_0 + t\omega_1$.

Because $\omega_t|_{p_0} = \omega_0|_{p_0}$ is non degenerate for all t there is some neighborhood U_1 of p_0 contained in U_0 such that $\omega_t : TU_1 \to T^*U_1$ is an isomorphism for all t.

Define a time-dependent vector field by $V : J \times U_1 \to TU_1$ by $V_t = \omega_t^{-1}(\alpha)$.

Note: $\frac{d\omega_t}{dt} = \omega_1 - \omega_0 = \eta$. Note: $-\eta = d\alpha = d(\omega(V_t, -) = d\iota_{V_t}(\omega_t) = d\iota_{V_t}(\omega_t) + \iota_{V_t}(d\omega_t) = \mathcal{L}_{V_t}\omega_t$. Thus $\mathcal{L}_{V_t}\omega_t + \frac{d\omega_t}{dt} = 0$.

Associated to a time dependent vector field V_t there exists a family of diffeomorphisms $\theta_t : U_1 \to U_1$ called a time dependent flow such that $v_t = \frac{d}{dt}\theta_t^* \circ \theta_t^{-1}$.

Since $0 = \theta_t^*(\mathcal{L}_{V_t}\omega_t + \frac{d\omega_t}{dt}) = \frac{d}{dt}(\theta_t^*(\omega_t))$ then $\theta_t^*(\omega_t) = \theta_0^*(\omega_0) = \omega_0$.
In particular $\theta^*(\omega_1) = \omega_0$. □

References

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